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AN INVARIANTIVE INVESTIGATION OF IRREDUCIBLE BINARY MODULAR FORMS*

BY

LEONARD EUGENE DICKSON

1. A fundamental system of invariants of the group of all binary linear transformations in a finite field is shown in §§ 2–6 to consist of two invariants, one the product of the distinct linear forms and the other the product of the distinct irreducible quadratic forms, where in each case no two factors have a constant ratio. The product π_m of the irreducible forms of degree m can be expressed in terms of the fundamental invariants; this is accomplished in §§ 7–9 by means of the remarkable three-term recursion formula (15).

Two forms shall be said to belong to the same class if and only if one of them can be transformed into a constant multiple of the other by a linear transformation of determinant unity in the field. It is shown in § 10 that there are as many classes of irreducible binary forms of degree m as there are irreducible factors of π_m when expressed as a function of a certain pair of invariants. The choice of the latter is different in the two cases $p = 2$, $p > 2$, where p is the modulus of the field; this is due to the fact that, in the respective cases, there are one or two similarity transformations of determinant unity. The investigation is completed for the values of m less than 8. The difficulties encountered increase as the number of factors of m increases. Certain problems arise for which the present invariantive theory affords an indirect solution, whereas a direct solution appears to be quite difficult (cf. end of § 10, and end of § 17). For $m = 6$, it was necessary to enumerate the irreducible cubics in the $GF[p^n]$ whose roots are squares in the $GF[p^{3n}]$ and have a given sum.

Determination of a Fundamental System of Invariants, §§ 2–6.

2. Let G be a group of finite order g composed of linear homogeneous transformations on $m\,(m > 1)$ variables with coefficients in any given field F. The term point will be used in the sense of homogeneous coördinates, so that $(x_1, \cdots, x_m)$ is identified with $(\mu x_1, \cdots, \mu x_m)$, while $(0, \cdots, 0)$ is excluded.

* Presented to the Society, September 7, 1910.

A transformation which leaves every point unaltered is necessarily a similarity transformation

$$x_i' = ax_i \qquad (i=1, \cdots, m). \tag{1}$$

In fact, by employing the points all but one of whose coördinates are zero, we see that the transformation is of the form

$$x_i' = a_i x_i \qquad (i=1, \cdots, m).$$

Then by employing the points all but two of whose coördinates are zero, we see that the a_i are equal. Let γ be the number of transformations (1) contained in G, and set $\omega = g/\gamma$. We assume that $\omega > 1$. Any point is one of at most ω distinct conjugates under G. A point is called special if it is one of fewer than ω conjugates, that is, if it is invariant under at least one transformation other than (1) of the group G. If a point P is invariant under T and if S replaces P by P', then P' is invariant under $S^{-1}TS$. To determine all special points it therefore suffices to employ a representative of each set of conjugate transformations other than (1), obtain the points invariant under the representative, select the distinct points so obtained, and find the conjugates to each under G.

3. Let G be the group of all binary linear homogeneous transformations of determinant unity in the Galois field $GF[p^n]$ of order p^n. The order of G is

$$g = p^n(p^{2n}-1). \tag{2}$$

Within G any transformation with irreducible characteristic determinant $\Delta(\rho)$ is equivalent to a transformation *

$$T = \begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}, \qquad \Delta(\rho) = \rho^2 - \rho\alpha + 1.$$

The only points invariant under T are $(\rho, 1)$, where $\Delta(\rho) = 0$. The transformations of G which leave $(\rho, 1)$ unaltered are

$$\begin{pmatrix} b & -c \\ c & b - c\alpha \end{pmatrix}, \qquad b^2 - \alpha bc + c^2 = 1,$$

namely, the transformations commutative with T. The quadratic condition may be written in the form

$$(b - \rho c)^{p^n+1} = 1,$$

where ρ is a root of $\Delta(\rho) = 0$, and hence has $p^n + 1$ sets of solutions in the $GF[p^n]$. Thus $(\rho, 1)$ is one of $p^{2n} - p^n$ conjugate points under G. The resulting system of special points is therefore $(e\rho + f, 1)$, where e is any element

* These Transactions, vol. 2 (1901), p. 117.

$\neq 0$ of the $GF[p^n]$ and f is any element. Such a point together with $(e\bar{\rho}+f, 1)$, where $\bar{\rho}$ is the second root of $\Delta(\rho)=0$, determine a quadratic form $x^2+\cdots$ which vanishes only for these two points. Hence a relative invariant under G is given by the product Q of the $\frac{1}{2}(p^{2n}-p^n)$ irreducible quadratic forms $x^2+\cdots$. We have *

$$Q=\frac{x^{p^{2n}-1}-y^{p^{2n}-1}}{x^s-y^s}=\sum_{j=0}^{p^n} x^{s(p^n-j)}y^{sj} \qquad (s=p^n-1). \tag{3}$$

Consider next a transformation S for which

$$\Delta(\rho)=(\rho-\kappa)(\rho-\kappa^{-1}),$$

where κ is in the $GF[p^n]$. There exists a linear function l which S replaces by κl. If $\kappa^2\neq 1$, S is therefore equivalent within G with $\left(\begin{smallmatrix}\kappa & 0\\ 0 & \kappa^{-1}\end{smallmatrix}\right)$, which leaves only $(1, 0)$ and $(0, 1)$ unaltered. If $\kappa^2=1$, S is equivalent to

$$S_\beta=\begin{pmatrix}\pm 1 & 0\\ \beta & \pm 1\end{pmatrix}.$$

Since S_0 is of type (1), we may set $\beta\neq 0$; then S_β leaves only $(0, 1)$ invariant. Hence when $\Delta(\rho)$ is reducible, the resulting system of special points is composed of the p^n+1 points $(1, 0)$ and $(a, 1)$, where a ranges over the field. There results the relative invariant

$$L=y\prod_a(x-ay)=x^{p^n}y-xy^{p^n}. \tag{4}$$

4. **Lemma.** *If an integral function is invariant under the group G of all m-ary linear homogeneous transformations of determinant unity in the $GF[p^n]$, it is an absolute invariant of G.*

The group G is generated † by transformations of the type $B_{rs\lambda}$, which alters only one variable x_r, replacing it by $x_r+\lambda x_s$. If B multiplies a function by μ, B^p multiplies it by μ^p. Now B is of period p. Hence $\mu^p=1$. Employing the power p^{n-1}, we get

$$\mu^{p^n}=\mu=1.$$

5. We may now prove that any integral invariant I of the binary group G is an integral function of the invariants Q and L. If I vanishes for a special point it has the factor Q or L. It remains to investigate the invariants I which vanish for no special point. The number of transformations (1) contained in G is

$$\gamma=1 \text{ if } p=2, \qquad \gamma=2 \text{ if } p>2. \tag{5}$$

* Dickson, *Linear Groups* (Leipzig, 1903), page 17.

† *Linear Groups*, page 78.

Any non-special point is one of $\omega = g/\gamma$ conjugates under G. Now the forms

$$q = Q^{(p^n+1)/\gamma}, \qquad l = L^{p^n(p^n-1)/\gamma} \tag{6}$$

are of degree ω. Let F_{p^n} be the field composed of all roots of all algebraic equations with coefficients in the $GF[p^n]$. The invariant I vanishes for a set of ω conjugate points; if (c, d) is one of these points, the ratio $c:d$ belongs to the field F_{p^n}. Hence we can determine τ in the latter field such that $q(c, d) + \tau l(c, d) = 0$. Thus I has the factor $q + \tau l$, and I is a product of such linear functions of q, l. If I has its coefficients in the $GF[p^n]$, and τ is a root of an equation of degree t irreducible in the $GF[p^n]$, the presence of the factor $q + \tau l$ in I implies that of the conjugate factors $q + \tau^{p^{nt}} l$ and hence of a function of degree t in q and l with coefficients in the $GF[p^n]$.

Theorem. *Any integral invariant with coefficients in the $GF[p^n]$ of the binary group G is an integral function of Q and L with coefficients in the $GF[p^n]$.*

6. The transformation which leaves y unaltered and replaces x by Dx, where $D^s = 1$, leaves Q unaltered and replaces L by DL. Hence Q is an absolute invariant and L a relative invariant of weight unity under the group G' of all binary transformations in the $GF[p^n]$. Hence *any absolute invariant of G' is expressible in terms of Q and L^s, where $s = p^n - 1$.*

The invariants π_m $(m > 2)$ investigated in § 7 are absolute invariants under G' which vanish for no special points. Hence they are expressible in terms of invariants (6) in such a manner that, for $p > 2$, l occurs only to even powers, and in view of the homogenity q occurs only to even powers. Hence, for any p, they are expressible in terms of J and K, where

$$J = Q^{p^n+1}, \qquad K = L^{s p^n}. \tag{7}$$

The Invariants π_m as Functions of the Fundamental Invariants, §§ 7–9.

7. For $m > 1$, let π_m denote the product of all binary forms of degree m which are irreducible in the $GF[p^n]$ and have unity as the coefficient of x^m. Let

$$m = q_1^{r_1} q_2^{r_2} \cdots q_\mu^{r_\mu}, \tag{8}$$

where $q_1, \cdots, q_\mu$ are the distinct prime factors > 1 of m. Set

$$F_t = \frac{x^{p^{nt}-1} - y^{p^{nt}-1}}{x^s - y^s} \qquad (s = p^n - 1). \tag{9}$$

Thus $F_1 = 1$, $F_2 = Q$. It follows readily from *Linear Groups*, page 18, that

$$\pi_m = \frac{F_m \Pi F_{m/q_i q_j} \Pi F_{m/q_i q_j q_k q_l} \cdots}{\Pi F_{m/q_i} \Pi F_{m/q_i q_j q_k} \cdots}, \tag{10}$$

in which the first product in the numerator extends over the $\frac{1}{2}\mu(\mu-1)$ combinations of $q_1, \cdots, q_\mu$ two at a time, and similarly for the remaining products. By § 6, $\pi_m (m>2)$ is an integral function of J and K of degree

$$d_m = \frac{p^{nm} - \sum p^{\frac{nm}{q_i}} + \sum p^{\frac{nm}{q_i q_j}} - \sum p^{\frac{nm}{q_i q_j q_k}} + \cdots}{p^n(p^{2n}-1)}, \tag{11}$$

For example, if m is an odd prime,

$$\pi_m = F_m, \quad d_m = \frac{p^{n(m-1)}-1}{p^{2n}-1} \qquad (m \text{ an odd prime}). \tag{12}$$

8. Since $d_3 = 1$, $\pi_3 = F_3$ is a linear function of J and K. By (3),

$$Q^{p^n} = \sum_{k=0}^{p^n} x^{sp^n(p^n-k)} y^{skp^n}.$$

Hence by (5),

$$J = \sum_{j,\,k=0}^{p^n} x^{s(p^{2n}+p^n-t)} y^{st} \qquad (t = kp^n + j). \tag{13}$$

The terms of (13) given by $j = p^n$ and $k < p^n$ are

$$\sum_{k=0}^{s} x^{s(p^{2n}-kp^n)} y^{sp^n(1+k)} = \sum_{k=0}^{s} x^{p^{2n}(s-k)+kp^n} y^{kp^{2n}+p^n(s-k)}.$$

But every coefficient in the expansion of $(a-b)^s$ is congruent to unity modulo p. Hence the preceding sum equals, in the field,

$$(x^{p^n} y^{p^n} - x^{p^n} y^{p^{2n}})^s = (L^{p^n})^s = K. \tag{14}$$

Next the terms of (13) with $k = p^n$ are

$$\sum_{j=0}^{p^n} x^{s(p^n-j)} y^{s(p^{2n}+j)} = \sum_{\tau=p^{2n}}^{p^{2n}+p^n} x^{s(p^{2n}+p^n-\tau)} y^{s\tau}.$$

When the last sum is combined with the terms of (13) given by $j < p^n$, $k < p^n$, and hence by $t = 0, 1, \cdots, p^{2n}-1$, we get

$$\sum_{t=0}^{p^{2n}+p^n} x^{s(p^{2n}+p^n-t)} y^{st} = \frac{x^{p^{3n}-1} - y^{p^{3n}-1}}{x^s - y^s} = F_3.$$

It follows that $\pi_3 = J - K$.

9. We make use of the following identity:

$$(x^{sp^n} - y^{sp^n})(x^{p^{tn}-1} - y^{p^{tn}-1}) = (x^{p^{2n}-1} - y^{p^{2n}-1})(x^{p^{tn}-p^n} - y^{p^{tn}-p^n}) - (xy)^{sp^n}(x^s - y^s)(x^{p^{tn}-p^{2n}} - y^{p^{tn}-p^n}),$$

where $s = p_n - 1$. We divide by $(x^s - y^s)(x^{sp^n} - y^{sp^n})$ and insert the factor $x^{sp^{2n}} - y^{sp^{2n}}$ in the numerator and denominator of the last term. By (14),

$$\frac{(xy)^{sp^n}(x^{sp^{2n}} - y^{sp^{2n}})}{x^{sp^n} - y^{sp^n}} = (xy)^{sp^n}(x^{sp^n} - y^{sp^n})^{p^n-1} = (x^{p^{2n}}y^{p^n} - x^{p^n}y^{p^{2n}})^s = K.$$

Hence we obtain the recursion formula

$$F_t = QF_{t-1}^{p^n} - KF_{t-2}^{p^{2n}} \qquad (F_1 = 1,\ F_2 = Q). \tag{15}$$

Taking $t = 3$ and applying (7) and (12), we get

$$\pi_3 = F_3 = J - K, \tag{16}$$

in agreement with § 8. By (10) and (15) for $t = 4$,

$$\pi_4 = F_4/Q, \qquad F_4 = QF_3^{p^n} - KQ^{p^{2n}}.$$

Applying (16) and (7) we get

$$\pi_4 = J^{p^n} - J^{p^n-1}K - K^{p^n}. \tag{17}$$

In a similar manner,

$$\pi_5 = J^{p^{2n}+1} - J^{p^{2n}}K - J^{p^{2n}-p^n+1}K^{p^n} - JK^{p^{2n}} + K^{p^{2n}+1}, \tag{18}$$

$$\pi_6 = J^{p^{3n}+p^n-1} - J^{p^{3n}-p^{2n}+p^n-1}K^{p^{2n}} - J^{p^n-1}K^{p^{3n}} - K^{p^n}\left(\frac{J^{p^{3n}} - K^{p^{3n}}}{J - K}\right), \tag{19}$$

$$\pi_7 = J^{p^{2n}-p^n+1}(J^{p^n} - K^{p^n})(J^{p^{4n}} - J^{p^{4n}-p^{3n}}K^{p^{3n}} - K^{p^{4n}}) - JK^{p^{2n}}(J^{p^{3n}} - K^{p^{4n}}) - K\pi_5^{p^{2n}}. \tag{20}$$

The Number of Classes of Irreducible Forms of Degree m, §§ 10–17.

10. Let $\phi(x, y) = x^m + \cdots$ be an irreducible binary form in the $GF[p^n]$. Let $\phi_1 = \phi, \phi_2, \cdots, \phi_k$ be the distinct forms having unity as the coefficient of x^m which are equivalent to a constant multiple of ϕ under the group G of determinant unity. By § 4, the product $P_m = \phi_1 \cdots \phi_k$ is an absolute invariant of G. By § 5, if $m > 2$, P_m is an integral function with coefficients in the $GF[p^n]$ of the invariants q and l, defined by (6). The function $P_m(q, l)$ is obviously irreducible in the field.

Two binary forms shall be said to belong to the same class if and only if one of them is equivalent to a constant multiple of the other under the group G of all binary transformations in the $GF[p^n]$ of determinant unity.

The preceding remarks lead to the

Theorem. *There are as many classes of irreducible binary forms of degree m as there are irreducible factors in the $GF[p^n]$ of the invariant $\pi_m(q, l)$.*

To decide whether or not two given forms ϕ and ψ belong to the same class, we employ a root $\rho = x/y$ of $\phi = 0$ and a root $\sigma = x/y$ of $\psi = 0$, each belonging to the $GF[p^{nm}]$, and determine the values of q/l for $x/y = \rho$ and σ, respectively. According as these values of q/l are the roots of the same factor or different irreducible factors of $\pi_m(q, l)$, the given forms ϕ and ψ belong to the same or different classes.

Since these values of q/l belong to the $GF[p^{nm}]$, *the irreducible factors of* $\pi_m(q, l)$ *are of degree* m *or a divisor of* m.

11. We proceed to the investigation of the irreducible factors of the function $\pi_m(J, K)$ when $p = 2$, and those of this function with J and K replaced by q^2 and l^2 when $p > 2$.

From (16) we conclude that *the cubic forms fall into a single class if* $p = 2$, *and into two classes if* $p > 2$.

For $m = 4$, set $K = \rho J$ in (17). Thus π_4 vanishes only if

$$\rho^{p^n} = 1 - \rho.$$

Then ρ belongs to the $GF[p^{2n}]$ since

$$\rho^{p^{2n}} = 1 - \rho^{p^n} = \rho.$$

For $p = 2$, ρ does not belong to the $GF[2^n]$. Hence π_4 is the product of 2^{n-1} irreducible quadratic factors

$$(K - \rho J)(K - \rho^{p^n} J) = K^2 - KJ + tJ^2,$$

where

$$t = \rho^{p^n+1} = \rho - \rho^2. \tag{21}$$

Hence,* for $p = 2$, t must be a root of

$$t + t^2 + t^4 \cdots + t^{2^{n-1}} = 1. \tag{22}$$

The latter has 2^{n-1} roots in the $GF[2^n]$. Thus, *for* $p = 2$, *the quartic forms fall into* 2^{n-1} *classes differentiated by the roots of* (22).

Next, let $p > 2$. The only value of ρ in the $GF[p^n]$ is $\rho = \frac{1}{2}$. The corresponding factor $q^2 - 2l^2$ is reducible if $p^n = 8l \pm 1$, irreducible if $p^n = 8l \pm 3$. For the remaining $p^n - 1$ values of ρ, $\rho^2 - \rho + t$ is irreducible, so that t has the $\frac{1}{2}(p^n - 1)$ values in the $GF[p^n]$ for which $1 - 4t$ is a not-square. For $l/q = \lambda$, we have $\lambda^2 = \rho$. Now λ will belong to the $GF[p^n]$ only when ρ is a square in the latter field and hence, by (21), only when

$$t^{(p^n-1)/2} = \rho^{(p^{2n}-1)/2} = 1.$$

* In accord with *Linear Groups*, p. 29, formula (23) for $\lambda = \nu = 1$, $p = 2$.

The condition is therefore that t be a square in the $GF[p^n]$. The number of squares t for which $1-4t$ is a not-square is $\frac{1}{4}(p^n-1)$ or $\frac{1}{4}(p^n-3)$ according as -1 is a square or a not-square. Indeed,* $1-\xi^2=\mu\eta^2$, where μ is a not-square, has $p^n\pm 1$ sets of solutions according as -1 is a square or a not-square. The sets $\xi=\pm 1$, $\eta=0$ are here excluded. We have now shown that, when t has any one of the $\frac{1}{2}(p^n-1)$ values for which $\rho^2-\rho+t$ is irreducible in the $GF[p^n]$, $f(\lambda)=\lambda^4-\lambda^2+t$ is irreducible when t is a not-square, but is reducible when t is a square. In the latter case, set $t=\tau^2$. Then the factors of $f(\lambda)$ are

$$\lambda^2\pm\lambda\sqrt{1+2\tau}+\tau, \tag{23}$$

where τ is that square root of t for which $1+2\tau$ is a square. A partial summary of our results for $p>2$ is given in the following table giving the number of irreducible factors of $\pi_4(q,l)$:

p^n	linear	quadratic	quartic	total number
$8k+1$	2	$\frac{1}{2}(p^n-1)$	$\frac{1}{4}(p^n-1)$	$\frac{1}{4}(3p^n+5)$
$8k-1$	2	$\frac{1}{2}(p^n-3)$	$\frac{1}{4}(p^n+1)$	$\frac{1}{4}(3p^n+3)$
$8k+3$	0	$\frac{1}{2}(p^n-1)$	$\frac{1}{4}(p^n+1)$	$\frac{1}{4}(3p^n-1)$
$8k-3$	0	$\frac{1}{2}(p^n+1)$	$\frac{1}{4}(p^n-1)$	$\frac{1}{4}(3p^n+1)$.

In the following examples we give all the irreducible factors:

$p^n=3$. $\quad q^2-2l^2, \quad l^4-l^2q^2-q^4$.

$p^n=5$. $\quad q^2-2l^2, \quad l^2\pm 2lq-q^2, \quad l^4-l^2q^2+2q^4$.

$p^n=7$. $\quad q\pm 3l, \quad l^2\pm 2lq-2q^2, \quad l^4-l^2q^2+tq^4 \quad (t=3,6)$.

$p^n=9$, $i^2\equiv i+1 \pmod 3$. $\quad q\pm(i+1)l, \quad l^2\pm ilq-iq^2, \quad l^2\pm(2i+1)lq-(2i+1)q^2,$

$l^4-l^2q^2+tq^4 \quad (t=i, 2i+1)$.

$p^n=11$. $\quad q^2-2l^2, \quad l^2\pm 5lq+q^2, \quad l^2\pm 4lq+2q^2,$

$l^4-l^2q^2+tq^4 \quad (t=6,7,8)$.

For $p>2$, the irreducible quartic forms fall into $6k+2$, $6k$, $6k+2$, or $6k-2$ classes, according as $p^n=8k+1$, $8k-1$, $8k+3$, or $8k-3$, respectively.

Classes of Irreducible Quintic Forms.

12. For quintic forms, (18) vanishes for $K=\rho J$ if

$$\rho^{p^{2n}+1}-\rho^{p^{2n}}-\rho^{p^n}-\rho+1=0. \tag{24}$$

* *Linear Groups*, page 46.

As remarked at the end of § 10, each root of (24) belongs to the $GF[p^{5n}]$. We may give a direct proof as follows.

$$\rho^{p^{2n}} = 1 + \frac{\rho^{p^n}}{\rho - 1}, \qquad \rho^{p^{3n}} = 1 + \frac{1}{\rho - 1} + \frac{\rho}{(\rho - 1)(\rho^{p^n} - 1)},$$

$$\rho^{p^{4n}} = 1 + \frac{\rho}{\rho^{p^n} - 1}, \quad \rho^{p^{5n}} = 1 + \frac{\rho^{p^n}}{\rho^{p^n}/(\rho - 1)} = \rho.$$

If a root of (24) belongs to the $GF[p^n]$, it satisfies the equation

$$\rho^2 - 3\rho + 1 = 0. \tag{25}$$

For $p = 2$, a root of (25) satisfies the equation $\rho^3 \equiv 1$ and hence belongs to the $GF[2^n]$ if and only if n is even. *If $p = 2$, the quintic forms fall into $\frac{1}{5}(2^{2n} + 1)$ or $2 + \frac{1}{5}(2^{2n} - 1)$ classes, according as n is odd or even.* For example, the 6 irreducible quintic forms in the $GF[2]$ are equivalent.

For $p > 2$, we must consider (24) for ρ replaced by λ^2. By § 10, the resulting equation for λ has all its roots in the $GF[p^{5n}]$. If one of its roots belongs to the $GF[p^n]$, it must satisfy the equation

$$\lambda^4 - 3\lambda^2 + 1 = (\lambda^2 + \lambda - 1)(\lambda^2 - \lambda - 1) = 0. \tag{25'}$$

Hence $(2\lambda \pm 1)^2 = 5$, so that $p = 5$ or $p^n = 5k \pm 1$. In the respective cases, we obtain the factors $l \pm 2q$ or

$$l + \tfrac{1}{2}(1 \pm \sqrt{5})q, \qquad l - \tfrac{1}{2}(1 \pm \sqrt{5})q.$$

Hence, if $p > 2$, the irreducible factors are of the following types:

p^n	linear	quintic
5^n	2	$2 \cdot 5^{2n-1}$
$5k \pm 1$	4	$\frac{2}{5}(p^{2n} - 1)$
$5k \pm 2$	0	$\frac{2}{5}(p^{2n} + 1)$

For example, if $p^n = 3$, the factors are

$$\lambda^5 - \lambda^4 - \lambda^3 + \lambda^2 + 1, \ \lambda^5 + \lambda^4 - \lambda^3 - \lambda^2 - 1, \ \lambda^5 - \lambda^4 + 1, \ \lambda^5 + \lambda^4 - 1.$$

For $p > 2$, the irreducible quintic forms fall into

$$2(5^{2n-1} + 1), \ 4 + \tfrac{2}{5}(p^{2n} - 1), \ \tfrac{2}{5}(p^{2n} + 1)$$

classes, according as $p = 5$, $p^n = 5k \pm 1$, or $p^n = 5k \pm 2$.

Classes of Irreducible Sextic Forms.

13. For sextic forms, (19) vanishes for $K = \rho J$ if

$$1-\rho^{p^{2n}}-\rho^{p^{3n}}-\rho^{p^{2n}}\left(\frac{1-\rho^{p^n}}{1-\rho}\right)=0. \tag{26}$$

For $\rho=1$ the final term is a multiple of p, so that $\rho=1$ is not a root. For $p=2$ or $p=3$, there is no root in the $GF[p^n]$, while for $p>3$, the only root is $\rho=\frac{1}{3}$.

We determine the roots of (26) which belong to the $GF[p^{2n}]$ by a simple device. Any such root satisfies the equation

$$1-\rho-\rho^{p^n}=\rho^{p^n}\left(\frac{1-\rho^{p^n}}{1-\rho}\right). \tag{27}$$

The first member equals its p^n-th power. Hence

$$\rho^{p^n}=1-\rho-s, \tag{28}$$

where s is an element of the $GF[p^n]$. Hence (27) gives

$$s=\frac{(1-\rho-s)(\rho+s)}{1-\rho}$$

$$\rho^2+\rho(s-1)+s^2=0. \tag{29}$$

Hence any root of (26) which belongs to the $GF[p^{2n}]$, but not to the $GF[p^n]$, satisfies an equation of type (29) irreducible in the $GF[p^n]$. Conversely, any root ρ of an irreducible equation (29) is a root of (26). Indeed, the second root is ρ^{p^n} and the sum of the roots is $1-s$; thus (28) and hence also (27) is satisfied. Now, for $p>2$, (29) is irreducible if and only if $(1+s)(1-3s)$ is a not-square in the $GF[p^n]$. We readily verify* that this is the case for the following number of values of s:

$$\tfrac{1}{2}(p^n-3) \text{ if } p^n=3l+2, \qquad \tfrac{1}{2}(p^n-1) \text{ if } p=3 \text{ or } p^n=3l+1. \tag{30}$$

For $p>2$, we must investigate the reducibility of (29) when ρ is replaced by λ^2. By the product of the roots,

$$\rho^{p^n+1}=s^2, \qquad \rho^{(p^{2n}-1)/2}=+1,$$

so that ρ is a square in the $GF[p^{2n}]$. Hence the quartic in λ is reducible. Its factors are

$$\lambda^2+a\lambda\pm s, \qquad \lambda^2-a\lambda\pm s,$$

where $a^2=1-s\pm 2s$. The product of the two values for a^2 is the not-square $(1+s)(1-3s)$, so that one value is a square.

* By means of *Linear Groups*, p. 48, § 66. Note that -3 is a square or not-square according as $p^n=3l+1$ or $3l+2$.

Examples. If $p^n = 3$, then $s = 1$ and

$$\lambda^4 + 1 = (\lambda^2 + \lambda - 1)(\lambda^2 - \lambda - 1).$$

If $p^n = 5$, then $s = -2$, and

$$\lambda^4 + 2\lambda^2 - 1 = (\lambda^2 + 2\lambda - 2)(\lambda^2 - 2\lambda - 2).$$

If $p = 2$, we multiply (29) by $(s-1)^{-2}$ and set

$$\sigma = \rho(s-1)^{-1}, \qquad t = s^2(s^2+1)^{-1}.$$

Then $\sigma^2 + \sigma + t = 0$. This is irreducible if and only if t is one of the 2^{n-1} roots of (22) in the $GF[2^n]$. Unless $t = 1$, the corresponding value of s is uniquely determined in the field. But $t = 1$ is a root of (22) if and only if n is odd. Hence the number of values of s for which (29) is irreducible in the $GF[2^n]$ is

(31) $\qquad 2^{n-1} - 1$ for n odd, $\qquad 2^{n-1}$ for n even.

The number is zero for $n = 1$; for $n = 2, 3, 4$, s is a root of

$$s^2 + s + 1 = 0, \qquad s^3 + s + 1 = 0, \qquad (s^4 + s + 1)(s^4 + s^3 + 1) = 0.$$

It remains to determine the roots of (26) which belong to the $GF[p^{3n}]$. These must satisfy the equation

$$\rho + \rho^{p^n} + \rho^{p^{2n}} = 1. \tag{32}$$

Conversely, any root of the latter belongs to the $GF[p^{3n}]$ and, with the exception of $\rho = 1$ when $p = 2$, satisfies (26). A root of (32) belongs to the $GF[p^n]$ only when $3\rho = 1$. Hence the irreducible cubic factors of (26) are of the form $\rho^3 - \rho^2 + \cdots$ and their number is

(33) $\qquad 3^{2n-1}$ if $p = 3$, $\qquad \frac{1}{3}(p^{2n} - 1)$ if $p \neq 3$.

By § 15, these factors give all the existing irreducible cubics $x^3 - x^2 + \cdots$.

The irreducible factors of (26) are of degree 6 or a divisor of 6. Denote by L, Q, C the number of linear, quadratic and cubic factors, respectively. Then the number of degree 6 is

$$S = \tfrac{1}{6}(p^{3n} + p^n - 1 - L - 2Q - 3C).$$

By the above results we have, whether $p = 2$ or $p > 2$,

$$\begin{aligned} &S = \tfrac{1}{6}(p^{3n} - p^{2n} + 2) \text{ for } p^n = 3l + 2, \\ &S = \tfrac{1}{6}(p^{3n} - p^{2n}) \text{ for } p = 3 \text{ or } p^n = 3l + 1. \end{aligned} \tag{34}$$

In view of (31), (33), (34), we have the

Theorem. *If $p = 2$, the number of classes of irreducible sextic forms is $\frac{1}{6}(2^{3n} + 2^{2n} + 3\cdot 2^n - 6)$ for n odd, $\frac{1}{6}(2^{3n} + 2^{2n} + 3\cdot 2^n - 2)$ for n even.*

For $p^n = 2$ this number is 2. The irreducible factors of (26) are then

$$\rho^3 + \rho^2 + 1, \qquad \rho^6 + \rho^5 + 1.$$

Irreducible Cubics whose Roots are Squares with a given Sum.

14. For sextic forms when $p > 2$ there remains the difficult problem: to determine which of the cubics

$$C = \rho^3 - \rho^2 + a\rho - b \tag{35}$$

are irreducible and give irreducible sextics when ρ is replaced by λ^2, and which give reducible sextics. The first or second case arises according as ρ is a not-square or a square in the $GF[p^{3n}]$, namely, according as b is a not-square or a square in the $GF[p^n]$. In fact,

$$b = \rho\cdot\rho^{p^n}\cdot\rho^{p^{2n}}, \qquad b^{(p^n-1)/2} = \rho^{(p^{3n}-1)/2}.$$

In the second case, $b = e^2$, and $\lambda^6 - \lambda^4 + a\lambda^2 - e^2$ has the factors

$$\lambda^3 \pm c\lambda^2 + \tfrac{1}{2}(c^2 - 1)\lambda \pm e, \tag{36}$$

where

$$c^4 - 2c^2 - 8ec + 1 - 4a = 0.$$

This quartic has the resolvent cubic

$$y^3 + 2y^2 - 4(1 - 4a)y - 8(1 - 4a) - 64e^2 = 0.$$

We set $y = 2z$ and multiply the resulting equation by 8, and get

$$z^3 + z^2 + (4a - 1)z - 1 + 4a + 8b = 0.$$

If in the cubic $8C = 0$ we replace ρ by $\frac{1}{2}(z + 1)$, we obtain the preceding cubic. Hence the resolvent cubic is irreducible, so that* the quartic has one and but one root c in the $GF[p^n]$. The unique pair of factors of the sextic in λ are therefore of the form (36).

To determine the irreducible cubics (35) in which $b = e^2$,

$$C = \rho^3 - \rho^2 + a\rho - e^2 \qquad (e \neq 0), \tag{37}$$

we proceed indirectly and find those which are reducible.

(a) First, let C have a linear factor $\rho - k$ and an irreducible quadratic factor. The latter must be of the form $\rho^2 + (k-1)\rho + ks^2$, where $s^2 = e^2/k^2 \neq 0$.

* Bulletin of the American Mathematical Society, vol. 13 (1906), p. 5, middle of p. 7.

The quadratic is irreducible if and only if $(k-1)^2-4ks^2$ is a not-square. Let ν be a fixed not-square and let $p^n=4l\pm 1$, the upper or lower sign holding according as -1 is a square or a not-square. Then

$$(k-1)^2-4ks^2=\nu z^2$$

has $p^n\mp 1$ sets of solutions s, z when k is any chosen not-square, $p^n\pm 1$ sets when k is any chosen square $\neq 1$, while for $k=0$ or 1 the solutions are obvious. After excluding the sets in which either $s=0$ or $z=0$, and dividing by 4, we obtain the number of values of $s^2\neq 0$ for which $(k-1)^2-4ks^2$ is a not-square. If -1 is a square the number of sets s^2, k, and hence the number of irreducible quadratics, is

$$\tfrac{1}{4}(p^n-1)\cdot\tfrac{1}{2}(p^n-1)+\tfrac{1}{4}(p^n-1)\cdot\tfrac{1}{2}(p^n-3)=\tfrac{1}{4}(p^n-1)(p^n-2).$$

If -1 is a not-square, the number is

$$\tfrac{1}{4}(p^n+1)\cdot\tfrac{1}{2}(p^n-1)+\tfrac{1}{4}(p^n-3)\cdot\tfrac{1}{2}(p^n-3)+\tfrac{1}{2}(p^n-1)=\tfrac{1}{2}(p^{2n}-p^n+2).$$

In the following examples we give all the irreducible quadratics: For $p^n=3$, $s^2=1$, $k=\pm 1$. For $p^n=5$, $s^2=1$, $k=2, 3, 4$. For $p^n=7$, $s^2=1$, $k=1, 3, 5$; $s^2=2$, $k=1, 2, 4, 6$; $s^2=4$, $k=1, 3, 5, 6$. For $p^n=11$, $s^2=1$, $k=1, 5, 7, 8, 9, 10$; $s^2=3$, $k=1, 2, 6, 7, 8$; $s^2=4$, $k=1, 2, 5, 6, 9$; $s^2=5$, $k=1, 3, 4, 7, 8, 10$; $s^2=9$, $k=1, 2, 3, 4, 6, 10$. For $p^n=9$, $i^2\equiv -1\pmod 3$, $s^2=1$, $k=\pm i+1, \pm i-1$; $s^2=\pm i$, $k=-1, i, -i, \mp i+1, \mp i-1$.

(b) Next let (37) have three roots in the $GF[p^n]$. Either all three are squares or only one root is a square.

(b_1) Let the three roots be squares. We seek the number of systems of three squares s_i with the sum unity, two systems being identified if they are composed of the same s_i in different orders.

If t is the number of systems with $s_1=s_2=s_3$, we have

$$(38)\qquad t=0 \text{ if } p=3 \text{ or } p^n=12k\pm 5, \qquad t=1 \text{ if } p^n=12k\pm 1.$$

Let μ be the number of sets with $s_1=s_2$. Now $x^2+2y^2=1$ has p^n-1 or p^n+1 sets of solutions according as -2 is a square or a not-square. Of these, 4 or 2 have $xy=0$, according as 2 is a square or a not-square, namely, according as $p^n=8l\pm 1$ or $8l\pm 3$. Hence

$$\mu=\tfrac{1}{4}(p^n-5) \text{ if } p^n=8l+1, \qquad \tfrac{1}{4}(p^n-3) \text{ if } p^n=8l-1 \text{ or } 8l+3,$$
$$\tfrac{1}{4}(p^n-1) \text{ if } p^n=8l+5.$$

The number of sets s_i in which exactly two are equal is therefore $3(\mu-t)$.

Next, the total number of sets of solutions of *

$$x^2 + y^2 + z^2 = 1$$

is $p^{2n} \pm p^n$ according as $p^n = 4l \pm 1$. These include $p^n \mp 1$ sets with $z = 0$. Hence the numbers of sets of squares s_i whose sum is unity is

$$k = \tfrac{1}{8}\{ p^{2n} \pm p^n - 3(p^n \mp 1 - 4) - 6 \}.$$

Hence there are $\frac{1}{6}\{k - 3(\mu - t) - t\}$ systems in which s_1, s_2, s_3 are all distinct. As shown above, there are $\mu - t$ systems in which exactly two of the s_i are equal, and t systems with all three equal. The total number of systems is therefore $\frac{1}{6}(k + 3\mu + 2t)$.

(b_2) Let one of the roots be a square s and the other two not-squares n_1, n_2. If the latter are equal, we employ the $p^n - 1$ or $p^n + 1$ sets of solutions of $x^2 + 2vy^2 = 1$, according as -2 is a not-square or a square, namely, according as $p^n = 8l - 1$, $8l - 3$ or $p^n = 8l + 1$, $8l + 3$. Now there are two sets of solutions with $x = 0$ if 2 is a not-square, namely, if $p^n = 8l \pm 3$. Hence the number of sets s, $n_1 = n_2$ is

$$N = \tfrac{1}{4}(p^n - 1) \text{ if } p^n = 8l + 1, \qquad \tfrac{1}{4}(p^n - 3) \text{ if } p^n = 8l - 1 \text{ or } 8l + 3,$$
$$\tfrac{1}{4}(p^n - 5) \text{ if } p^n = 8l + 5.$$

The total number of sets of solutions of $x^2 + vy^2 + vz^2 = 1$ is $p^{2n} \pm p^n$ according as $p^n = 4l \pm 1$. These include $p^n \mp 1$ sets with $x = 0$ and $p^n \pm 1$ sets with $y = 0$. Hence the total number of sets s, n_1, n_2 is

$$M = \tfrac{1}{8}\{ p^{2n} \pm p^n - (p^n \mp 1) - 2(p^n \pm 1 - 2) - 2 \}.$$

The total number of systems is therefore $\frac{1}{2}(M - N) + N = \frac{1}{2}M + \frac{1}{2}N$. Now

$$\tfrac{1}{6}k + \tfrac{1}{2}M = \tfrac{1}{12}(p^{2n} \pm p^n - 3p^n + 3),$$

while $N + \mu = \frac{1}{2}(p^n - 3)$ for all values of p^n. Hence the number of cubics (37) with three roots in the $GF[p^n]$ is $\frac{1}{12}(p^{2n} \pm p^n - 6) + \frac{1}{3}t$, according as $p^n = 4l \pm 1$.

Combining this number with the number obtained in case (a), we get

$$\tfrac{1}{3}(p^{2n} - 2p^n + t) \text{ if } p^n = 4l + 1, \qquad \tfrac{1}{3}(p^{2n} - p^n + t) \text{ if } p^n = 4l - 1.$$

The total number of cubics (37) is $\frac{1}{2}(p^n - 1)p^n$. Hence, by (38), we obtain the

Theorem. *The number of irreducible cubics* (37) *is*

* *Linear Groups*, p. 48.

$$(39)\qquad \begin{aligned} &\tfrac{1}{6}(p^n \mp 1)(p^n \pm 2) \text{ if } p^n = 12k \pm 1, \\ &\tfrac{1}{6}p^n(p^n \pm 1) \text{ if } p^n = 12k \pm 5, \text{ or if } p = 3 \text{ and } \pm 1 = (-1)^n. \end{aligned}$$

A check upon the above results is afforded by the following examples which give all systems of three elements in the $GF[p^n]$ whose sum is unity and whose product is a square:

$p^n = 3$, none; $p^n = 5$, $1, 1, 4; 1, 2, 3$; $p^n = 7$, $4, 2, 2; 2, 3, 3; 4, 5, 6$;

$p^n = 3^2$, $i^2 \equiv -1 \pmod 3$, $1, 1, 2; i, -i, 1; 1, i \pm 1, -i \mp 1$;

$-1, i+1, -i+1; \pm i, \pm i - 1, \pm i - 1$;

$p^n = 11$, $4, 4, 4; 3, 4, 5; 5, 9, 9; 3, 2, 7; 3, 10, 10$;

$4, 2, 6; 5, 8, 10; 9, 7, 7; 9, 6, 8$;

$p^n = 13$, $1, 1, 12; 1, 3, 10; 1, 4, 9; 3, 12, 12; 9, 9, 9; 1, 2, 11; 1, 5, 8$;

$1, 6, 7; 3, 5, 6; 4, 2, 8; 4, 5, 5; 9, 7, 11; 10, 2, 2; 10, 6, 11; 12, 7, 8$.

By excluding the resulting cubics and those given by the examples under case (a), we find the only irreducible cubics (37) are those in which a, e^2 is one of the following sets:

$p^n = 3$, $2, 1$; $p^n = 5$, $1, -1; 3, -1; 0, 1; 4, 1; 3, 1$;

$p^n = 7$, $0, 1; 3, 1; 5, 1; 1, 2; 5, 2; 2, 4; 5, 4$;

$p^n = 9$, $i^2 \equiv -1 \pmod 3$, $0, -1; 1, -1; \pm i, -1; -1, 1$;

$\pm i + 1, 1; \pm i - 1, 1; 0, \pm i; 1, \pm i; \pm i - 1, \mp i$;

$p^n = 11$, $3, 1; 7, 1; 0, 3; \pm 1, 3; 2, 3; \pm 2, 4; 6, 4; 7, 4$;

$8, 4; 0, 5; 7, 5; 10, 5; 1, 9; 2, 9; 3, 9; 5, 9$.

15. The investigation made in § 14 enables us to determine the number N_{gs} of irreducible cubics $x^3 - gx^2 + hx + k$, in which g is a given element and k ranges over the squares, and the corresponding number N_{gn} when k ranges over the not-squares.

Let C_g be the number of all the irreducible cubics $x^3 - gx^2 + \cdots$ in which g is a given element. If ρ is a root of such a cubic,

$$(40)\qquad \rho + \rho^{p^n} + \rho^{p^{2n}} = g, \qquad \rho^{p^{3n}} = \rho.$$

Since the latter equation follows from the former, we conclude that (40_1) has p^{2n} roots in the $GF[p^{3n}]$. If such an element ρ is distinct from ρ^{p^n} it is a root of a cubic irreducible in the $GF[p^n]$. Now a common root of $\rho = \rho^{p^n}$ and (40_1)

makes $3\rho = g$, so that the two equations have a single solution if $p \neq 3$, no solution if $p = 3$, $g \neq 0$, and 3^n solutions if $p = 3$, $g = 0$. Hence

(41) $C_g = \frac{1}{3}(p^{2n}-1)$ if $p \neq 3$, $C_0 = \frac{1}{3}(3^{2n}-3^n)$ if $p=3$, $C_{g'} = 3^{2n-1}$ if $p=3$, $g' \neq 0$.

For any p, the total number of irreducible cubics is therefore $\frac{1}{3}p^n(p^{2n}-1)$, in agreement with § 7. Conversely, if * $p \neq 3$, the latter result implies (41), since $x' = x + \frac{1}{3}(g+g')$ transforms each $x^3 - gx^2 + \cdots$ into a cubic $x^3 - g'x^2 + \cdots$.

If ν is a fixed not-square in the $GF[p^n]$, (40_1) gives

$$\nu\rho + (\nu\rho)^{p^n} + (\nu\rho)^{p^{2n}} = \nu g.$$

First, let ρ be a square in the $GF[p^{3n}]$. Since ν is a not-square in the $GF[p^{3n}]$, $\nu\rho$ is a not-square. Hence

$$N_{0s} = N_{0n}, \qquad N_{gs} = N_{g'n},$$

where one of the elements g, g' is any square and the other any not-square. Thus *if σ is any square and ν any not-square in the $GF[p^n]$,*

(42) $N_{0s} = N_{0n} = \frac{1}{2}C_0, \qquad N_{\nu n} = N_{\sigma s} = N_{1s}, \qquad N_{\nu s} = N_{\sigma n} = C_1 - N_{1s},$

where C_0 and C_1 are given by (41) *and N_{1s} is given by* (39).

16. We may now enumerate the irreducible factors in the $GF[p^n]$, $p > 2$, of the function of degree $d = 2(p^{3n} + p^n - 1)$ derived from (26) by replacing ρ by λ^2. First, if $p \neq 3$, $\lambda^2 - \frac{1}{3}$ has $t+1$ factors, where t is defined by (38). Next, there are $2r$ further irreducible quadratic factors, where r is defined by (30). Again, there are $2N_{1s}$ irreducible cubic factors and N_{1n} irreducible sextic factors, all derived from the cubic functions of ρ (§ 14). The remaining factors are irreducible sextics; their number is therefore

$$\tfrac{1}{6}(d - e - 4r - 6N_{1s} - 6N_{1n}) \quad (e = 0 \text{ if } p = 3,\ e = 2 \text{ if } p \neq 3).$$

Thus if $\beta = 0$ when $p = 3$, $\beta = t + \frac{2}{3}$ if $p \neq 3$, the total number of factors is

$$\tfrac{1}{6}d + \beta + \tfrac{4}{3}r + N_{1s}.$$

Hence we may state the following

Theorem. *The number of classes of irreducible sextics in the $GF[p^n]$, $p > 2$, is $\frac{1}{6}(2p^{3n} + p^{2n} + 7p^n - a)$, $a = -2$ if $p^n = 12k+1$, $a = 10$ if $p^n = 12k+5$, $a = 6$ if $p = 3$, n even; $\frac{1}{6}(2p^{3n} + p^{2n} + 5p^n - b)$, $b = 2$ if $p^n = 12k-5$, $b = 6$ if $p^n = 12k-1$ or if $p = 3$, n odd.*

* By means of $x' = gx$ we conclude that, for any p, $C_g = C_1$ if $g \neq 0$. The value of C_0 was determined otherwise in Bulletin of the American Mathematical Society, l. c., pp. 3, 4.

For $p^n = 3$, the 12 irreducible factors * are

$$\lambda^2 \pm \lambda - 1,\ \lambda^3 \pm \lambda^2 \mp 1,\ \lambda^6 - \lambda^4 + 1,\ \lambda^6 - \lambda^4 + \lambda^2 + 1,$$

$$\lambda^6 \pm \lambda^5 - \lambda^4 + \lambda^2 - 1,\ \lambda^6 \pm \lambda^5 \pm \lambda^3 - 1,\ \lambda^6 \pm \lambda^5 \mp \lambda^3 + 1.$$

For $p^n = 5$, the 50 irreducible factors are 34 sextics † and

$$\lambda^2 - 2,\ \lambda^2 \pm 2\lambda - 2,\ \lambda^3 \pm \lambda^2 \pm 2,\ \lambda^3 \pm 2\lambda^2 - \lambda \pm 2,\ \lambda^3 \mp 2\lambda^2 - \lambda \pm 1,$$

$$\lambda^3 + 2\lambda \pm 1,\ \lambda^3 \pm \lambda^2 \pm 1,\ \lambda^6 - \lambda^4 - \lambda^2 - 3,\ \lambda^6 - \lambda^4 + \lambda^2 - 3,\ \lambda^6 - \lambda^4 - 2.$$

For $p^n = 7$, the 128 irreducible factors are 98 sextics † and

$$\lambda^2 - 5,\ \lambda^2 \pm 3\lambda + 1,\ \lambda^2 \pm 3\lambda - 2,\ \lambda^2 \pm 2\lambda + 3,\ \lambda^3 \pm 2\lambda^2 - 2\lambda \pm 1,\ \lambda^3 \pm \lambda^2 \pm 1,$$

$$\lambda^3 \pm 3\lambda^2 + 4\lambda \pm 1,\ \lambda^3 \pm \lambda^2 \pm 3,\ \lambda^3 \pm 3\lambda^2 + 4\lambda \pm 3,\ \lambda^3 + 3\lambda \pm 2,\ \lambda^3 \pm 2\lambda^2 - 2\lambda \mp 2,$$

$$\lambda^6 - \lambda^4 + 4\lambda^2 - 6,\ \lambda^6 - \lambda^4 + 6\lambda^2 - 3,\ \lambda^6 - \lambda^4 - 3,\ \lambda^6 - \lambda^4 + \lambda^2 - 5,\ \lambda^6 - \lambda^4 + 5\lambda^2 - 3,$$

$$\lambda^6 - \lambda^4 + 2\lambda^2 - 6,\ \lambda^6 - \lambda^4 + 3\lambda^2 - 5,\ \lambda^6 - \lambda^4 + 4\lambda^2 - 3,\ \lambda^6 - \lambda^4 + 6\lambda^2 - 5.$$

Classes of Irreducible Septic Forms.

17. For septic forms we set $K = \rho J$ in (20) and obtain an equation of degree $d = p^{4n} + p^{2n} + 1$ in ρ, having d roots in the $GF[p^{7n}]$. For $\rho^{p^n} = \rho$, the equation becomes

$$1 - 5\rho + 6\rho^2 - \rho^3 = 0. \tag{43}$$

If $p = 2$, a root of (43) belongs to the exponent 7 and hence belongs to the $GF[2^n]$ if and only if 7 is a divisor of $2^n - 1$, namely, if n is a multiple of 3. In the latter case there are 3 linear and $\frac{1}{7}(d-3)$ irreducible septic factors. But if n is prime to 3, all the factors are septics. For example, if $n = 1$, 2, or 3, the number of septic factors is 3, 39, or 594, respectively. *If $p = 2$, the number of classes of irreducible septic forms is $3 + \frac{1}{7}(d-3)$ or $\frac{1}{7}d$, according as n is a multiple of 3 or prime to 3.*

If we set $\rho = \sigma + 2$ in (43), we get ‡

$$\sigma^3 - 7\sigma - 7 = 0. \tag{44}$$

Hence if $p = 7$, $\rho = 2$ is the only root in the $GF[p^n]$. Upon replacing ρ by λ^2 we obtain the linear factors $\lambda \pm 3$ and $\frac{1}{7}(2d-2)$ irreducible septic factors. *For $p = 7$, the number of classes of irreducible septic forms is* $2 \cdot 7^{4n-1} + 2 \cdot 7^{2n-1} + 2$.

* Two factors with an ambiguous sign are obtained from the same irreducible function of ρ.

† The omitted sextics are by pairs factors of an irreducible sextic in $\rho = \lambda^2$.

‡ This cubic is a resolvent for the seventh roots of unity. For its numerical solution see SERRET, *Algèbre Supérieure*, vol. 1, pp. 338, 343.

For $p \neq 2$, $p \neq 7$, we determine the number N of distinct roots of (43) in the $GF[p^n]$ by the following indirect process. Since the equation of degree d has $d - N$ roots in the $GF[p^{7n}]$, not belonging to the $GF[p^n]$, it has

$$E = \tfrac{1}{7}(p^{4n} + p^{2n} + 1 - N)$$

irreducible septic factors. Thus E must be an integer. Since the discriminant of (43) or (44) is the square 7^2, we have $N = 0$ or 3 (Bulletin, l. c., p. 1). Now p^{2n} is of the form $7l + s$, where $s = 1, 2$ or 4. Thus E is an integer only when $N = 3, 0$ or 0, respectively. Hence for $p \neq 2$, $p \neq 7$, the cubic (43) has 3 roots in the $GF[p^n]$ if $p^n = 7k \pm 1$, no root if $p^n = 7k \pm 2, \pm 3$. In the former case, each root r is a square in the $GF[p^n]$; for, if not, each root of $\lambda^2 = r$ would belong to the $GF[p^{2n}]$ and not to the $GF[p^n]$, whereas all the roots of the λ-equation belonging to the $GF[p^{7n}]$. Hence if $p^n = 7k \pm 1$, $p > 2$, there are six linear factors. For example,

$$p^n = 13,\ \lambda \pm 2,\ \lambda \pm 4,\quad \lambda \pm 8;\quad p^n = 29,\ \lambda \pm 4,\ \lambda \pm 8,\quad \lambda \pm 10;$$

$$p^n = 41,\ \lambda \pm 3,\ \lambda \pm 10,\ \lambda \pm 15;\ p^n = 43,\ \lambda \pm 9,\ \lambda \pm 16,\ \lambda \pm 20.$$

For $p \neq 2$, $p \neq 7$, the number of classes of irreducible septics is

$$6 + \tfrac{2}{7}(p^{4n} + p^{2n} - 2) \text{ or } \tfrac{2}{7}(p^{4n} + p^{2n} + 1),$$

according as p^n is or is not of the form $7k \pm 1$.

The University of Chicago,
July 14, 1910.

AN APPLICATION OF SYMBOLIC METHODS TO THE TREATMENT OF MEAN CURVATURES IN HYPERSPACE*

BY

WILLIAM HUNT BATES

This paper is an application of MASCHKE's symbolic method for discussing invariants of quadratic differential forms, as developed in his article, *A Symbolic Treatment of the Theory of Invariants of Quadratic Differential Quantics of n Variables.*† Extensive use is also made of results and methods contained in two later publications, *Differential Parameters of the First Order*,‡ and *The Kronecker-Gaussian Curvature of Hyperspace.* § Some familiarity with these three articles is implied.

Part I of the present paper is devoted to the study of the curvatures of an n-space R_n in an euclidean $(n+1)$-space S_{n+1}. In §§ 1–3 the equations and some of the properties of the lines of curvature of R_n in S_{n+1} are developed. In particular, equation (28) gives the n curvatures of the n lines of curvature through a given point of R_n. The coefficients $K_1, \cdots, K_n$ of this equation are the so-called curvatures of R_n in S_{n+1}, involving the coefficients a_{ik} and α_{ik} of the two fundamental forms of R_n. With the help of his symbolic method, ‖ MASCHKE has expressed K_n, when n is even, and K_n^2, when n is odd, as rational integral functions of the coefficients a_{ik} of the first fundamental form and their derivatives.

In §§ 4–6 similar expressions are derived for all the curvatures $K_{2\nu}$ of even index. It does not seem possible to obtain rational results for the curvatures $K_{2\nu+1}$ of odd index. In § 7, however, it is shown that, with the exception of K_1, these curvatures are expressible irrationally in terms of the first fundamental quantities and their derivatives.

The symbolic expressions for $K_{2\nu}$ and K_n^2 show at once that they are differential invariants of the first fundamental quadratic form for R_n, and they have meaning as invariants of any quadratic form in n variables. Part II of this paper considers a space R_λ defined in a space $R_n (n > \lambda)$, which is not neces-

* Presented to the Society December 31, 1910.

† These Transactions, vol. 4 (1903), pp. 445–469. This paper is referred to hereafter as M. I.

‡ *Ibid.*, vol. 7 (1906), pp. 69–80; referred to as D. P.

§ *Ibid.*, vol. 7 (1906), pp. 81–93; referred to as K.–G. C.

‖ In K.–G. C.

sarily euclidean. The invariants $K_{2\nu}$ and K_{λ}^{2} for R_{λ} are calculated in terms of the coefficients a_{ik} belonging to the length element of R_n and of the functions $U^{\lambda+1}, \cdots, U^n$ which determine the space R_{λ} in R_n.

PART I.

CURVATURES OF AN n-SPACE IN AN $(n+1)$-SPACE.

§ 1. *Parametric Representation for an n-space in an* $(n+1)$*-space.*

Let $z', z^2, \cdots, z^{n+1}$ be the coördinates* of an euclidean space S_{n+1} of $n+1$ dimensions, i. e., a space whose arc-element is of the form

$$ds^2 = \sum_{i=1}^{n+1} [dz^i]^2. \tag{1}$$

We define in S_{n+1} any hypersurface, or space R_n, of n dimensions, by expressing each z as a funcion of n independent variables $x_1, \cdots, x_n$:

$$\begin{aligned} z' &= z'(x_1, \cdots, x_n), \\ &\cdots\cdots\cdots\cdots \\ z^{n+1} &= z^{n+1}(x_1, \cdots, x_n). \end{aligned} \tag{2}$$

The arc-element of R_n is given by the equation

$$ds^2 = \sum_{i,k}^{1,\cdots,n} a_{ik}\, dx_i\, dx_k, \tag{3}$$

where

$$a_{ik} = \sum_{j=1}^{n+1} \frac{\partial z^j}{\partial x_i}\frac{\partial z^j}{\partial x_k} = \sum_{j=1}^{n+1} z^j_i z^j_k = f_i f_k, \tag{4}$$

differentiation with respect to x_i being indicated here, as in the following pages, by the lower index i.

A space of λ dimensions, $\lambda < n$, would be obtained by using in (2) only λ independent variables $x_1, \cdots, x_{\lambda}$. In particular, a curve in S_{n+1} is obtained by expressing each z as a function of one new variable x.

If in (2) one puts $x_2 = \cdots = x_n = 0$, the resulting curve is called the x_1-axis of a curvilinear system of coördinates on R_n. By letting $x_2, \cdots, x_n$ represent arbitrary constants, one gets the complete system of x_1-curves; and similarly for the other cases.

$$\begin{aligned} &x_1\text{-curves if } x_1 \text{ varies and } x_2, \cdots, x_n \text{ are constants}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &x_n\text{-curves if } x_n \text{ varies and } x_1, \cdots, x_{n-1} \text{ are constants}. \end{aligned} \tag{5}$$

Equations (3) and (5) give the elements of the new axes,

$$ds_1^2 = a_{11}\, dx_1^2, \cdots, ds_n^2 = a_{nn}\, dx_n^2, \tag{6}$$

* It is assumed that no confusion will arise from writing the upper index, as MASCHKE does, without parentheses. Exponents are only occasionally used, and will be easily recognized.

where ds_k is the element of the x_k-axis. Represent the direction cosines of the x_k-axis, in the old system, by $\cos(z', x_k), \cdots, \cos(z^{n+1}, x_k)$. Then

$$\cos(z^j, x_k) = \frac{dz^j}{ds_k} = \frac{z^j_\kappa dx_k}{\sqrt{a_{kk}}\, dx_k} = \frac{z^j_k}{\sqrt{a_{kk}}} \quad (j=1, \cdots, n+1; k=1, \cdots, n). \tag{7}$$

Let ω_{ik} be the angle between the x_i-axis and the x_k-axis. Then

$$\cos\omega_{ik} = \sum_{j=1}^{n+1} \cos(z^j x_i)\cos(z^j x_k) = \sum_{j=1}^{n+1} \frac{z^j_i z^j_k}{\sqrt{a_{ii}}\sqrt{a_{kk}}} = \frac{a_{ik}}{\sqrt{a_{ii}}\sqrt{a_{kk}}} \quad (i, k=1, \cdots, n), \tag{8}$$

so that *necessary and sufficient conditions for mutual orthogonality of the axes of the new system are*

$$a_{ik} = 0 \qquad (i, k=1, \cdots, n;\ i \neq k). \tag{9}$$

§ 2. *General Curves on R_n.*

A general curve on R_n may be defined by means of $n-1$ equations,

$$U^2(x_1, \cdots, x_n) = \text{const.}, \cdots, \quad U^n(x_1, \cdots, x_n) = \text{const.} \tag{10}$$

The differential equations of this curve, which we call the U-curve, are

$$\sum_{i=1}^{n} U^2_i dx_i = 0, \ \cdots, \ \sum_{i=1}^{n} U^n_i dx_i = 0. \tag{11}$$

Its direction is defined by the ratios of $dx_1, \cdots, dx_n$ in (11). In order to solve for these differentials, let p be any function of $x_1, \cdots, x_n$ which satisfies the condition *

$$D = (pU^2 \cdots U^n) = (pU) \neq 0.$$

If A^r denotes the cofactor of p_r in D, equations (11) are identically satisfied by

$$dx_1 = \rho A', \ \cdots, \ dx_n = \rho A^n, \tag{12}$$

where ρ is an arbitrary parameter.

The direction cosines $\xi', \cdots, \xi^{n+1}$ of the U-curve are found as follows. From (12),

$$\sum_{i=1}^{n} p_i dx_i = \rho \sum_{i=1}^{n} p_i A^i = \rho(pU). \tag{13}$$

Then

$$\xi^k = \frac{dz^k}{ds} = \frac{1}{ds}\sum_{i=1}^{n} z^k_i dx_i = \frac{\rho}{ds}(z^k U),$$

where ds is arc-element of the U-curve. Now

$$\sum_{k=1}^{n} [\xi^k]^2 = 1 = \sum_{k=1}^{n+1} \left[\frac{\rho}{ds}\right]^2 (z^k U)^2 = \left[\frac{\rho}{ds}\right]^2 (fU)^2.$$

* See M. I., § 2, for an explanation of this invariantive notation.

Hence

$$\frac{\rho}{ds} = \frac{1}{\sqrt{(fU)^2}}.$$

Then the direction cosines of the U-curve on R_n, referred to the original system of axes, are

$$\xi' = \frac{(z'U)}{\sqrt{(fU)^2}}, \quad \cdots, \quad \xi^{n+1} = \frac{(z^{n+1}U)}{\sqrt{(fU)^2}}. \tag{14}$$

If there is given also a V-curve on R_n by equations similar to (10), its direction cosines may be written

$$\eta' = \frac{(z'V)}{\sqrt{(fV)^2}}, \quad \cdots, \quad \eta^{n+1} = \frac{(z^{n+1}V)}{\sqrt{(fV)^2}}. \tag{15}$$

If ω is the angle between the two curves, we have from (14) and (15)

$$\cos\omega = \sum_{i=1}^{n+1} \xi^i \eta^i = \sum_{i=1}^{n+1} \frac{(z^iU)(z^iV)}{\sqrt{(fU)^2}\sqrt{(fV)^2}} = \frac{(fU)(fV)}{\sqrt{(fU)^2}\sqrt{(fV)^2}}. \tag{16}$$

Thus *a necessary and sufficient condition for orthogonality of the two curves is*

$$(fU)(fV) = 0. \tag{17}$$

Equation (17) also defines the orthogonal trajectories of a system of U-curves on R_n. An illustration is found in the case of curves on an ordinary surface.

§ 3. *Lines of Curvature on R.*

A line L drawn on R_n such that the normals to R_n along L (with respect to the enclosing space S_{n+1}) generate a developable surface is called* a line of curvature of R_n in S_{n+1}.

At a point P of R_n there is a unique normal to R_n in S_{n+1}. Let the direction cosines of this normal be $\zeta', \cdots, \zeta^{n+1}$. Choose P as origin of the system of x-axes on R_n. Then, since the normal to R_n at P is orthogonal to every direction on R_n at P, we have from (7)

$$\sum_{i=1}^{n+1} \zeta^i z^i_k = 0 \qquad (k = 1, \cdots, n). \tag{18}$$

The coefficients a_{ik} of the first fundamental form of R_n, given in (3) are the first fundamental quantities. The second fundamental quantities are defined by the equations

$$a_{ik} = \sum_{j=1}^{n+1} \zeta^j z^j_{ik} \qquad (i, k = 1, \cdots, n).$$

* Cf. BIANCHI, *Lezioni di Geometria Differenziale*, vol. I, p. 125.

By differentiating (18), one obtains

$$\alpha_{ik} = \sum_{j=1}^{n+1} \zeta^j z^j_{ik} = -\sum_{j=1}^{n+1} \zeta^j_i z^j_k \qquad (i, k = 1, \cdots, n). \tag{19}$$

The letters g and γ are to be used in this paper as symbols of the second fundamental quantities

$$\alpha_{ik} = g_i g_k = \gamma_i \gamma_k = \alpha_{ki}. \tag{20}$$

Let C be the curve of S_{n+1} which is the envelope of the normals along L. Let $M(z', \cdots, z^{n+1})$ be any point of L, and $\overline{M}(\overline{z}', \cdots, \overline{z}^{n+1})$ be the point where the normal at M meets C. Denote by r the distance $M\overline{M}$, which is positive or negative according to the direction of $\overline{M}$ from M. Then

$$\overline{z}' = z' - r\zeta', \ \cdots, \ \overline{z}^{n+1} = z^{n+1} - r\zeta^{n+1}. \tag{21}$$

Take derivatives of equations (21) with respect to the arc s of L. Then, since C is envelope of the normals along L,

$$\begin{aligned} \frac{d\overline{z}'}{ds} &= \frac{dz'}{ds} - r\frac{d\zeta'}{ds} - \zeta'\frac{dr}{ds} = q\zeta', \\ &\cdots\cdots\cdots\cdots\cdots \\ \frac{d\overline{z}^{n+1}}{ds} &= \frac{dz^{n+1}}{ds} - r\frac{dz^{n+1}}{ds} - \zeta^{n+1}\frac{dr}{ds} = q\zeta^{n+1}, \end{aligned} \tag{22}$$

where q is a factor of proportionality to be determined. Multiply equations (22) by $\zeta', \cdots, \zeta^{n+1}$ in order and add. We get

$$q\sum_{i=1}^{n+1}[\zeta^i]^2 = \sum_{i=1}^{n+1}\zeta^i\frac{dz^i}{ds} - r\sum_{i=1}^{n+1}\zeta^i\frac{d\zeta^i}{ds} - \frac{dr}{ds}\sum_{i=1}^{n+1}[\zeta^i]^2. \tag{23}$$

Now

$$\sum_{i=1}^{n+1}[\zeta^i]^2 = 1, \qquad \sum_{i=1}^{n+1}\zeta^i\frac{d\zeta^i}{ds} = 0, \qquad \sum_{i=1}^{n+1}\zeta^i\frac{dz^i}{ds} = 0,$$

since $\zeta', \cdots, \zeta^{n+1}$ are direction cosines of the normal and $dz'/ds, \cdots, dz^{n+1}/ds$ are direction cosines of L. Substituting these results in (23), one obtains

$$q = -\frac{dr}{ds}.$$

Then equations (22) give

$$\frac{dz'}{ds} = r\frac{d\zeta'}{ds}, \ \cdots, \ \frac{dz^{n+1}}{ds} = r\frac{d\zeta^{n+1}}{ds}, \tag{24}$$

or

$$\frac{dz'}{d\zeta'} = \frac{dz^2}{d\zeta^2} = \cdots = \frac{dz^{n+1}}{d\zeta^{n+1}} = r. \tag{25}$$

This result is expressed in curvilinear coördinates as follows:

$$z_1' dx_1 + \cdots + z_n' dx_n = r(\zeta_1' dx_1 + \cdots + \zeta_n' dx_n),$$
$$\cdots\cdots\cdots$$
$$z_1^{n+1} dx_1 + \cdots + z_n^{n+1} dx_n = r(\zeta_1^{n+1} dx_1 + \cdots + \zeta_n^{n+1} dx_n).$$

Multiply these equations in order by $z_k', \cdots, z_k^{n+1}$ and add. We find

$$\sum_{i=1}^{n+1} z_k^i z_1^i dx_1 + \cdots + \sum_{i=1}^{n+1} z_k^i z_n^i dx_n = r\left[\sum_{i=1}^{n+1} z_k^i \zeta_1^i dx_1 + \cdots + \sum_{i=1}^{n+1} z_k^i \zeta_n^i dx_n\right].$$

Then, by (4) and (19),

$$(26) \qquad a_{k1} dx_1 + \cdots + a_{kn} dx_n = - r[\alpha_{k1} dx_1 + \cdots + \alpha_{kn} dx_n] \quad (k = 1, \cdots, n).$$

Equations (26) hold for every line of curvature on R_n. Conversely, if equations (26) be true for any curve L on R_n, then there exists a curve C in S_{n+1} whose tangents are normal to R_n along L, so that L is a line of curvature on R_n by definition.

When equations (26) are written in the form

$$(27) \qquad \begin{array}{l} (a_{11} + \alpha_{11} r) dx_1 + \cdots + (a_{1n} + \alpha_{1n} r) dx_n = 0, \\ \cdots\cdots\cdots \\ (a_{n1} + \alpha_{n1} r) dx_1 + \cdots + (a_{nn} + \alpha_{nn} r) dx_n = 0, \end{array}$$

it is evident that the curvature $(1/r)$ of each line of curvature through a point P on R_n must satisfy the condition

$$(28) \qquad \begin{vmatrix} a_{11} + \alpha_{11} r & a_{12} + \alpha_{12} r & \cdots & a_{1n} + \alpha_{1n} r \\ a_{21} + \alpha_{21} r & a_{22} + \alpha_{22} r & \cdots & a_{2n} + \alpha_{2n} r \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} + \alpha_{n1} r & a_{n2} + \alpha_{n2} r & \cdots & a_{nn} + \alpha_{nn} r \end{vmatrix} = 0,$$

since otherwise equations (27) would have no solution except

$$dx_1 = dx_2 = \cdots = dx_n = 0.$$

Hence the reciprocals of the roots of (28) are exactly the curvatures of the n lines of curvature through a point P on R_n.

The coefficients of r in (28) are called the curvatures of R_n in S_{n+1} and are discussed in §§ 4–7. Before proceeding to that discussion, we derive an important property of lines of curvature.

Expressing (26) in symbolic notation, one finds

$$(29) \qquad f_k \sum_{i=1}^{n} f_i dx_i = - r g_k \sum_{i=1}^{n} g_i dx \qquad (k = 1, \cdots, n)$$

If now a line of curvature be represented as a U-curve (10), one gets from (13) and (29)

$$f_k(fU) = -rg_k(gU) \qquad (k=1, \cdots, n). \tag{30}$$

A symmetrical expression for r is obtained by multiplying equations (30) in order by the cofactors of $f_1, \cdots, f_n$ in (fU) and adding:

$$r = -\frac{(fU)^2}{(gU)^2}. \tag{31}$$

If any two lines of curvature through P be given as U and V-curves, and their respective curvatures be denoted by $1/r'$ and $1/r''$, one gets from (30)

$$g_1(gU) = -\frac{1}{r'}f_1(fU), \ \cdots, \ g_n(gU) = -\frac{1}{r'}f_n(fU),$$

$$g_1(gV) = -\frac{1}{r''}f_1(fV), \ \cdots, \ g_n(gV) = -\frac{1}{r''}f_n(fV).$$

Multiply the equations of the first line in order by the cofactors of $f_1, \cdots, f_n$ in (fV) and add. Also multiply the equations of the second line in order by the cofactors of $f_1, \cdots, f_n$ in (fU) and add. Then

$$(gU)(gV) = -\frac{1}{r'}(fU)(fV) = -\frac{1}{r''}(fU)(fV),$$

so that either $r' = r''$ or $(fU)(fV) = 0$. Hence by (17) we have

Theorem I. *Any two distinct lines of curvature through an ordinary point P of R_n are orthogonal to each other.*

If the lines of curvature through P be taken as parameter lines, then, by (9),

$$a_{ik} = 0 \qquad (i \neq k).$$

It follows at once from (26) that also

$$\alpha_{ik} = 0 \qquad (i, k=1, \cdots, n; i \neq k).$$

Theorem II. *If the lines of curvature at an ordinary (not umbilic) point of R_n be taken as parameter lines, then*

$$a_{ik} = 0, \qquad \alpha_{ik} = 0 \qquad (i, k=1, \cdots, n; i \neq k).$$

§ 4. *Definition of the Curvatures of R_n in S_{n+1}.*

Equation (28) may be written in the form

$$H_0 + H_1 r + \cdots + H_{n-1} r^{n-1} + H_n r^n = 0, \tag{32}$$

where*

$$H_0 = |a_{ik}| = 1/\beta^2, \qquad H_n = |\alpha_{ik}|,$$

* M. I., (9).

while for $j = 1, \cdots, n$, H_j is the sum of all the determinants obtained from $|a_{ik}|$ by replacing in all possible ways j columns of $|a_{ik}|$ by the corresponding columns of $|\alpha_{ik}|$. Dividing (32) by H_0 one obtains

$$1 + K_1 r + \cdots + K_{n-1} r^{n-1} + K_n r^n = 0. \tag{33}$$

The coefficient K_n (the product of all the curvatures) is the Kronecker-Gaussian curvature of hyperspace. It has been shown to be expressible in terms of the first fundamental quantities and their derivatives (cf. K–G. C.). In this paper the coefficients of (33) are called the n curvatures of R_n in S_{n+1}. By definition

$$K_1 = \beta^2 \sum_{i,k}^{1,\cdots,n} \alpha_{ik} A^i_k, \qquad K_2 = \beta^2 \sum_{i_1 i_2,\, k_1 k_2}^{1,\cdots,n} \begin{vmatrix} \alpha_{i_1 k_1} & \alpha_{i_1 k_2} \\ \alpha_{i_2 k_1} & \alpha_{i_2 k_2} \end{vmatrix} \cdot A^{i_1 i_2}_{k_1 k_2} = \sum_{i_1 i_2 k_1 k_2}^{1,\cdots,n} \Delta_{\substack{i_1 i_2 \\ k_2 k_2}} A^{i_1 i_2}_{k_1 k_2}, \cdots,$$

$$K_m = \beta^2 \sum_{i_1 \cdots i_m k_1 \cdots k_m}^{1,\cdots,n} \Delta_{\substack{i_1 \cdots i_m \\ k_1 \cdots k_m}} \cdot A^{i_1 \cdots i_m}_{k_1 \cdots k_m} \qquad (m = 1, \cdots, n), \tag{34}$$

where A^i_k is the cofactor of a_{ik} in $|a_{ik}|$, $A^{i_1 i_2}_{k_1 k_2}$ is the *algebraic complement* of

$$\begin{vmatrix} a_{i_1 k_1} & a_{i_1 k_2} \\ a_{i_2 k_1} & a_{i_2 k_2} \end{vmatrix}$$

in $|a_{ik}|$, while $\Delta_{\substack{i_1 i_2 \\ k_1 k_2}}$ is the *second minor* of $|\alpha_{ik}|$ indicated for K_2 above; and similarly for the A's and Δ's in K_m. Both sets $i_1, \cdots, i_m$ and $k_1, \cdots, k_m$ are considered as being in ascending numerical order.

§ 5. *Invariant Symbolic Forms of* $K_1, \cdots, K_n$.

If F^i_k be the cofactor of f^i_k in the functional determinant $\{f', \cdots, f^n\}$, Maschke * has shown that

$$A^i_k = \frac{1}{(n-1)!} F'_i F'_k.$$

Thus

$$K_1 = \beta^2 \sum_{i,k}^{i,\cdots,n} \alpha_{ik} A^i_k = \frac{\beta^2}{(n-1)!} \sum_{i,k}^{i,\cdots,n} g_i g_k F'_i F'_k$$

$$= \frac{\beta^2}{(n-1)!} \{gf^2 \cdots f^n\}^2 = \frac{1}{(n-1)!} (gf)^2.$$

This suggests a method for reducing all the curvatures to convenient invariant forms. Let $F^{1 \cdots m}_{i_1 \cdots i_m}$ be the algebraic complement of

$$\begin{vmatrix} f'_{i_1} & \cdots & f'_{i_m} \\ \cdot & \cdots & \cdot \\ f^m_{i_1} & \cdots & f^m_{i_m} \end{vmatrix}$$

* M. I., p. 450.

in $\{f' \cdots f^n\}$. Then the product $F^{1\cdots m}_{i_1\cdots i_m} \cdot F^{1\cdots m}_{k_1\cdots k_m}$ may be written

$$\begin{vmatrix} f_1^{m+1} \cdots f_{i_1-1}^{m+1} f_{i_1+1}^{m+1} \cdots f_{i_m-1}^{m+1} f_{i_m+1}^{m+1} \cdots f_{i_n}^{m+1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f_1^n \cdots f_{i_1-1}^n f_{i_1+1}^n \cdots f_{i_m-1}^n f_{i_m+1}^n \cdots f_{i_n}^n \end{vmatrix} F^{1\cdots m}_{k_1\cdots k_m}.$$

If the first determinant of this product be expanded, one finds $(n-m)!$ terms of the form

$$(-1)^\mu f_1 \cdots f_{i_1-1} f_{i_1+1} \cdots f_{i_m-1} f'_{i_m+1} \cdots f_n \cdot F^{1\cdots m}_{k_1\cdots k_m},$$

where the suppressed upper indices of the first factor are understood to be any permutation of the numbers $m+1, \cdots, n$, while μ represents the number of inversions in the permutation. Since the equivalent symbols $f^{m+1}, \cdots, f^n$ may be interchanged in all possible ways without altering the value of the term, let them be so interchanged for each term as to reduce the first factor to $(-1)^\mu$ times the principal diagonal term of $F^{1\cdots m}_{i_1\cdots i_m}$. This causes an interchange of rows in the second (determinant) factor $F^{1\cdots m}_{k_1\cdots k_m}$ so that it becomes in each case $(-1)^\mu$ times its original form. Hence the above product becomes

$$(n-m)! f_1^{m+1} \cdots f_{i_1-1}^{i_1+m-1} f_{i_1+1}^{i_1+m} \cdots f_{i_m-1}^{i_m} f_{i_m+1}^{i_m+1} \cdots f_n^n \cdot F^{1\cdots m}_{k_1\cdots k_m}.$$

Multiplying each f into the corresponding row of the determinant $F^{1\cdots m}_{k_1\cdots k_m}$ (which has a form similar to that given above for $F^{1\cdots m}_{i_1\cdots i_m}$), we have

$$F^{1\cdots m}_{i_1\cdots i_m} \cdot F^{1\cdots m}_{k_1\cdots k_m} = (n-m)! A^{i_1\cdots i_m}_{k_1\cdots k_m},$$

or

$$A^{i_1\cdots i_m}_{k_1\cdots k_m} = \frac{1}{(n-m)!} F^{1\cdots m}_{i_1\cdots i_m} \cdot F^{1\cdots m}_{k_1\cdots k_m}. \tag{35}$$

Also

$$\Delta^{i_1\cdots i_m}_{k_1\cdots k_m} = \begin{vmatrix} \alpha_{i_1 k_1} \cdots \alpha_{i_1 k_m} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \alpha_{i_m k_1} \cdots \alpha_{i_m k_m} \end{vmatrix} = \begin{vmatrix} g'_{i_1} g'_{k_1} \cdots g'_{i_1} g'_{k_m} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ g^m_{i_m} g^m_{k_1} \cdots g^m_{i_m} g^m_{k_m} \end{vmatrix} = g'_{i_1} \cdots g^m_{i_m} \begin{vmatrix} g'_{k_1} \cdots g'_{k_m} \\ \cdot \quad \cdot \quad \cdot \\ g^m_{k_1} \cdots g^m_{k_m} \end{vmatrix},$$

$$\Delta^{i_1\cdots i_m}_{k_1\cdots k_m} = \frac{1}{m!} \begin{vmatrix} g'_{i_1} \cdots g'_{i_m} \\ \cdot \quad \cdot \quad \cdot \\ g^m_{i_1} \cdots g^m_{i_m} \end{vmatrix} \cdot \begin{vmatrix} g'_{k_1} \cdots g'_{k_m} \\ \cdot \quad \cdot \quad \cdot \\ g^m_{k_1} \cdots g^m_{k_m} \end{vmatrix}. \tag{36}$$

Substituting (35) and (36) in (34), one finds, by a well-known theorem of determinants,

$$K_m = \frac{1}{m!(n-m)!} (g' \cdots g^m f^{m+1} \cdots f^n)^2 = \frac{1}{m!(n-m)!} (g' \cdots g^m f)^2. \tag{37}$$

In particular,

$$K_1 = \frac{1}{(n-1)!} (gf)^2 = \Delta_1 g,$$

where, by M. I. (22), $\Delta_1 g$ is the first differential parameter of the first quadratic form ( ).

Since the other coefficients are corresponding differential parameters (the number of g's being the same as the subscript of K), it would seem fitting to generalize the notation and set *

$$K_m = \frac{1}{m!\,(n-m)!}(g' \cdots g^m f)^2 = \Delta^m g, \tag{38}$$

with the note that $\Delta' g = \Delta_1 g$.

§ 6. *Expression of $K_{2\nu}$ in terms of the first Fundamental Quantities and Derivatives.*

The generalization of the Gauss equation shows that any second order determinant of the second fundamental quantities is equal to a Riemann quadruple index symbol, which is expressible in terms of the first fundamental quantities and derivatives.† By K.–G. C. (27),

$$\begin{vmatrix} \alpha_{i_1 k_1} & \alpha_{i_1 k_2} \\ \alpha_{i_2 k_1} & \alpha_{i_2 k_2} \end{vmatrix} = (i_1 i_2 k_1 k_2) = \frac{1}{(n-1)!} f'_{i_1} f^2_{i_2} \begin{vmatrix} (fa)'_{k_1} & (fa)'_{k_2} \\ (fa)^2_{k_1} & (fa)^2_{k_2} \end{vmatrix}.$$

By an easy induction, any even order determinant of the α's is expressed in terms of the symbols of the a's as follows: ‡

$$\Delta^{i_1 \cdots i_{2\nu}}_{k_1 \cdots k_{2\nu}} = \frac{\epsilon^\nu}{(2\nu)!} \begin{vmatrix} (fa)'_{i_1} & \cdots & (fa)'_{i_{2\nu}} \\ \cdot & \cdots & \cdot \\ (fa)^{2\nu}_{i_1} & \cdots & (fa)^{2\nu}_{i_{2\nu}} \end{vmatrix} \cdot \begin{vmatrix} f'_{k_1} & \cdots & f'_{k_{2\nu}} \\ \cdot & \cdots & \cdot \\ f^{2\nu}_{k_1} & \cdots & f^{2\nu}_{k_{2\nu}} \end{vmatrix}, \tag{39}$$

where $\epsilon = 1/(n-1)!$; the symbol $(fa)^j$ contains f^j and $a^2 \cdots a^n$, while the symbols a in every consecutive pair $(fa^2 \cdots a^n)^{2\lambda-1}$, $(fa^2 \cdots a^n)^{2\lambda}$ are equal when they have the same index, otherwise they are distinct but equivalent symbols of the first fundamental form (3).

Now from (34), (35), and (39),

$$K_{2\nu} = \frac{\beta^2 \epsilon^\nu}{(2\nu)!\,(n-2\nu)!} \sum_{i_1 \cdots i_{2\nu} k_1 \cdots k_{2\nu}}^{1 \cdots n} \begin{vmatrix} (fa)'_{i_1} & \cdots & (fa)'_{i_{2\nu}} \\ \cdot & \cdots & \cdot \\ (fa)^{2\nu}_{i_1} & \cdots & (fa)^{2\nu}_{i_{2\nu}} \end{vmatrix} \cdot \begin{vmatrix} (f'_{k_1} & \cdots & f'_{k_{2\nu}} \\ \cdot & \cdots & \cdot \\ f^{2\nu}_{k_1} & \cdots & f^{2\nu}_{k_{2\nu}} \end{vmatrix}$$

$$\times F^{1 \cdots 2\nu}_{i_1 \cdots i_{2\nu}} F^{1 \cdots 2\nu}_{k_1 \cdots k_{2\nu}},$$

or

$$K_{2\nu} = \frac{\epsilon^\nu}{(2\nu)!\,(n-2\nu)!} \big((fa)' \cdots (fa)^{2\nu} f \big)(f). \tag{40}$$

* The use of $\Delta_m g$ would conflict with the second differential parameter of ordinary differential geometry, which has an entirely different meaning. Cf. BIANCHI, *Lezioni di Geometria Differenziale*, vol. I, p. 67.

† M. I., (117)–(126).

‡ Cf. K.–G. C., (28).

This gives Maschke's expression * for K_n when n is even:

$$K_n = \frac{1}{n![(n-1)!]^{n/2}} \left((fa)' \cdots (fa)^n \right)(f) \qquad (n \text{ even}).$$

Theorem. *The mean curvatures $K_{2\nu}$, with even subscript, are represented in* (40) *as rational integral functions of the coefficients of the first fundamental form and their derivatives.*

§ 7. *Expression of $K_{2\nu+1}$ in terms of the first Fundamental Quantities and Derivatives, when ν is greater than zero.*

Use is made of the determinant theorem

$$(41) \quad \Delta^2_{\substack{i_1 \cdots i_{2\nu+1} \\ k_1 \cdots k_{2\nu+1}}} = \frac{1}{2} \sum_{j,r}^{1,\cdots,n} \begin{vmatrix} \alpha_{i_j k_s} & \alpha_{i_j k_t} \\ \alpha_{i_r k_s} & \alpha_{i_r k_t} \end{vmatrix} \cdot \begin{vmatrix} D_{i_j k_s} & D_{i_j k_t} \\ D_{i_r k_s} & D_{i_r k_t} \end{vmatrix} \quad (s, t = 1, \cdots, 2\nu+1;\ s \neq t),$$

where $\nu \neq 0$ and the D's are cofactors of the corresponding α's in $\Delta^{i_1 \cdots i_{2\nu+1}}_{k_1 \cdots k_{2\nu+1}}$ and are therefore all of even order and expressible by (39). The results are

$$D_{i_j k_s} = \frac{\epsilon^\nu}{(2\nu)!} F'_{i_j} (FA)'_{k_s}, \qquad D_{i_j k_t} = \frac{\epsilon^\nu}{(2\nu)!} \Phi'_{i_j} (\Phi B)'_{k_t},$$

$$D_{i_r k_s} = \frac{\epsilon^\nu}{(2\nu)!} F'_{i_r} (FA)'_{k_s}, \qquad D_{i_r k_t} = \frac{\epsilon^\nu}{(2\nu)!} \Phi'_{i_r} (\Phi B)'_{k_t},$$

where F'_{i_j} is the cofactor of f'_{i_j} in $\{ f'_{i_1} \cdots f^{2\nu+1}_{i_{2\nu+1}} \}$, $\cdots$, $(\Phi B)'_{k_t}$ is the cofactor of $(\phi b)'_{k_t}$ in $\{ (\phi b)'_{k_1} \cdots (\phi b)^{2\nu+1}_{k_{2\nu+1}} \}$. Also, by M. I. (120),

$$\begin{vmatrix} \alpha_{i_j k_s} & \alpha_{i_j k_t} \\ \alpha_{i_r k_s} & \alpha_{i_r k_t} \end{vmatrix} = \epsilon (fc)'_{k_s} (\phi c)'_{k_t} \begin{vmatrix} f'_{i_j} & \phi'_{i_j} \\ f'_{i_r} & \phi'_{i_r} \end{vmatrix},$$

Substituting in (41), we find

$$\Delta^2_{\substack{i_1 \cdots i_{2\nu+1} \\ k_1 \cdots k_{2\nu+1}}} = \frac{\epsilon^{2\nu+1}}{[(2\nu)!]^2} (fc)'_{k_s} (FA)'_{k_s} (\phi c)'_{k_t} (\Phi B)'_{k_t} \times \frac{1}{2} \sum_{j,r}^{1 \cdots 2\nu+1} \begin{vmatrix} f'_{i_j} & \phi'_{i_j} \\ f'_{i_r} & \phi'_{i_r} \end{vmatrix} \cdot \begin{vmatrix} F'_{i_j} & \Phi'_{i_j} \\ F'_{i_r} & \Phi'_{i_r} \end{vmatrix}.$$

This last sum expands into

$$\frac{1}{2} \sum_{j,r}^{1,\cdots,2\nu+1} [f'_{i_j} F'_{i_j} \phi'_{i_r} \Phi'_{i_r} - f'_{i_j} \Phi'_{i_j} \phi'_{i_r} F'_{i_r} - f'_{i_r} \Phi'_{i_r} \phi'_{i_j} F'_{i_j} + f'_{i_r} F'_{i_r} \phi'_{i_j} \Phi'_{i_j}] = \begin{vmatrix} \{ f'_{i_1} \cdots f^{2\nu+1}_{i_{2\nu+1}} \} & \{ f'_{i_1} \phi^2_{i_2} \cdots \phi^{2\nu+1}_{i_{2\nu+1}} \} \\ \{ \phi'_{i_1} f^2_{i_2} \cdots f^{2\nu+1}_{i_{2\nu+1}} \} & \{ \phi'_{i_1} \cdots \phi^{2\nu+1}_{i_{2\nu+1}} \} \end{vmatrix},$$

so that

$$\Delta^2_{\substack{i_1 \cdots i_{2\nu+1} \\ k_1 \cdots k_{2\nu+1}}} = \frac{\epsilon^{2\nu+1}}{[(2\nu)!]^2} (fc)'_{k_s} (FA)'_{k_s} (\phi c)'_{k_t} (\Phi B)'_{k_t} \times \begin{vmatrix} \{ f'_{i_1} \cdots f^{2\nu+1}_{i_{2\nu+1}} \} & \{ f'_{i_1} \phi^2_{i_2} \cdots \phi^{2\nu+1}_{i_{2\nu+1}} \} \\ \{ \phi'_{i_1} f^2_{i_2} \cdots f^{2\nu+1}_{i_{2\nu+1}} \} & \{ \phi'_{i_1} \cdots \phi^{2\nu+1}_{i_{2\nu+1}} \} \end{vmatrix}.$$

* K.-G. C., (29).

By (41) this equation holds for all values of s and t from 1 to $2\nu+1$ except $s=t$. When $s=t$, the second member vanishes. Sum the equations given by using all values of s and t from 1 to $2\nu+1$ and divide by $(2\nu+1)2\nu$; also multiply by β^i. Then

$$\beta^i \Delta^2_{\substack{i_1 \cdots i_{2\nu+1} \\ k_1 \cdots k_{2\nu+1}}} = \frac{\varepsilon^{2\nu+1}}{(2\nu+1)(2\nu)[(2\nu)!]^2} \left((fc)'_{k_1} (fa)^2_{k_2} \cdots (fa)^{2\nu+1}_{k_{2\nu+1}} \right)$$

(42)

$$\times \left((\phi c)'_{k_1} (\phi b)^2_{k_2} \cdots (\phi b)^{2\nu+1}_{k_{2\nu+1}} \right) \begin{vmatrix} \{ f'_{i_1} \cdots f^{2\nu+1}_{i_{2\nu+1}} \} & \{ f'_{i_1} \phi^2_{i_2} \cdots \phi^{2\nu+1}_{i_{2\nu+1}} \} \\ \{ \phi'_{i_1} f^2_{i_2} \cdots f^{2\nu+1}_{i_{2\nu+1}} \} & \{ \phi'_{i_1} \cdots \phi^{2\nu+1}_{i_{2\nu+1}} \} \end{vmatrix}.$$

And by (34)

$$K_{2\nu+1} = \sum_{i_1 \cdots i_{2\nu+1} k_1 \cdots k_{2\nu+1}}^{1, \ldots, n} \left[\beta^i \Delta^2_{\substack{i_1 \cdots i_{2\nu+1} \\ k_1 \cdots k_{2\nu+1}}} \right]^{\frac{1}{2}} A_{\substack{i_1 \cdots i_{2\nu+1} \\ k_1 \cdots k_{2\nu+1}}} \qquad (\nu > 0).$$

Thus by (34) and (42) we have $K_{2\nu+1} (\nu > 0)$ expressed in terms of the first fundamental quantities and derivatives (but only in the irrational form of a sum of square roots).

The case of K_1 presents special difficulty:

$$K_1 = \beta^2 \sum_{ik}^{1, \ldots, n} \alpha_{ik} A^i_k.$$

In K.–G. C. (p. 24), Maschke suggests a method for expressing the α's in terms of the a's when n is odd. His formula (24) should, however, be written,

$$\alpha_{11} \Delta^{n-2} = \begin{vmatrix} A_{22} \cdots A_{2n} \\ A_{n2} \cdots A_{nn} \end{vmatrix}. \tag{43}$$

If n is odd, the elements of the second member of (43) are of even order, and therefore expressible by (39), and similarly for every α. But Δ itself is of odd order, and is raised to an odd power ($n-2$ instead of $n-1$).* Equation (43) is true also for even values of n, so that the α's are always expressible by (43) in terms of the first fundamental quantities and derivatives (if $n > 2$), but in all cases irrationally.

Using (43), the author has calculated irrational values of K_1 when n is greater than two; but the notation is so complicated that the presentation of the results seems impracticable, if not also useless.†

If $2\nu+1=n$, the sum reduces to a single term and formulas (34) and (42)

*Cf. BÔCHER, *Introduction to Higher Algebra*, § 11.

† In a recent paper the author has calculated the value of K_1 as well as of the other curvatures of odd subscript, for a space of $n-1$ dimensions defined in R_n by the equation $U(x_1 \cdots x_n) = 0$. These values involve only the coefficients of the first fundamental form of R_n and their derivatives, together with the function U.

give a rational value for K_n^2,

$$(44)\quad K_n^2=\beta^4\Delta^2=\frac{\epsilon^{n+2}}{n(n-1)}\big((fc)'(fa)^2\cdots(fa)^n\big)\big((\phi c)'(\phi b)^2\cdots(\phi b)^n\big)\begin{vmatrix}(f) & (f'\phi)\\ (\phi' f) & (\phi)\end{vmatrix}.$$

By the method used in K.–G. C. (p. 86), this may be reduced to Maschke's form (31): *

$$(45)\quad K_n^2=\frac{1}{n[(n-1)!]^{n+2}}\big((fc)'(fa)^2\cdots(fa)^n\big)\big((\phi c)'(\phi b)^2\cdots(\phi b)^n\big)(f'\phi' f)(f^2\phi^2\phi).$$

The rather unsatisfactory results of this section are then as follows:

If n is odd, K_n^2 is expressed by (45) *as a rational function of the first fundamental quantities and their derivatives. Equations* (34) *and* (42) *give irrational expressions for the curvatures of odd index except K_1, for which no expression is here given.*

PART II.

Invariants of R_λ in R_n.

The quantities $K_{2\nu}$ and K_n^2, for n odd, are by their forms (40) and (45) differential invariants of the first fundamental quadratic form (3). When (3) defines the arc-element of a space R_n of n dimensions contained in an euclidean space S_{n+1} of $n+1$ dimensions, these K's have the geometric meaning already assigned to them. It is our object † to find corresponding invariants of a space R_λ of λ dimensions, represented as differential parameters of a general space R_n of higher dimensions containing R_λ.

§ 1. *Definitions and Preliminary Formulas.*

In the general space R_n, of n dimensions, whose coördinates are $x_1, \cdots, x_n$ and whose arc-element is defined by equation (3), let the space R_λ of λ dimensions ($\lambda < n$) be defined by the $n-\lambda$ equations

$$(46)\qquad U^{\lambda+1}(x_1, \cdots, x_n)=\text{const.}, \cdots, U^n(x_1, \cdots, x_n)=\text{const.}$$

If λ other arbitrarily chosen functions of $x_1, \cdots, x_n$, say $u', \cdots, u^\lambda$, such that

$$\Delta=(u'\cdots u^\lambda U^{\lambda+1}\cdots U^n)\neq 0,$$

are adjoined to these, the space R_λ may also be represented in parametric form

$$(47)\qquad x_1=x_1(u', \cdots, u^\lambda), \cdots, x_n=x_n(u', \cdots, u^\lambda),$$

* In Maschke's reduction there are two slight numerical errors which balance each other. His equation (30) differs from (44) above in that he has divided by n^2 instead of by $n(n-1)$; while in his reduction of (30) there are $n-1$ of the terms which become equal, instead of n.

† Cf. K.–G. C., § 5.

by solving the $n - \lambda$ equations (46) with the λ equations

$$u'(x_1, \cdots, x_n) = u', \cdots, u^\lambda(x_1, \cdots, x_n) = u^\lambda. \tag{48}$$

Any n differentials satisfying the $n - \lambda$ equations, found by differentiating (46),

$$\sum_{i=1}^{n} U_i^{\lambda+1} dx_i = 0, \cdots, \quad \sum_{i=1}^{n} U_i^n dx_i = 0$$

determine a certain direction in R_λ. In order to find these differentials in terms of $du', \cdots, du^\lambda$, we differentiate also equations (48) and solve the set

$$\begin{aligned} u_1' dx_1 + \cdots + u_n' dx_n &= du', \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ u_1^\lambda dx_1 + \cdots + u_n^\lambda dx_n &= du^\lambda, \\ U_1^{\lambda+1} dx_1 + \cdots + U_n^{\lambda+1} dx_n &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ U_1^n dx_1 + \cdots + U_n^n dx_n &= 0. \end{aligned}$$

If A^{kr} be the cofactor of u_r^k in Δ, then

$$dx_r = \frac{1}{\Delta} \sum_{k=1}^{\lambda} A^{kr} du^k$$

and therefore,

$$\sum_{r=1}^{n} p_r dx_r = \frac{1}{\Delta} \sum_{k=1}^{\lambda} \{u' \cdots u^{k-1} p u^{k+1} \cdots u^\lambda U\} du^k, \tag{49}$$

where p is any ordinary function of $x_1, \cdots, x_n$.

In order to find the expression for ds in terms of $u', \cdots, u^\lambda$, we introduce for the differential quantic (3) the symbolic form

$$ds^2 = \sum_{i,k}^{1,\cdots,n} a_{ik} dx_i dx_k = \left[\sum_{i=1}^{n} f_i dx_i \right]^2.$$

Then (49) gives for the length element in R_λ

$$\begin{aligned} ds^2 &= \frac{1}{\Delta^2} \left[\sum_{i=1}^{\lambda} \{u' \cdots u^{i-1} f u^{i+1} \cdots u^\lambda U\} du_i \right]^2 \\ &= \frac{1}{\beta^2 \Delta^2} \left[\sum_{i=1}^{\lambda} (u' \cdots u^{i-1} f u^{i+1} \cdots u^\lambda U) du_i \right]^2. \end{aligned} \tag{50}$$

We may also introduce for ds^2, as given in terms of $u', \cdots, u^\lambda$, the symbolic form

$$ds^2 = \left[\sum_{i=1}^{\lambda} \mathfrak{f}_i du^i \right]^2. \tag{51}$$

By comparing (50) and (51) we find

$$\mathfrak{f}_i = \frac{1}{\Delta} \{u' \cdots u^{i-1} f u^{i+1} \cdots u^\lambda U\} = \frac{1}{\beta\Delta} (u' \cdots u^{i-1} f u^{i+1} \cdots u^\lambda U). \tag{52}$$

If we use the symbols of form (51), the invariants $K_{2\nu}$ and K_λ^2 (λ odd) of R_λ may be written, by (40) and (45),

$$(53)\quad (2\nu)!\,(\lambda-2\nu)!\,[(\lambda-1)!]^\nu K_{2\nu} = G_{2\nu} = \big((\mathfrak{f}a)'\cdots(\mathfrak{f}a)^{2\nu}\mathfrak{f}^{2\nu+1}\cdots\mathfrak{f}^\lambda\big)(\mathfrak{f}'\cdots\mathfrak{f}^\lambda),$$

$$(54)\quad \begin{aligned}\lambda\,[(\lambda-1)!]^{\lambda+2} K_\lambda^2 = G_\lambda^2 = \big((\mathfrak{f}c)'(\mathfrak{f}a)^2\cdots(\mathfrak{f}a)^\lambda\big)\\ \times\big((\mathfrak{g}c)'(\mathfrak{g}b)^2\cdots(\mathfrak{g}b)^\lambda\big)(\mathfrak{f}'\mathfrak{g}'\mathfrak{f}^3\cdots\mathfrak{f}^\lambda)(\mathfrak{f}^2\mathfrak{g}^2\cdots\mathfrak{g}^\lambda),\end{aligned}$$

where $G_{2\nu}$ and G_λ^2 are introduced merely for convenience. In all invariantive brackets containing the new symbols, of the quadratic form (51), the differentiation is with respect to the λ variables $u', \cdots, u^\lambda$. This is indicated sufficiently by the German type and the number of symbols inside the brackets. β_u is defined by the equation

$$(\mathfrak{f}'\cdots\mathfrak{f}^\lambda) = \beta_u\{f'\cdots f^\lambda\}.$$

We now proceed to compute the values of the invariantive expressions used in (53) and (54) in terms of the symbols of the first fundamental form (3), of R_n and the functions $U^{\lambda+1}, \cdots, U^n$ which define R_λ in R_n.

By means of (52) and D. P. (3), we obtain

$$\{\mathfrak{f}'\cdots\mathfrak{f}^\lambda\} = \frac{1}{\Delta^\lambda}\{f'\cdots f^\lambda U\}\{u'\cdots u^\lambda U\}^{\lambda-1} = \frac{1}{\Delta}\{f'\cdots f^\lambda U\},$$

so that

$$(55)\quad \frac{1}{\beta_u}(\mathfrak{f}'\cdots\mathfrak{f}^\lambda) = \frac{1}{\beta\Delta}(f'\cdots f^\lambda U).$$

To calculate the value of β_u, square (55) and simplify the result by placing $(\mathfrak{f}'\cdots\mathfrak{f}^\lambda)^2 = \lambda!$, according to M. I. (17), and $(f'\cdots f^\lambda U)^2 = \lambda!\,(n-\lambda)!\,\Delta^{n-\lambda}U$ by (38). This gives

$$(56)\quad \beta_u = \omega\beta\Delta, \qquad \omega = \sqrt{\frac{1}{(n-\lambda)!\,\Delta^{n-\lambda}U}}.$$

Then

$$(\mathfrak{f}'\cdots\mathfrak{f}^\lambda) = \omega(f'\cdots f^\lambda U).$$

The other invariantive forms in (53) and (54) are reduced by the same method, and by interchanging equivalent symbols, giving*

$$(57)\quad \begin{aligned}&(\mathfrak{f}'\cdots\mathfrak{f}^\lambda) = \omega(f'\cdots f^\lambda U),\\ &(\mathfrak{f}'\mathfrak{g}'\mathfrak{f}^3\cdots\mathfrak{f}^\lambda) = \omega(f'\phi' f^3\cdots f^\lambda U), \qquad (\mathfrak{f}^2\mathfrak{g}^2\cdots\mathfrak{g}^\lambda) = \omega(f^2\phi^2\cdots\phi^\lambda U)\\ &\big((\mathfrak{f}a)'\cdots(\mathfrak{f}a)^{2\nu}\mathfrak{f}^{2\nu+1}\cdots\mathfrak{f}^\lambda\big)\\ &\qquad = \omega\big(\omega(faU)', \omega(faU)^2, \cdots, \omega(faU)^{2\nu}, f^{2\nu+1}\cdots f^\lambda U\big),\end{aligned}$$

*Inside the invariantive brackets, we have followed MASCHKE's custom of omitting commas between symbols, except where ambiguity might occur. Cf. M. I., p. 448.

$$(57)\qquad \begin{aligned}\big((\mathfrak{f}\mathfrak{c})'(\mathfrak{f}\mathfrak{a})^2\cdots(\mathfrak{f}\mathfrak{a})^\lambda\big) &= \omega\big(\omega(fcU)',\ \omega(faU)^2,\ \cdots,\ \omega(faU)^\lambda U\big),\\ \big((\mathfrak{g}\mathfrak{c})'(\mathfrak{g}\mathfrak{b})^2\cdots(\mathfrak{g}\mathfrak{b})^\lambda\big) &= \omega\big(\omega(\phi cU)',\ \omega(\phi bU)^2,\ \cdots,\ \omega(\phi bU)^\lambda U\big).\end{aligned}$$

§ 2. *Expression for* $K_{2\nu}$.

By (53) and (57),

$$G_{2\nu} = \omega^2(\omega(faU)',\ \cdots,\ \omega(faU)^{2\nu}, f^{2\nu+1}\cdots f^\lambda U)(f'\cdots f^\lambda U).$$

Applying D. P. (4) to the second member, we get

$$G_{2\nu}=\omega^{2\nu+2}\big((faU)'\cdots(faU)^{2\nu}f^{2\nu+1}\cdots f^\lambda U\big)(f'\cdots f^\lambda U)+\omega^{2\nu+1}(f'\cdots f^\lambda U)$$
$$\times\sum_{k=1}^{2\nu}(faU)^k\big((faU)'\cdots(faU)^{k-1},\ \omega,\ (faU)^{k+1}\cdots(faU)^{2\nu}f^{2\nu+1}\cdots f^\lambda U\big).$$

It will now be shown that each term of this last sum vanishes. Aside from the factor $\omega^{2\nu+1}$, each odd term of this sum may be written in the form

$$T=(-1)^{\lambda-1}(f^{k+1}\cdots f^\lambda f'\cdots f^k U)(faU)^k$$
$$\times\big((faU)'\cdots(faU)^{k-1},\ \omega,\ (faU)^{k+1}\cdots(faU)^{2\nu}f^{2\nu+1}\cdots f^\lambda U\big).$$

Applying D. P. (1) to the first two brackets of the second member, we obtain

$$T=(-1)^{\lambda-1}\left[\begin{array}{l}(f^k f^{k+2}\cdots f^\lambda f'\cdots f^k U)(f^{k+1}a^2\cdots a^\lambda U)\\ +(a^2 f^{k+2}\cdots f^\lambda f'\cdots f^k U)(f^k f^{k+1}a^3\cdots a^\lambda U)\\ +(a^3 f^{k+2}\cdots f^\lambda f'\cdots f^k U)(f^k a^2 f^{k+1}a^4\cdots a^\lambda U)\\ +\ \cdots\cdots\cdots\cdots\\ +(a^\lambda f^{k+2}\cdots f^\lambda f'\cdots f^k U)(f^k a^2\cdots a^{\lambda-1}f^{k+1}U)\end{array}\right]$$
$$\times\big((faU)'\cdots(faU)^{k-1},\ \omega,\ (faU)^{k+1}\cdots(faU)^{2\nu}f^{2\nu+1}\cdots f^\lambda U\big).$$

Of these λ terms, the first vanishes because of two identical rows in the first bracket, while the others become equal to each other if we interchange f^{k+1} with $a^2\cdots a^\lambda$ in turn and in each case restore the original order in $(faU)^{k+1}$ by the interchange of two rows.* Thus

$$T=(-1)^{\lambda-1}(1-\lambda)(f^{k+1}\cdots f^\lambda f'\cdots f^k U)(f^k a^2\cdots a^\lambda U)$$
$$\times\big((faU)'\cdots(faU)^{k-1},\ \omega,\ (faU)^{k+1}\cdots\big)=(-1)^{\lambda-1}(1-\lambda)\,T.$$

Hence $T=0$ for odd values of k.

If k is even, each term T may be written

$$T=(f^{k-1}\cdots f^\lambda f'\cdots f^{k-2}U)(faU)^k$$
$$\times\big((faU)'\cdots(faU)^{k-1},\ \omega,\ (faU)^{k+1}\cdots(faU)^{2\nu}f^{2\nu+1}\cdots f^\lambda U\big).$$

*Cf. K.-G. C., p. 92.

By applying D. P. (1) to the first two brackets, and proceeding as above, one finds $T = 0$ also for even values of k.

With the help of these results (58) becomes

$$G_{2\nu} = \omega^{2\nu+2}\big((faU)' \cdots (faU)^{2\nu} f^{2\nu+1} \cdots f^{\lambda} U\big)(f' \cdots f^{\lambda} U).$$

Then, by (53) and (56),

$$(59) \quad K_{2\nu} = \frac{(\lambda-1)!\big((faU)' \cdots (faU)^{2\nu} f^{2\nu+1} \cdots f^{\lambda} U\big)(f' \cdots f^{\lambda} U)}{(2\nu)!(\lambda-2\nu)!\,[(\lambda-1)!(n-\lambda)!\Delta^{n-\lambda}U]^{\nu+1}}.$$

If $2\nu = \lambda$, (59) becomes

$$(60) \qquad K_{\lambda} = \frac{\big((faU)' \cdots (faU)^{\lambda} U\big)(f' \cdots f^{\lambda} U)}{\lambda[(\lambda-1)!(n-\lambda)!\Delta^{n-\lambda}U]^{(\lambda+2)/2}},$$

which agrees with Maschke's form, K.–G. C. (60). The symbols f and a belong to the quadratic form (3), expressing the length element of R_n. Further, $(faU)^{i} = (f^{i}a^{2} \cdots a^{\lambda} U^{\lambda+1} \cdots U^{n})$, in which f^{i} is equal to f' in $(f' \cdots f^{\lambda} U)$, while the sets of symbols $a^{2} \cdots a^{\lambda}$ are equal in any two consecutive brackets $(faU)^{2k-1}$, $(faU)^{2k}$ and otherwise distinct.

The result is then that $K_{2\nu}$, for the space R_{λ}, is expressible rationally in terms of the coefficients of the first fundamental form of R_n and their derivatives, together with the functions $U^{\lambda+1}, \cdots, U^{n}$ (which define R_{λ} in R_n) and their derivatives.

§ 3. *Expression for K_{λ}^{2} when λ is odd.**

The invariant K_{λ}^{2} (λ odd) can be expressed in a manner similar to the above. Substituting from (57) into (54), one gets

$$(61) \quad \begin{aligned} G_{\lambda}^{2} = {} & \omega^{4}\big(\omega(fcU)', \omega(faU)^{2}, \cdots, \omega(faU)^{\lambda}U\big) \\ & \times\big(\omega(\phi cU)', \omega(\phi bU)^{2}, \cdots, \omega(\phi bU)^{\lambda}U\big)\big(f'\phi' f^{3} \cdots f^{\lambda}U)(f^{2}\phi^{2} \cdots \phi^{\lambda}U\big). \end{aligned}$$

By D. P. (4),

$$\begin{aligned} \big(\omega(fcU)', \omega(faU)^{2}, \cdots, \omega(faU)^{\lambda}U\big) &= \omega^{\lambda}\big((fcU)'(faU)^{2} \cdots (faU)^{\lambda}U\big) \\ &+ \omega^{\lambda-1}(fcU)'\big(\omega, (faU)^{2} \cdots (faU)^{\lambda}U\big) \\ &+ \omega^{\lambda-1}\sum_{i=2}^{\lambda}(faU)^{i}\big((fcU)'(faU)^{2} \cdots (faU)^{i-1}, \omega, (faU)^{i+1} \cdots (faU)^{\lambda}U\big) \\ &\equiv \omega^{\lambda}\alpha_{1} + \omega^{\lambda-1}\alpha_{2} + \omega^{\lambda-1}\alpha_{3}. \end{aligned}$$

$$\begin{aligned} \big(\omega(\phi cU)', \omega(\phi bU)^{2}, \cdots, \omega(\phi bU)^{\lambda}U\big) &= \omega^{\lambda}\big((\phi cU)'(\phi bU)^{2} \cdots (\phi bU)^{\lambda}U\big) \\ &+ \omega^{\lambda-1}(\phi cU)'\big(\omega, (\phi bU)^{2} \cdots (\phi bU)^{\lambda}U\big) \\ &+ \omega^{\lambda-1}\sum_{k=2}^{\lambda}(\phi bU)^{k}\big((\phi cU)'(\phi bU)^{2} \cdots (\phi bU)^{k-1}, \omega, (\phi bU)^{k+1} \cdots (\phi bU)^{\lambda}U\big) \\ &\equiv \omega^{\lambda}\beta_{1} + \omega^{\lambda-1}\beta_{2} + \omega^{\lambda-1}\beta_{3}. \end{aligned}$$

* See K.–G. C., p. 93.

The notations $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ are used for brevity to represent the expressions whose relative places they occupy. If we also use

$$\gamma = (f'\phi' f^3 \cdots f^\lambda U), \qquad \delta = (f^2\phi^2 \cdots \phi^\lambda U),$$

then

$$G_\lambda^2 = \omega^{2\lambda+2}[\omega\alpha_1 + \alpha_2 + \alpha_3][\omega\beta_1 + \beta_2 + \beta_3]\gamma\delta. \tag{62}$$

The nine terms of this product (omitting powers of ω) will now be considered in the following order:

1) $\alpha_1\beta_1\gamma\delta$, 4) $\alpha_3\beta_2\gamma\delta$, 7) $\alpha_1\beta_3\gamma\delta$,

2) $\alpha_1\beta_2\gamma\delta$, 5) $\alpha_2\beta_1\gamma\delta$, 8) $\alpha_3\beta_3\gamma\delta$,

3) $\alpha_2\beta_2\gamma\delta$, 6) $\alpha_2\beta_3\gamma\delta$, 9) $\alpha_3\beta_1\gamma\delta$.

For the first we have $\alpha_1\beta_1\gamma\delta = L$, where

$$\begin{aligned} L = {} & \big((fcU)'(faU)^2 \cdots (faU)^\lambda U\big) \\ & \times \big((\phi cU)'(\phi bU)^2 \cdots (\phi bU)^\lambda U\big)(f'\phi' f^3 \cdots f^\lambda U)(f^2\phi^2 \cdots \phi^\lambda U). \end{aligned} \tag{63}$$

The second is shown to vanish as follows:

$$\begin{aligned} 2)\ \alpha_1\beta_2\gamma\delta &= (f'\phi' f^3 \cdots f^\lambda U)(\phi' c^2 \cdots c^\lambda U) \\ &\quad \times \big((fcU)'(faU)^2 \cdots (faU)^\lambda U\big)(f^2\phi^2 \cdots \phi^\lambda U)\big(\omega, (\phi bU)^2 \cdots (\phi bU)^\lambda U\big) \\ &= \begin{bmatrix} (\phi'\phi' f^3 \cdots f^\lambda U)(f'c^2 \cdots c^\lambda U) \\ +(c^2\phi' f^3 \cdots f^\lambda U)(\phi' f' c^3 \cdots c^\lambda U) \\ +(c^3\phi' f^3 \cdots f^\lambda U)(\phi' c^2 f' c^4 \cdots c^\lambda U) \\ + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ +(c^\lambda\phi' f^3 \cdots f^\lambda U)(\phi' c^2 \cdots c^{\lambda-1} f' U) \end{bmatrix} \big((fcU)'(faU)^2 \cdots (faU)^\lambda U\big) \cdots \qquad \text{[by D. P. (1)]} \\ &= (1-\lambda)(f'\phi' f^3 \cdots f^\lambda U)(\phi' c^2 \cdots c^\lambda U)\big((fcU)'(faU)^2 \cdots (faU)^\lambda U\big) \cdots \\ &= (1-\lambda)\alpha_1\beta_2\gamma\delta. \end{aligned}$$

Hence the second vanishes. The third and fourth are shown to vanish by applying D. P. (1) to exactly the same expressions.

For the fifth term,

$$\begin{aligned} 5)\ \alpha_2\beta_1\gamma\delta &= (f''\phi' f^3 \cdots f^\lambda U)(f'c^2 \cdots c^\lambda U)\big((\phi cU)'(\phi bU)^2 \cdots (\phi bU)^\lambda U\big) \\ &\quad \times \big(\omega, (faU)^2 \cdots (faU)^\lambda U\big)(f^2\phi^2 \cdots \phi\ U) \\ &= (\phi' f^3 \cdots f^\lambda f'' U)(f'c^2 \cdots c^\lambda U)\big((\phi cU)'(\phi bU)^2 \cdots (\phi bU)^\lambda U\big) \cdots . \end{aligned}$$

By applying D. P. (1) to the first two forms and simplifying as for 2), we find

$$\alpha_2\beta_1\gamma\delta = (1-\lambda)\alpha_2\beta_1\gamma\delta.$$

Hence 5) vanishes, and the sixth term is shown to vanish by applying D. P. (1) to the same forms.

For the seventh term,

$$7)\quad \alpha_1\beta_3\gamma\delta = \sum_{k=2}^{\lambda} (\phi^k b^2 \cdots b^\lambda U)(f^2\phi^2 \cdots \phi^\lambda U) \times \big((\phi c U)'(\phi b U)^2 \cdots (\phi b U)^{k-1}, \omega, \cdots\big) \times (f'\phi' f^3 \cdots f^\lambda U)\big((f c U)'(f a U)^2 \cdots (f a U)^\lambda U\big).$$

This sum is shown to vanish for all values of k by the method used for (58), and the vanishing of 8) follows by the same method.

For the last term,

$$9)\quad \alpha_3\beta_1\gamma\delta = \sum_{i=2}^{\lambda} (f a U)^i (f'\phi' f^3 \cdots f^\lambda U) \times \big((f c U)'(f a U)^2 \cdots (f a U)^{i-1}, \omega, (f a U)^{i+1} \cdots (f a U)^\lambda U\big) \times (f^2\phi^2 \cdots \phi^\lambda U)\big((\phi c U)'(\phi b U)^2 \cdots (\phi b U)^\lambda U\big).$$

The terms in which $i > 3$ vanish by the methods used for (58), but the terms T_2 (for $i = 2$) and T_3 (for $i = 3$) do not vanish and require special treatment.

We have $T_2 = (f^3 \cdots f^\lambda f'\phi' U)(f^2 a^2 \cdots a^\lambda U) N$, where

$$N = \big((f c U)', \omega, (f a U)^3 \cdots (f a U)^\lambda U\big)(f^2\phi^2 \cdots \phi^\lambda U) \times \big((\phi c U)'(\phi b U)^2 \cdots (\phi b U)^\lambda U)\big),$$

$$T_2 = \begin{bmatrix} (f^2 f^4 \cdots f^\lambda f'\phi' U)(f^3 a^2 \cdots a^\lambda U) \\ +(a_2 f^4 \cdots f^\lambda f'\phi' U)(f^2 f^3 a^3 \cdots a^\lambda U) \\ +(a^3 f^4 \cdots f^\lambda f'\phi' U)(f^2 a^2 f^3 a^4 \cdots a^\lambda U) \\ + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ +(a^\lambda f^4 \cdots f^\lambda f'\phi' U)(f^2 a^2 \cdots a^{\lambda-1} f^3 U) \end{bmatrix} N \qquad \text{[by D. P. (1)]}$$

$$= (f^2 f^4 \cdots f^\lambda f'\phi' U)(f^3 a^2 \cdots a^\lambda U) N + (1-\lambda) T_2.$$

Hence

$$T_2 = \frac{1}{\lambda}(f^2 f^4 \cdots f^\lambda f'\phi' U)(f^3 a^2 \cdots a^\lambda U) N.$$

If now, we interchange f^2 and f^3 and then restore the regular order of symbols, we get

$$T_2 = -\frac{1}{\lambda}(f'\phi' f^3 \cdots f^\lambda U)(f a U)^2\big((f c U)', (f a U)^2, \omega, (f a U)^4 \cdots (f a U)^\lambda U\big) \times (f^3\phi^2 \cdots \phi^\lambda U)\big((\phi c U)'(\phi b U)^2 \cdots (\phi b U)^\lambda U\big).$$

Next, T_3 may be written as $(f^2\phi^2 \cdots \phi^\lambda U)(f^3 a^2 \cdots a^\lambda U) \cdot P$, where

$$P = \big((fcU)'(faU)^2,\ \omega,\ (faU)^4 \cdots (faU)^\lambda U\big)$$
$$\times\ (f'\phi' f^3 \cdots f^\lambda U)\big((\phi cU)'(\phi bU)^2 \cdots (\phi bU)^\lambda U\big)\cdot$$

$$T_3 = \begin{bmatrix} (f^3\phi^2 \cdots \phi^\lambda U)(f^2a^2 \cdots a^\lambda U) \\ +\,(a^2\phi^2 \cdots \phi^\lambda U)(f^3f^2a^3 \cdots a^\lambda U) \\ +\,(a^3\phi^2 \cdots \phi^\lambda U)(f^3a^2f^2a^4 \cdots a^\lambda U) \\ +\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \\ +\,(a^\lambda\phi^2 \cdots \phi^\lambda U)(f^3a^2 \cdots a^{\lambda-1}f^2U) \end{bmatrix} P \qquad \text{[by D. P. (1)]}$$

$$= (f^3\phi^2 \cdots \phi^\lambda U)(f^2a^2 \cdots a^\lambda U)\,P + (1-\lambda)\,T_3,$$

so that $T_3 = -T_2$.

Thus all nine terms in the second member of (62) vanish except the first, whence

$$G^2_\lambda = \omega^{2\lambda+4} L. \tag{64}$$

Then, by (54) and (56),

$$K^2_\lambda = \frac{L}{\lambda\,[(\lambda-1)!(n-\lambda)!\,\Delta^{n-\lambda}U]^{\lambda+2}}, \tag{65}$$

where L is given by (63), in which the symbols f, ϕ, a, b, c belong to the quadratic form (3); the form $(faU)^k = (f^k a^2 \cdots a^\lambda U^{\lambda+1} \cdots U^n)$; the f's (also ϕ's and c's) with same index are equal; the sets of symbols $a^2, \cdots, a^\lambda$ (also $b^2, \cdots, b^\lambda$) are equal in any two consecutive brackets of which the first has even index, and otherwise distinct.

Hence K^2_λ (λ odd), for the space R_λ, is expressible rationally in terms of the coefficients of the first fundamental form and their derivatives, together with the functions $U^{\lambda+1}, \cdots, U^n$ (which define R_λ in R_n) and their derivatives.

ON THE ORDER OF LINEAR HOMOGENEOUS GROUPS*

(FOURTH PAPER)

BY

H. F. BLICHFELDT

1. In the writer's theorems† on finite groups of linear homogeneous substitutions of determinant unity, a group of special nature called *a self-conjugate* (or *invariant*) *subgroup* H plays an important rôle. There is a lack of completeness to these theorems due to the fact it has not been proved that H is actually less than the transitive (irreducible) group G in which, under certain conditions, it is contained; i. e., the groups G and H may be identical so far as the theorems are concerned. Account had to be taken of this fact in constructing the collineation-groups for the plane and space.‡

The relation between G and H is as follows. Let the number of variables be n, and let V be any substitution of G. The sum of the multipliers of V (weight, characteristic) we shall indicate by (V), which, therefore, represents the sum of n roots of unity:

$$(V) = \sum_{i=1}^{n} \alpha_i \beta_i .$$

Each of these roots we write as the product of one α_i, whose index is prime to a given prime number p, and one β_i, whose index is a power of p. By $(V)_p$ we indicate the quantity obtained by replacing in (V) every root β_i by unity:

$$(V)_p = \sum_{i=1}^{n} \alpha_i .$$

Then, under certain conditions, as mentioned above, there is in G a self-conjugate subgroup H whose substitutions T have the property

$$(VT)_p \equiv (V)_p \pmod{p};$$

from which, in particular,

$$(T)_p \equiv n \pmod{p}.$$

* Presented to the Society (San Francisco), September 24, 1910.

† The theorems chiefly concerned are as follows: Theorems 10 and 11 of the article *On the Order of Linear Homogeneous Groups (Second Paper)*, these Transactions, vol. 5 (1904), pp. 315, 317; Theorem 14 of the article *On the Order of Linear Homogeneous Groups (Supplement)*, *ibid.*, vol. 7 (1906), p. 523. These two articles will be referred to by *LGII* and *LGS* respectively.

‡ Mathematische Annalen, vol. 60 (1905), p. 214, § 16; vol. 63 (1907), p. 563, § 15.

2. The writer has recently been able to prove that H (or more strictly, a modified form of it) is actually less than G when this group is primitive, so that in all such cases G cannot be a simple group. Furthermore, it follows that

$$n \equiv 0 \pmod{p}.$$

Theorem 14 of LGS may be modified to read:

THEOREM 14′. *If for $n > 1$ a primitive group G in n variables has an abelian subgroup K of order $p^a \geqq p^n$, then G will have an invariant subgroup H which contains at least p^{a-n+1} substitutions of K, but which does not contain K entirely. If T be any substitution of H, and V any substitution of G whose weight (V) is zero,* then*

$$(VT)_p \equiv 0 \pmod{p}.$$

Moreover,

$$(T)_p \equiv 0, \qquad \text{and} \qquad n \equiv 0 \pmod{p}.$$

Let the substitutions of K be represented by $S_0 = \text{identity}, S_1, \cdots, S_{p^a-1}$.

In proving Theorem 14, LGS, it was first proved that for at least p^{a-n+1} substitutions of K,† say

$$S_0, S_1, \cdots, S_i, \cdots, \tag{1}$$

we have equations of the form

$$(VS_i) - (V) + (1-\theta)X_i = 0,$$

V being any substitution of G, θ a root of the equation

$$\theta^{p^a} - 1 = 0,$$

and X_i a sum of the weights

$$(VS_j) \qquad (j = 0, 1, \cdots, p^a - 1), \tag{2}$$

multiplied by integral functions of θ, the numerical coefficients entering being integers or fractions whose denominators are prime to p.

Now let V be any substitution of G such that (V) is zero. Then

$$(VS_i) + (1-\theta)X_i = 0 \tag{3}$$

for all substitutions S_i of the series (1).

Consider all the weights (2).

* BURNSIDE has proved the following theorem: "In any irreducible group of linear substitutions of finite order, other than a cyclical group in a single variable, at least one of its characteristics is zero," Proceedings of the London Mathematical Society, ser. 2, vol. 1 (1903), p. 115.

† Attention is here called to an omission in LGS, page 524. First line below the matrix reads, "Now, to this matrix may be added p^{a-m+1} rows $\cdots$." This should read: "Now, to this matrix may be added $p^{a-m+1} - 1$ rows $\cdots$."

First, they may all vanish, whatever be the substitution V chosen, so long as (V) vanishes. By the arguments of § 7, *LGII*, all the substitutions of K and all further substitutions T of G for which (VT) vanishes, form a group H, self-conjugate in G. If H had any weight (W) which vanished, then every (WT) would vanish, and therefore every (T). But this is impossible, since $(S_0) = n$. Accordingly, every weight of H is non-vanishing, and therefore H is intransitive, by Burnside's Theorem. It follows by Theorem 8, *LGII*, and by Burnside's Theorem that H is composed of similarity-substitutions.

Second, the weights (2) do not all vanish. By (3), some of them are divisible by $1 - \theta$, the quotient being expressible as a linear function of a finite number of roots of unity, no numerical coefficient entering having a denominator which is divisible by p. Assume that all the weights (2) are divisible by $(1 - \theta)^k$, whenever V represents a substitution of G whose weight is divisible by $(1 - \theta)^k$, k ranging through the values $0, 1, 2, \cdots, m$; but that the weights (2) are not all divisible by $(1 - \theta)^{m+1}$ whenever (V) is divisible by $(1 - \theta)^{m+1}$. In general, we should not expect m to be greater than zero. Now, all the substitutions T of G for which

$$(VT) = (1 - \theta)^{m+1} X,$$

whenever

$$(V) = (1 - \theta)^{m+1} Y,$$

form an invariant subgroup H. To H belongs the series (1), but not the entire group K.

Since G is assumed to be primitive, H, if not composed of similarity-substitutions, must contain a substitution W whose weight vanishes (Theorem 8, *LGII*, and Burnside's Theorem). Then

$$(WT) = (1 - \theta)^{m+1} Z$$

for every substitution T of H; i. e., every

$$(WT)_p \equiv 0 \pmod{p},$$

and therefore every

$$(T)_p \equiv 0 \pmod{p}.$$

The Theorems 10 and 11, *LGII*, may be modified in like manner to read that the self-conjugate subgroup H is less than G, the latter being primitive. In addition,

$$(T)_p \equiv 0 \pmod{p}$$

for every substitution of H.

3. One of the most important problems in the theory of linear homogeneous groups is the determination of the maximum order; i. e., the fixing of a superior limit to λ in Jordan's Theorem, the number of variables being given. The

limit known * can now be greatly reduced in special cases. Let G be a primitive group, n the number of variables, and p a prime. Then, using Theorem 14′ in conjunction with Theorem 9, $LGII$, we can prove the

COROLLARY. *If p and n are prime to each other, the highest power of p which divides the order of G must divide $n!\ p^{n-1}$.*

If $n = p$, an invariant subgroup H (assumed to contain and to be greater than the group of similarity-substitutions of order p) must be of order p^k and cannot be abelian, G being assumed primitive. Writing H in monomial form we readily find that, if $k > 3$, it possesses one, and only one, invariant of degree p which can be factored into p linear factors. In such a case G cannot be primitive. Hence, H is of order p^3 (or p^2 when considered as a collineation-group), being generated by the substitutions

$$A\colon\ x_1' = x_1,\ x_2' = \theta x_2,\ \cdots,\ x_p' = \theta^{p-1}a_p;\ \frac{\theta^p - 1}{\theta - 1} = 0\,;$$

$$B\colon\ x_1' = x_2,\ x_2' = x_3,\ \cdots,\ x_p' = x_1.$$

The order of G is a factor of $(p^2 - 1)p \cdot p^2$ when considered as a collineation-group. The corollary above is therefore true also when $n = p > 2$.

THEOREM 17. *Let G be a primitive collineation-group in n variables, n being a prime > 1. Then the order of G is a factor of*

$$n!\ (2 \cdot 3 \cdot \cdots \cdot p \cdot \cdots)^{n-1}, \tag{4}$$

$2, 3, \cdots, p, \cdots$ being the different primes not greater than the greater of the numbers $4n - 3$, $(n - 2)(2n + 1)$.† The only exception is the octahedral group of order 24 in two variables.

It will be noticed that a transitive group in a prime number of variables n is either primitive or of monomial type. In either case, the corresponding Jordan factor λ must divide the number (4) if $n > 2$.

*See *LGII*, pp. 310, 320-321, and *On Imprimitive Linear Homogeneous Groups*, these Transactions, vol. 6 (1905), p. 232, for an upper limit for all transitive groups. SCHUR has given an upper limit when the weights belong to a given field: Sitzungsberichte der Kgl. Preussischen Akademie der Wissenschaften, 1905, p. 77 ff.

† *LGS*, p. 528.

THE METRICAL ASPECT OF THE LINE-SPHERE TRANSFORMATION*

BY

JULIAN LOWELL COOLIDGE

The line-sphere transformation of Sophus Lie, first described in his remarkable article, "Ueber Complexe, insbesondere Linien und Kugelcomplexe, mit Anwendung auf die Theorie partieller Differentialgleichungen"† derives its interest from two causes. Firstly, lines and spheres are the simplest four-dimensional geometrical manifolds with which we have to deal; it is interesting to establish connections between them. Secondly, intersecting lines correspond to tangent spheres. The consequence usually drawn from this is that the line-sphere transformation falls under the general type of contact transformation, and has the beautiful property of carrying asymptotic lines over into lines of curvature. These facts are of capital importance when the subject is approached from the point of view of Pfaff equations. But we may adopt a different point of view, and then the question naturally arises: "Can not the correspondence of intersecting lines with tangent spheres be made to appear as a special case of some more general relation connecting the distances of lines with the angles of intersection of spheres?" It is the object of the present paper to answer this question affirmatively, to develop as far as possible the relations between the metrical properties of line and sphere systems, and to bring the two into the closest possible connection.‡

Before proceeding to our task, two words of caution are necessary. First, as the angles between spheres appear most naturally through their trigonometric functions, the corresponding line-functions are found most elegantly in non-euclidean space, where distances usually appear in trigonometric form. Secondly, the group of contact transformations that carries lines into lines and keeps distances invariant depends upon six parameters, the group that leaves the angles of spheres intact is a ten parameter group, isomorphic with the quinary orthogonal one. We must, therefore, introduce some seemingly arbitrary restrictions, i. e.,

* Presented to the Society December 29, 1910.

† Mathematische Annalen, vol. 5 (1872).

‡ Since this paper was written my attention has been called to one written in a similar spirit by SEGRE, *Sulle geometrie metriche dei complessi e delle sfere.* Atti della R. Accademia delle Scienze di Torino, vol. 19 (1884). It is to be noted that SEGRE deals with non-oriented spheres, and euclidean line space, so that he is only able to establish a certain similarity in the formulæ, not an identity as in the present case.

assume the invariance of certain additional relations in our sphere system, in order to scale our group down to the proper size.

§ 1. The Line Space $\sum$.

In a three-dimensional space $\sum$ let a point have four homogeneous coördinates *

$$\xi_0 : \xi_1 : \xi_2 : \xi_3 .$$

The system of measurement shall be of the elliptic type, the equation of the Absolute

$$\sum_i^{0\ldots3} \xi_i^2 = 0 . \tag{1}$$

The unit of measure shall be so chosen that the distance d of (ξ) and (η) is given by

$$\cos d = \frac{\sum_i^{0\ldots3} \xi_i \eta_i}{\sqrt{\sum_i^{0\ldots3} \xi_i^2}\sqrt{\sum_i^{0\ldots3} \eta_i^2}} . \tag{2}$$

In like manner the angle θ of the two planes

$$\sum_i^{0\ldots3} u_i \xi_i = 0 , \qquad \sum_i^{0\ldots3} v_i \xi_i = 0 ,$$

will have as cosine

$$\cos \theta = \frac{\sum_i^{0\ldots3} u_i v_i}{\sqrt{\sum_i^{0\ldots3} u_i^2}\sqrt{\sum_i^{0\ldots3} v_i^2}} . \tag{3}$$

A line (p) shall be given by its six Plücker coördinates

$$p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}, \qquad \rho p_{ij} = \xi_i \eta_j - \xi_j \eta_i = u_k v_l - u_l v_k , \tag{4}$$

where the numbers k, l are complementary to i, j in the system $0, 1, 2, 3$. The polar of (p) with regard to the Absolute, or as we shall say, its *absolute polar* will have the coördinates

$$p'_{ij} = \sigma p_{kl} . \tag{5}$$

The condition for the intersection of two lines (p) and (q) will be

$$\sum_{ijkl}^{0\ldots3} p_{ij} q_{kl} = 0 . \tag{6}$$

Each will intersect the polar of the other with regard to the Absolute if

$$\sum_{ij}^{0\ldots3} p_{ij} q_{ij} = 0 . \tag{7}$$

* The developments of the present section will be found at greater length in chapters IX, X of the author's *Elements of Non-euclidean Geometry*, Oxford, 1909.

Both expressions will vanish if they intersect orthogonally. A line will touch the Absolute if

$$\sum_{ij}^{0\ldots3} p_{ij}^2 = 0 . \tag{8}$$

The expression

$$J = \frac{\sum_{ijkl}^{0\ldots3} p_{ij} q_{kl}}{\sum_{ij}^{0\ldots3} p_{ij} q_{ij}} \tag{9}$$

will be an absolute invariant under the group of all congruent transformations of space. What will be its meaning?

Two lines in elliptic space have two distances, the distances of their intersections with either of the two lines that cut them both orthogonally. They have also two angles, the angles of the pairs of planes determined by the given lines and by each of these perpendiculars. Let one perpendicular cut the two lines in (ξ) and (ξ'), the other cutting them in (η) and (η'). We have for the product of the cosines of the distances d_1 and d_2 of the lines

$$\cos d_1 \cos d_2 = \frac{\sum_{i}^{0\ldots3} \xi_i \xi'_i \sum_{i}^{0\ldots3} \eta_i \eta'_i}{\sqrt{\sum_{i}^{0\ldots3} \xi_i^2} \sqrt{\sum_{i}^{0\ldots3} \eta_i^2} \sqrt{\sum_{i}^{0\ldots3} \xi_i'^2} \sqrt{\sum_{i}^{0\ldots3} \eta_i'^2}} .$$

Now, since the pair $(\xi)(\xi')$ are orthogonal to $(\eta)(\eta')$, i. e., each point lies in the absolute polar plane of each point of the other pair; therefore

$$\sum_{i}^{0\ldots3} \xi_i \eta_i = \sum_{i}^{0\ldots3} \xi_i \eta'_i = \sum_{i}^{0\ldots3} \xi'_i \eta_i = \sum_{i}^{0\ldots3} \eta'_i \eta'_i = 0 ,$$

$$\sum_{ij}^{0\ldots3} p_{ij} p'_{ij} = \sum_{i}^{0\ldots3} \xi_i \xi'_i \sum_{i}^{0\ldots3} \eta_i \eta'_i ,$$

$$\sum_{ij}^{0\ldots3} p_{ij}^2 = \sum_{i}^{0\ldots3} \xi_i^2 \sum_{i}^{0\ldots3} \eta_i^2 ,$$

$$\cos d_1 \cos d_2 = \frac{\sum_{ij}^{0\ldots3} p_{ij} p'_{ij}}{\sqrt{\sum_{ij}^{0\ldots3} p_{ij}^2} \sqrt{\sum_{ij}^{0\ldots3} p_{ij}'^2}} .$$

For the product of the sines of the distances we have likewise

$$\sin d_1 \sin d_2 = \frac{\sqrt{\sum_{i}^{0\ldots3} \xi_i^2 \sum_{i}^{0\ldots3} \xi_i'^2 - \left(\sum_{i}^{0\ldots3} \xi_i \xi'_i\right)^2} \sqrt{\sum_{i}^{0\ldots3} \eta_i^2 \sum_{i}^{0\ldots3} \eta_i'^2 - \left(\sum_{i}^{0\ldots3} \eta_i \eta'_i\right)^2}}{\sqrt{\sum_{i}^{0\ldots3} \xi_i^2} \sqrt{\sum_{i}^{0\ldots3} \eta_i^2} \sqrt{\sum_{i}^{0\ldots3} \xi_i'^2} \sqrt{\sum_{i}^{0\ldots3} \eta_i'^2}} ,$$

$$\left[\sum_{ijkl}^{0\ldots3}(p_{ij}p'_{kl})\right]^2 = |\,\xi\eta\xi'\eta'\,|^2$$

$$= \left[\sum_{i}^{0\ldots3}\xi_i^2\sum_{i}^{0\ldots3}\xi_i'^2 - \left(\sum_{i}^{0\ldots3}\xi_i\xi_i'\right)^2\right]\left[\sum_{i}^{0\ldots3}\eta_i^2\sum_{i}^{0\ldots3}\eta_i'^2 - \left(\sum_{i}^{0\ldots3}\eta_i\eta_i'\right)^2\right],$$

therefore,

$$J \equiv \frac{\sin d_1 \sin d_2}{\cos d_1 \cos d_2} = \tan d_1 \tan d_2. \tag{10}$$

We may say, in particular: for oblique intersection, J vanishes; for the intersection of each line with the absolute polar of the other, J is infinite; for orthogonal intersection, J is indeterminate; when the two distances of the lines are complementary, J equals unity.

The angles of two planes are equal to the distances of their absolute polar lines. But our expression (9) is unaltered if each Plücker coördinate be replaced by its complementary. Hence also

$$J = \tan\theta_1 \tan\theta_2. \tag{11}$$

The first interpretation is, however, the one which we shall follow in the present paper.

Two lines shall be said to be *parallel* * when they intersect in a point of the Absolute, they are *pseudo-parallel* † if they lie in a plane tangent to the Absolute. The necessary and sufficient condition for the existence of one of these relations is that the lines should intersect, and that there should be only one line of their pencil tangent to the Absolute,

$$\sum_{ijkl}^{0\ldots3} p_{ij}q_{kl} = \sum_{ij}^{0\ldots3} p_{ij}^2 \sum_{ij}^{0\ldots3} q_{ij}^2 - \left(\sum_{ij}^{0\ldots3} p_{ij}q_{ij}\right)^2 = 0. \tag{12}$$

Two lines shall be said to be *paratactic* ‡ if they intersect the same two generators of the same system of the Absolute. What are the Plücker coördinates of such a generator?

The coördinates of a point of the Absolute may be expressed thus,

$$\xi_0 : i\xi_1 : \xi_2 : i\xi_3 = (\lambda_1\mu_1 - \lambda_2\mu_2) : (\lambda_1\mu_1 + \lambda_2\mu_2) : (\lambda_1\mu_2 + \lambda_2\mu_1) : (\lambda_1\mu_2 - \lambda_2\mu_1).$$

The Plücker coördinates of the generators of one set, which, for definiteness

* This is Lobachevski's form of parallelism.

† This word seems to have been coined by STUDY, *Ueber Nicht-Euklidische und Linien Geometrie*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 15 (1906), pp. 489–490.

‡ The more usual name is *Clifford Parallel*. For "parataxy" see STUDY, loc. cit., or another article of same title, ibid., vol. 8 (1902), p. 317.

we shall call the *left*, will be

$$(13)\quad l_{01}=l_{23}=2\lambda_1\lambda_2,\qquad l_{02}=l_{31}=i(\lambda_1^2+\lambda_2^2),\qquad l_{03}=l_{12}=(\lambda_1^2-\lambda_2^2);$$

the *right* generators may similarly be written

$$(14)\quad r_{01}=-r_{23}=2\mu_1\mu_2,\qquad r_{02}=-r_{31}=i(\mu_1^2+\mu_2^2),\qquad r_{03}=-r_{12}=-(\mu_1^2-\mu_2^2).$$

Two lines (p) and (p') will be *left* paratactic if

$$(15)\quad \begin{aligned} p_{01}+p_{23}&=\rho(p'_{01}+p'_{23}),\\ p_{02}+p_{31}&=\rho(p'_{02}+p'_{31}),\\ p_{03}+p_{12}&=\rho(p'_{03}+p'_{12}). \end{aligned}$$

For *right* parataxy we have similarly

$$(16)\quad \begin{aligned} p_{01}-p_{23}&=\sigma(p'_{01}-p'_{23}),\\ p_{02}-p_{31}&=\sigma(p'_{02}-p'_{31}),\\ p_{03}-p_{12}&=\sigma(p'_{03}-p'_{12}). \end{aligned}$$

For parataxy, right or left, we have

$$(17)\quad \left[\sum_{ijkl}^{0\ldots3} p_{ij}p'_{kl}\pm\sum_{ij}^{0\ldots3}p_{ij}p'_{ij}\right]^2=\left[\sum_{ij}^{0\ldots3}p_{ij}^2\right]\left[\sum_{ij}^{0\ldots3}p'^2_{ij}\right],$$

and from this equation it will appear that when two lines are paratactic, their distances and angles are equal. Conversely, when the distances are equal

$$\frac{\sum_{i}^{0\ldots3}\xi_i\xi'_i}{\sqrt{\sum_{i}^{0\ldots3}\xi_i^2}\sqrt{\sum_{i}^{0\ldots3}\xi'^2_i}}=\frac{\sum_{i}^{0\ldots3}\eta_i\eta'_i}{\sqrt{\sum_{i}^{0\ldots3}\eta_i^2}\sqrt{\sum_{i}^{0\ldots2}\eta'^2_i}}=\frac{\sum_{i}^{0\ldots3}(\rho\xi_i+\sigma\eta_i)(\rho\xi'_i+\sigma\eta'_i)}{\sqrt{\sum_{i}^{0\ldots2}(\rho\xi_i+\sigma\eta_i)^2}\sqrt{\sum_{i}^{0\ldots2}(\rho\xi'^2_i+\sigma\eta'_i)^2}}.$$

There are thus ∞ pairs of points whose distances are the distances of the lines, the lines have ∞^1 common perpendiculars, and so are paratactic.

It is better for our present purpose not to use Plücker coördinates, but the Klein coördinates related to them as follows:

$$(18)\quad \begin{aligned} p_{01}&=\rho(x_0+ix_3), & p_{23}&=\rho(x_0-ix_3),\\ p_{01}&=\rho(x_1+ix_4), & p_{31}&=\rho(x_1-ix_4),\\ p_{03}&=\rho(x_2+ix_5), & p_{12}&=\rho(x_2-ix_5). \end{aligned}$$

Here i is a square root of -1 whose value is supposed to be well determined.

We have then

$$x_0 = \sigma(p_{01} + p_{23}), \qquad x_3 = \sigma i(p_{01} - p_{23}),$$
$$x_1 = \sigma(p_{02} + p_{31}), \qquad x_4 = \sigma i(p_{02} - p_{31}), \tag{19}$$
$$x_2 = \sigma(p_{03} + p_{12}), \qquad x_5 = \sigma i(p_{03} - p_{12}).$$

The coördinates $x_0 : x_1 : x_2$ are proportional to the three non-vanishing Plücker coördinates of the lines through $(1, 0, 0, 0)$ left paratactic to the given line, while $x_3 : x_4 : x_5$ are thenon-vanishing coördinates of the right paratactic through the same point. We get the Absolute polar of a given line by changing the signs of the first three, or last three Klein coördinates; it will be more convenient to adopt the latter alternative. Let us write our general scheme of formulæ in the new coördinates. Intersection of two lines:

$$x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5 \equiv (xy) = 0, \qquad (xx) \equiv (yy) \equiv 0.$$

Each intersects the absolute polar of the other if

$$x_0y_0 + x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4 - x_5y_5 \equiv (x|y) = 0,$$

$$J = \frac{(xy)}{(x|y)}.$$

Orthogonal intersection:

$$(xy) = (x|y) = 0.$$

Parallelism or pseudo-parallelism:

$$(xy) = (x|x)(y|y) - (x|y)^2 = 0.$$

Right parataxy:

$$x_3 = y_3, \qquad x_4 = y_4, \qquad x_5 = y_5.$$

Left parataxy:

$$x_0 = y_0, \qquad x_1 = y_1, \qquad x_2 = y_2.$$

Parataxy in general:

$$[(xy) \pm (x|y)]^2 = (x|x)^2(y|y)^2.$$

Left generators of Absolute:

$$x_3 = x_4 = x_5 = 0.$$

Right generators of Absolute:

$$x_0 = x_1 = x_2 = 0.$$

We shall refer to these formulæ as set (20).

§ 2. Sphere Space S.

In a three-dimensional euclidean space S let a point have the homogeneous rectangular cartesian coördinates $x : y : z : t$. The equation of any sphere may be written

$$ix_1(x^2+y^2+z^2+t^2) + x_2(x^2+y^2+z^2-t^2) + x_3(2x) + x_4(2y) + x_5(2z) = 0. \tag{21}$$

The radius will be

$$r = \frac{\sqrt{\sum_i^{1\cdots5} x_i^2}}{ix_1 + x_2}. \tag{22}$$

The centre will have coördinates

$$x : y : z : t = -x_3 : -x_4 : -x_5 : ix_1 + x_2. \tag{23}$$

Let us next assume that the radius may be either positive or negative, and say that a sphere has been *oriented* when the sign of the radius has been determined. A plane shall be said to be *oriented* if the normals have an assigned sense. Hereafter the words *sphere* and *plane* shall be taken to mean *oriented sphere* and *oriented plane*. Let us write

$$x_0^2 = -(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2), \tag{24}$$

so that

$$r = \frac{ix_0}{ix_1 + x_2}.$$

We see that an oriented sphere has six homogeneous coördinates

$$x_0 : x_1 : x_2 : x_3 : x_4 : x_5, \qquad (xx) \equiv 0.$$

The angle of intersection θ of two non-oriented spheres being given by

$$\cos\theta = \frac{\sum_i^{1\cdots5} x_i y_i}{\sqrt{\sum_i^{1\cdots5} x_i^2}\sqrt{\sum_i^{1\cdots5} y_i^2}},$$

we shall define as the angle of intersection of two oriented spheres (x) and (y)

$$\theta = \cos^{-1}\frac{\sum_i^{1\cdots5} x_i y_i}{-x_0 y_0}. \tag{25}$$

They shall be said to be *tangent* if $\cos^2\theta = 1$ and to *touch* if $\cos\theta = 1$. Two tangent spheres whose radii have like signs will touch if their contact be internal.

If one of the oriented spheres be a plane, (25) will give the angle between the directed normal and the internal direction of the radius, if the latter be positive, otherwise the external direction. If both spheres be planes we have the angle of their directed normals. Let us replace (25) by the more convenient

$$\sin^2\tfrac{1}{2}\theta = \frac{(xy)}{2x_0 y_0}. \tag{26}$$

The condition that two spheres should touch is

$$(xy) = 0. \tag{27}$$

If we reverse the signs of $x_3 : x_4 : x_5$ we get the reflection of the given oriented sphere in the origin. We shall understand by *reflection* of a sphere the operation which replaces each point by its reflection and does not alter the orientation of the radius. If θ' be the angle which each of our spheres makes with the reflection of the other in the origin, we get the fundamental formulæ

$$\sin^2 \tfrac{1}{2}\theta' = \frac{(x \mid y)}{x_0 y_0}, \qquad J = \frac{(xy)}{(x \mid y)} = \frac{\sin^2 \tfrac{1}{2}\theta}{\sin^2 \tfrac{1}{2}\theta'}. \tag{28}$$

This last equation has a meaning even in the case of null spheres. If two proper spheres have radii r, r_1, while the distance of their centres is d and their angle of intersection is θ,

$$d^2 = r^2 + r_1^2 - 2rr_1 \cos\theta, \qquad d^2 - r^2 = -\frac{x_0^2}{(ix_1 + x_2)^2} + \frac{2\sum_i^{1\ldots5} x_i y_i}{(ix_0 + x_1)(iy_0 + y_1)}.$$

When the sphere (x) becomes null with a finite centre, the expression on the left is the power of that centre with regard to the other sphere; on the right the first term disappears. Hence when (x) is a null sphere with finite centre, and (y) is a sphere of finite radius, our invariant J is the ratio of the power of the centre of (x) with regard to (y) to its power with regard to the reflection of (y) in the origin. When (y) also becomes a null sphere J is the ratio of the square of the distance of centres of (x) and (y) to the square of the distance of each from the reflection of the other.

This deduction is still at fault if (x) be a null sphere, and (y) a plane. Starting with the case where (x) is a null sphere, and (y) a sphere of finite radius, the power of the centre of (x) with regard to (y) divided by the radius of the latter is

$$\frac{2(xy)}{(ix_1 + x_i)y_0}.$$

The limit of this expression as $iy_1 + y_2 \doteq 0$ is the distance from the centre of (x) to the plane (y). Our invariant J will thus give the distance from the centre of (x) to the plane (y) divided by its distance from the reflection of (y) in the origin. The only cases where there is no geometrical interpretation for J are where either

$$x_0 = ix_1 + x_2 = 0 \qquad \text{or} \qquad y_0 = iy_1 + y_2 = 0.$$

Here one at least of the two spheres is a self perpendicular plane (or the plane at infinity) which does not make any particular angle with any thing, and is not at any particular distance from any point not in itself.

§ 3. Corresponding Expressions.

Σ	S
Line.	Sphere.
Polarization in the Absolute.	Reflection in the origin.

Definition. The non-special linear complex in Σ given by the equation

$$x_0 = 0$$

shall be called the *notable complex*. The line whose Klein coördinates are

$$(0 : i : 1 : 0 : 0 : 0)$$

shall be called the *notable line*. Notice that the notable line belongs to the notable complex.

Line of the notable complex.	Null sphere.
Polar lines with regard to the notable complex.	Spheres differing only in orientation.
Notable line.	Plane at infinity.
Lines meeting notable line.	Planes.
Lines of notable complex meeting notable line.	Self-perpendicular planes.
Product of tangents of distances of two lines, neither of which belongs to notable complex.	*Ratio of square of sine of half-angle of two not null spheres to square of sine of half angle which each makes with the reflection of the other in the origin.*
Product of tangents of distances of line belonging to notable complex but not intersecting notable line, and line neither belonging to complex nor meeting notable line.	*Ratio of power of point with regard to a proper sphere, to its power with regard to the reflection of that sphere in the origin.*
Product of tangents of distances of two lines of notable complex neither of which meets notable line.	*Ratio of square of distance of two points to square of distance of each from the reflection of the other in the origin.*
Product of tangents of distances of line meeting notable line but not belonging to notable complex and line belonging to complex but not meeting notable line.	*Ratio of distance of point from plane to its distance from the reflection of plane in the origin.*

These four correspondences are the most important established in the present paper.

Intersecting lines.	Touching spheres.
Orthogonally intersecting lines.	Touching spheres, each of which touches the reflection of the other.

This last correspondence holds only in the general case. We leave to the reader the task of working out the special cases. Touching planes have the same orientation.

Pencil of lines not belonging to the notable complex.	Pencil of touching spheres.
Pencil of lines of notable complex not meeting notable line.	Pencil of null spheres through an isotropic line.

It will be convenient to call an isotropic or self-perpendicular line an *isotropic* merely. The center of the null sphere on the right will lie on the isotropic. The infinite point of the isotropic will correspond to the line of the pencil that meets the notable line.

Pencil of lines of the notable complex which includes notable line.	Pencil of parallel self-perpendicular planes.

If we consider a point in Σ every line through it intersects every line of the notable complex through that point, and the same is true of the lines in a plane.

Point and polar plane in null system of notable complex.	Spheres containing isotropic.

It is more usual to say that a point in Σ corresponds to an isotropic in S. If this mean that there is a one to one correspondence between them, well and good. But if it mean that the lines through a point are in one to one correspondence with the oriented spheres through an isotropic, it is certainly incorrect. Incidentally let us give the correspondence previously mentioned.

Asymptotic line of a surface.	Line of curvature of a surface.
Quadratic Regulus in notable complex whose lines do not meet notable one.	Points of a circle.
Conjugate regulus.	Spheres through same circle.
Conjugate reguli, not in notable complex.	Conjugate generations of a Dupin cyclide.

There are certain special lines worth remarking.

Left generator of Absolute.	Sphere whose centre is origin.
Right generator of Absolute.	Self-perpendicular plane through origin.
Left paratactic lines.	Spheres of equal radius and centres equidistant from origin.
Right paratactic lines.	Spheres whose centres are collinear with origin.

Theorem 1).

If two not mutually absolute polar lines intersect orthogonally two other such lines, they will be paratactic and have an infinite number of common perpendiculars which are paratactic in the other sense. The latter will generate a regulus, whose conjugate consists of paratactic lines including the original two.

Theorem 1′).

If two spheres, not the reflections of one another in the origin, touch two other such spheres, and their reflections, then either their radii are equal, and their centres equidistant from the origin, or they have their centres collinear with the origin. They will touch an infinite number of other spheres and their reflections in the origin. These will be one generation of an anchor ring whose center is the origin, the conjugate generation will include the original two spheres.

The theorem on the right is intuitively evident if we draw any sort of a picture, that on the left is one of the fundamental propositions in the theory of Clifford parallels. If two quadrics intersect in a pair of generators of one set, they will intersect in a pair of the other set also. If one be the Absolute, we have

Theorem 2).

If one set of generators of a quadric be paratactic, the other set are paratactic in the opposite sense.

Theorem 2′).

If one set of generating spheres of a Dupin cyclide have equal radii, and their centres equidistant from a fixed point, the other set have their centres collinear with this point and vice-versa. The cyclide is, under these circumstances, an anchor ring.

The surface on the left is named after the mathematician to whom we owe parataxy.

Clifford surface.

Anchor ring whose axis passes through the origin.

Theorem 3).

Through a given point and in a given plane there will be, in general, one left and one right paratactic to a given line.

Theorem 3′).

Through a given isotropic two spheres can, in general, be passed having their centres on a given line, and two of equal radius whose centres are equidistant from a given point.

Line tangent to Absolute not at a point of notable line.

Sphere whose centre is at null distance from the origin.

Pencil of tangents to Absolute not at point of notable line.	Pencil of spheres through two isotropics which are in a self-perpendicular plane and reflections of each other in the origin.
Line tangent to Absolute at point of notable line.	Self-perpendicular plane.
Pencil of such lines.	Pencil of parallel self-perpendicular planes.
Point of Absolute and its polar in the notable complex, or plane tangent to Absolute and its pole in notable complex.	Isotropic whose plane with the origin is self-perpendicular.
Parallel or pseudo-parallel lines.	Touching spheres one of whose common isotropics determines, with the origin, a self-perpendicular plane.

We get a good deal of light upon our correspondence by considering the corresponding transformation groups.

Group $g_{15}h_{15}$ of collineations and correlations.	Group $g_{15}h_{15}$ of contact sphere transformations.
Group g_6h_6, where g_6 includes all congruent transformations, and h_6 is product of g_6 and absolute polarization.	Group g_6h_6 keeping invariant the assemblage of all spheres which either have their centres at the origin, or are self-perpendicular planes through the origin.
Group g_6 is composed of sub-group γ_6 which permutes the members of each set of generators of the absolute, and the assemblage η_6 which interchanges the two sets.	Group g_6 is composed of sub-group γ_6 which permutes the spheres whose centres are the origin, and the self-perpendicular planes through that point, and η_6 which interchanges the two systems.
The group γ_6 is the product of two invariant subgroups $\gamma_3\gamma_3'$. The former is the group of left translations, which keep all left generators of the absolute in place, the latter is the group of right translations.	The group γ_6 is the product of two invariant subgroups $\gamma_3\gamma_3'$. The former is the group of all rotations about the origin, the latter carries each sphere into another whose centre is collinear with the given centre and the origin, and leaves each self-perpendicular plane through the origin in place.

§ 4. The Linear Complex.

Suppose that we have in $\sum$ a linear complex,

$$(ax) = 0. \tag{29}$$

The corresponding complex in S may be classified as follows: *

a) $$\sum_i^{1\cdots5} a_i^2 \neq 0,$$

here we may write

$$\frac{a_1\dot{x}_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5}{-ix_0\sqrt{\sum_i^{1\cdots5} a_i^2}} = \frac{a_0}{i\sqrt{\sum_i^{1\cdots5} a_i^2}}.$$

This shows that the spheres of the complex cut the *fundamental sphere* (*or plane*) with coördinates

$$i\sqrt{\sum_i^{1\cdots5} a_i^2} : a_1 : a_2 : a_3 : a_4 : a_5$$

at the fixed angle

$$\cos^{-1}\frac{a_0}{i\sqrt{\sum_i^{1\cdots5} a_i^2}}.$$

b) $$a_0 = 0.$$

The spheres are orthogonal to the fundamental sphere, which is null if $\sum_i^{1\cdots5} a_i^2 = 0$.

c) $$a_0 \neq 0, \qquad ia_1 + a_2 \neq 0, \qquad \sum_i^{1\cdots5} a_i^2 = 0.$$

The ratio of the power of a fixed point to the radius of the sphere is a constant for all spheres of the complex. The fixed point may be taken as the centre of a fundamental null sphere.

d) $$a_0 \neq 0, \qquad ia_1 + a_2 = 0, \qquad \sum_i^{1\cdots5} a_i^2 = 0.$$

Spheres are invariantly related to a certain self-perpendicular plane. It is not easy to give a description of this invariant relation.

e) $$a_0 = a_3 = a_4 = a_5 = 0.$$

The complex of all oriented planes.

f) $$a_1 = a_2 = a_3 = a_4 = a_5 = 0.$$

The complex of all null spheres.

* Conf. Snyder, *Über die linearen Complexe der Lies'chen Kugelgeometrie*, Göttingen, 1895, and P. F. Smith, *On the Surfaces enveloped by Spheres belong into a linear Spherical Complex*, Transactions of the American Mathematical Society, vol. 1 (1900).

This complete classification is by means of invariants of the group of motions and inversions, and not by invariants of our group $g_6 h_6$. Let us seek for another definition of the complex.

In our space $\sum$ let us take the non-special complex (29) and find the polar with regard to the corresponding null system of a line (y). This will be

$$\rho y_i' = (aa) y_i - 2(ay) a_i. \tag{30}$$

When will the lines (y) and (y') also be mutually absolute polar? Clearly they must be polar with regard to the absolute polar complex

$$(a \mid x) = 0$$

and so the directrices of the linear congruence determined by the two complexes.

Let (r) be the coördinates of a line sought. Then if we write

$$(aa) = h(a \mid a), \qquad -1 \pm \sqrt{1 - h^2} = l. \tag{31}$$

$$r_0 : r_1 : r_2 : r_3 : r_4 : r_5 = (h+l) a_0 : (h+l) a_1 : (h+l) a_2 : (h-l) a_3 : (h-l) a_4 : (h-l) a_5.$$

The two lines found from these equations shall be called the *axes* of the complex, the corresponding spheres, the *axial spheres*. Certain special cases must be taken up at once, as (31) will not always hold.

If the complex be self-polar with regard to the absolute, our method of finding the axes breaks down. We see, in fact

$$a_0 = a_1 = a_2 = 0 \qquad \text{or} \qquad a_3 = a_4 = a_5 = 0.$$

In the former case, from (30) and the relation of the absolute polar lines,

$$y_i - \frac{2(ay) a_i}{(aa)} = -y_i \qquad (i = 3, 4, 5),$$

$$(aa) y_i = (ay) a_i, \qquad y_i = a_i.$$

We may take as an axis any line $(r_0 r_1 r_2 a_3 a_4 a_5)$; in the latter case any line $(a_0 a_1 a_2 r_3 r_4 r_5)$ in other words, any line of a definite paratactic system. If

$$(aa) = 0, \qquad (a \mid a) \neq 0,$$

the complex is special, and the axes are the line met by all lines of the complex, and its absolute polar. If

$$(aa) \neq 0, \qquad (a \mid a) = 0, \qquad 1/h = 0,$$

the equations for the axis become

$$r_0 : r_1 : r_2 : r_3 : r_4 : r_5 = a_0 : a_1 : a_2 : i a_3 : i a_4 : i a_5$$

the quantity l/h is replaced by i. If

$$(aa) = (a \mid a) = 0,$$

the axes are (a) and its absolute polar, two lines touching the absolute at the same point. If $h = 1$, the axes fall together into a right generator of the Absolute. If $h = -1$, the axes fall together into a left generator of the absolute.

We shall mean by a *general* linear complex one which has a single definite pair of axes, not touching the absolute, nor lying therein. In such a case we may write

$$
\begin{aligned}
(rx) &= h(ax) + l(a|x), \\
(r|x) &= h(a|x) + l(ax), \\
(r|r) &= 2l\,[h(aa) - (a|a)], \\
\frac{(rx)}{(r|x)} &= \frac{l}{h} \text{ if } (ax) = 0.
\end{aligned} \tag{32}
$$

Theorem 4).

*A general linear complex may be generated in two different ways by a line so moving that the product of the tangents of its distances from a fixed line is constant. The two fixed lines which may be so used are mutually absolute polar, and the two constants are reciprocals.**

Theorem 4′).

A general linear complex may be generated in two different ways by a sphere so moving that the sine of its half-angle with a given sphere bears a fixed ratio to the sine of its half angle with an equal sphere.

The general linear complex has an interesting sub-class.

General linear complex whose axes belong to the notable complex but do not intersect the notable line.

Complex of spheres with regard to which two fixed points have powers in a constant ratio.

The absolute invariant l/h has, in Σ a most interesting significance. Suppose that the axes are the two lines $\xi_0 = \xi_1 = 0$ and $\xi_2 = \xi_3 = 0$. Let us take an infinitesimal screw motion about them

$$
\begin{aligned}
\xi_0' &= \xi_0 + \xi_1 d\theta, \\
\xi_1' &= -\xi_0 d\theta + \xi_1, \\
\xi_2' &= \xi_2 + \xi_3 d\phi, \\
\xi_3' &= -\xi_2 d\phi + \xi_3.
\end{aligned}
$$

The expression $d\phi/d\theta$ will be the *pitch* of the screw with regard to $\xi_0 = \xi_1 = 0$.

* A similar theorem will hold for the linear complex in euclidean space, see SEGRE loc. cit.

If (η) be a point of the absolute polar of the line $(\xi)(\xi')$

$$\sum_{i}^{0\ldots3} \xi_i \eta_i = \sum^{0\ldots3} \xi'_i \eta_i = 0,$$

$$(\xi_0\eta_1 - \xi_1\eta_0)d\theta + (\xi_2\eta_3 - \xi_3\eta_2)d\phi = 0.$$

A line through (ξ) perpendicular to the direction of instantaneous motion will belong to the linear complex

$$p_{01}d\theta + p_{23}d\phi = 0, \qquad (d\theta + d\phi)x_0 + i(d\theta - d\phi)x_3 = 0.$$

This axis of the complex has the coördinates

$$(1:0:0:-i:0:0), \qquad \frac{(rx)}{(r\,|\,x)} = -\frac{d\phi}{d\theta}.$$

Theorem 5). *In a general linear complex the product of the tangents of the distances of the lines from an axis is equal numerically to the pitch, with regard to that axis, of the instantaneous screw motion associated with the complex.*

It will thus be convenient to define our invariant as the *pitch* of the complex. Let us revert to (32). These equations were independent of the relation $(xx) \equiv 0$.

$$\begin{aligned} (ar) &= h(aa) + l(a\,|\,a), \\ (a\,|\,r) &= h(a\,|\,a) + l(aa), \\ (r\,|\,r) &= 2l[h(aa) - (a\,|\,a)]. \end{aligned} \tag{33}$$

Now let (y) and (y') be two lines mutually polar with regard to the complex (i. e., polar with regard to the accompanying null system). We get from (30)

$$\begin{aligned} (ry') &= (aa)(ry) - 2(ay)(ar) \\ &= -(ay)[2l(a\,|\,a) + h(\alpha a)] + l(aa)(a\,|\,y), \\ (r\,|\,y') &= (aa)(r\,|\,y) - 2(ay)(r\,|\,r) \\ &= -(ay)[l(aa) + 2h(a\,|\,a)] + h(aa)(a\,|\,y), \\ \frac{(ry)}{(r\,|\,y)} &= \frac{h(ay) + l(a\,|\,y)}{l(ay) + h(a\,|\,y)}, \\ \frac{(ry')}{(r\,|\,y')} &= \frac{[2l(a\,|\,a) + h(aa)](ay) - l(aa)(a\,|\,y)}{[2h(a\,|\,a) + l(aa)](ay) - h(aa)(a\,|\,y)}. \end{aligned}$$

Eliminating $(ay)/(a\,|\,y)$, and remembering that

$$(aa) = h(a\,|\,a), \qquad h^2 + l^2 = -2l,$$

we have

$$\frac{(ry')}{(r\,|\,y')}\cdot\frac{(ry)}{(r\,|\,y)}=\frac{l^2}{h^2}. \tag{34}$$

Let us also remember that two lines which are mutually polar with regard to a linear complex lie with the axes upon a quadratic regulus.

Theorem 6).

The polar of a line with regard to a general linear complex lies with it and the axes on a quadratic regulus, and the product of the tangents of the distances from an axis to two such polar lines is the square of the corresponding pitch of the complex.

Theorem 6′).

The polar of a sphere with regard to a general linear complex lies with it and the axial spheres in one generation of a Dupin cyclide, and the product of the squared sines of the half angles which two such polar spheres make with one axial sphere divided by the corresponding product for the other axial sphere is the square of the pitch of the complex.

Unfortunately a line is not uniquely determined if we know a regulus whereon it lies, and the product of the tangents of its distances from a line of the regulus. Nevertheless, this theorem enables us quickly to determine the coördinates of the polar of a line with regard to a linear complex. If the axes be (r) and (r') any regulus containing them may be written

$$x_i=\rho^2 r_i+2\rho\sigma s_i+\sigma^2 r'_i.$$

$$(rr)=(r'r')=(rs)=(r's)=(ss)-(rr')=0,\qquad (rr')=(r\,|\,r).$$

If (y) and (y') be polar lines with regard to the complex

$$\frac{(ry')}{(r\,|\,y')}\cdot\frac{(ry)}{(r\,|\,y)}=\frac{\sigma'^2}{\rho'^2}\cdot\frac{\sigma^2}{\rho^2}=\frac{l^2}{h^2},\qquad \frac{\sigma'}{\rho'}\cdot\frac{\sigma}{\rho}=\pm\frac{l}{h}.$$

We must choose the upper sign, as we see by a continuous change from the case where (y) and (y') fall together. When all quantities and all lines here involved are real, the two polar lines will be separated in the regulus by the axes if $l/h<0$ and not separated if $l/h>0$.

Suppose that we have two linear complexes

$$(ax)=0,\qquad (a'x)=0.$$

They have several absolute simultaneous invariants of which the simplest is

$$\frac{(aa')}{(a\,|\,a')},$$

$$r_0=(h+l)a_0,\ \cdots,\ r_3=(h-l)a_3,$$

$$r_0' = (h' + l')a_0' \cdots r_3' = (h' - l')a_3',$$

$$(rr') = (hh' + ll')(ab) + (hl' + lh')(a\,|\,b),$$

$$(r\,|\,r') = (hh' + ll')(a\,|\,b) + (hl' + lh')(ab),$$

$$\frac{(aa')}{(a\,|\,a')} = \frac{\dfrac{(rr')}{(r\,|\,r')}(hh' + ll') - (hl' + lh')}{(hh' + ll') - \dfrac{(rr')}{(r\,|\,r')}(hl' + lh')}.$$

If we write

$$\frac{l}{h} = p, \qquad \frac{l'}{h'} = p',$$

$$\frac{(ab)}{(a\,|\,b)} = \frac{\dfrac{(rr')}{(r\,|\,r')}(1 + pp') - (p + p')}{(1 + pp') - \dfrac{(rr')}{(r\,|\,r')}(p + p')}.$$

Theorem 7)
Two general linear line complexes are in involution if the product of the tangents of the distances of two axes is equal to the sum of pitches divided by the product thereof plus one.

Theorem 7′)
Two general linear sphere complexes are in involution if the square of the sine of the half angle formed by an axial sphere in each, divided by the square of the sine of the half angle of the other axial spheres, is equal to the sum of the corresponding pitches divided by the product thereof plus one.

Let us look briefly at the hitherto excluded case of the complex which is its own absolute polar. Here in S we have ∞^2 axial spheres which either have the same radius and their centers at the same distance from the origin, or all have their centers on one line through the origin. In the former case the complex is transformed into itself by any rotation about the origin, in the latter by any transformation of γ_3'. In the former case we shall have a pitch -1. For if the axis will be $a_0 : a_1 : a_2 : r_3 : r_4 : r_5$

$$(rx) + (r\,|\,x) = 2(ax) = 0.$$

If (x) be a line of the complex

$$\frac{(rx)}{(r\,|\,x)} = -1.$$

The centre of a radical sphere will have the coördinates

$$\frac{-r_3}{ia_1+a_1}, \qquad \frac{-r_4}{ia_1+a_1}, \qquad \frac{-r_5}{ia_1+a_1}.$$

The square of its distance from the origin will be

$$\frac{-(a_0^2+a_1^2+a_2^2)}{(ia_0+a_1)^2} \qquad \text{since} \qquad (rr)=0.$$

The radius of this sphere is

$$\frac{ia_0}{(ia_1+a_2)}.$$

The power of the origin with regard to this sphere is

$$\frac{-(a_1^2+a_2^2)}{(ia_1+a_2)^2},$$

and this is the square of the radius of the fundamental sphere.

Theorem 8). *When the centre of the fundamental sphere is midway between the centres of the axial spheres, the former cuts the latter orthogonally, and the pitch of the complex is equal to* -1.

Suppose next that

$$a_0=a_1=a_2=0.$$

The axial spheres will be any two spheres, symmetrical with regard to the origin,

$$r_0:r_1:r_2:a_3:a_4:a_5, \qquad (rr)=0,$$

$$\frac{(rx)}{(r\,|\,x)}=1.$$

The radical plane of the two axial spheres is $0:0:0:a_3:a_4:a_5$.

Theorem 9). *When the fundamental sphere is the radical plane of the axial spheres, the latter may be replaced by any other two spheres symmetrical with regard to the origin, and having the same line of centers, and the pitch of the complex equals* 1.

As a matter of fact we have a great deal more freedom in choosing our axial spheres in the present case, for, as the complex consists in the assemblage of all spheres orthogonal to the fundamental plane, it will be carried into itself by every translation parallel to this plane. We see, more generally, that when the spheres of the complex cut the fundamental sphere orthogonally they make equal angles with any two spheres mutually inverse with regard to the fundamental sphere.

Any linear complex which is not special may be carried into the complex $x_0=0$ by a transformation of the group g_{15}.

Theorem 10).
The intersection of two linear complexes is a linear congruence whose directrices are mutually polar with regard to both.

Theorem 10′).
The locus of points whose powers with regard to two equal spheres have a constant ratio is two spheres differing only in orientation.

The theorem on the right is, of course, true when the two spheres are unequal in radius. It is interesting that there should be a connection between two such well known, but, apparently dissimilar theorems.

There are ∞^1 linear complexes which contain all the lines of a linear congruence.

Theorem 11).
If a linear congruence be given whose directrices are not mutually absolute polar, we may in ∞^2 ways find two lines such that the product of the tangents of the distances from each to the lines of the congruence is constant.

Theorem 11′).
If a system of spheres be given touching two spheres, we may in ∞^5 ways find two pairs of spheres symmetrical with regard to a fixed point so that the ratio of the sines of the half-angles made by the spheres of the system with the two spheres of each pair is constant.

The increased freedom on the right is owing to our ability to choose the origin at will. Let us see if we can not make use of our freedom so that the two constants shall both be equal to 1. Let the given complexes be

$$(ax) = 0, \qquad (bx) = 0.$$

The roots of the quadratic equation

$$(35) \quad \lambda^2[(aa) + (a|a)] + 2\lambda\mu[(ab) + (a|b)] + \mu^2[(bb) + (b|b)] = 0,$$

will, in general, be distinct and give two complexes with different axes.

Theorem 12).
If a linear congruence be given we may, in general, find two not mutually absolute polar lines such that the two distances from each to every line of the congruence are complementary.

Theorem 12′).
If a system of spheres be given touching two spheres we may find two pairs of spheres symmetrical with regard to any point in general position, so that the spheres of the system make equal angles with the two spheres of each pair.

On the right a point "in general position" means one not on the locus that would give the quadratic equation (35) equal roots.

If we have a quadratic regulus given by

$$(ax) = (bx) = (cx) = 0,$$

we may in ∞^3 ways find three sets of roots for the equation

$$\begin{aligned}(36)\qquad &\lambda^2[(aa)+(a|a)]+\mu^2[(bb)+(b|b)]+\nu^2[(cc)+(c|0)]\\ &+2\mu\nu[(bc)+(b|c)]+2\nu\lambda[(ca)+(c|a)]+2\lambda\mu[(ab)+(a|b)]=0.\end{aligned}$$

Theorem 13).

A regulus may be generated in ∞^3 ways by lines whose two distances from each of three fixed lines, no two of which are mutually absolute polar, are complementary.

Theorem 13′).

A generation of a Dupin cyclide may be formed in ∞^6 ways by spheres cutting at equal angles the two spheres of each of three pairs symmetrical with regard to a fixed point.

§ 5. The Dual Projective Geometry.

Let us return to our fundamental equations (31) which may be rewritten

$$r_0 : r_1 : r_2 : r_3 : r_4 : r_5 = a_0 : a_1 : a_2 : \frac{h-l}{h+l}a_3 : \frac{h-l}{h+l}a_4 : \frac{h-l}{h+l}a_5,$$

$$(37)\qquad \frac{h-l}{h+l} = \frac{(aa)+(a|a) \mp \sqrt{(a|a)^2-(aa)^2}}{(aa)-(a|a) \pm \sqrt{(a|a)^2-(aa)^2}} = \pm\frac{\sqrt{(a|a)+(aa)}}{\sqrt{(a|a)-(aa)}}.$$

If, now, we leave $a_0 : a_1 : a_2$ invariant, but change $a_3 : a_4 : a_5$ into $\rho a_3 : \rho a_4 : \rho a_5$, we see that the two axes (r) are unaltered. We have thus ∞^1 complexes which have the same axes or axial spheres, and are said to be *coaxal*. We may thus conceive the two axes or axial spheres as a new space element, determined by two triads of separately homogeneous coördinates.

Theorems 14 and 14′). *There is a one to one correspondence between the pairs of points of two planes (projectively defined), the pairs of lines mutually polar with regard to a quadric and the pairs of spheres symmetrical with regard to a fixed point.*

The system of geometry with any one of these as element may be called *dual projective*. The case where the element is the pair of axes of a coaxal system of

complexes has already been treated at length.* Let us therefore confine ourselves to the relation between the geometry of point-pairs in two projective planes, and pairs of spheres symmetric with regard to a fixed point.

We must begin by taking account of the lapses in the one to one correspondence described in theorem 14. Let us call the plane $a_0 : a_1 : a_2$ the *left plane* while $a_3 : a_4 : a_5$ is the *right plane*. The conics

$$a_0^2 + a_1^2 + a_2^2 = 0, \qquad a_3^2 + a_4^2 + a_5^2 = 0,$$

shall be called the *fundamental conics*. Suppose, first, that the spheres coalesce into a self-perpendicular plane through the origin

$$r_0 = r_1 = r_2 = (rr) = 0.$$

To this will correspond every pair of points

$$(a_0 a_1 a_2), \qquad (r_3 r_4 r_5), \qquad a_0^2 + a_1^2 + a_2^2 \neq 0.$$

We see thus, that to a self-perpendicular plane through the origin will correspond any point in the left plane, not on the fundamental conic, and a definite point of the fundamental conic in the right plane. In the same way, to a sphere whose center is the origin will correspond a definite point of the fundamental conic in the left plane, and any point not on the fundamental conic in the right.

Suppose, thirdly, that we have

$$r_0^2 + r_1^2 + r_2^2 = r_3^2 + r_4^2 + r_5^2 = 0.$$

Every complex of the coaxal system is special, and the spheres form a pencil of tangent spheres through two isotropics, the reflections of each other in the origin, their plane being self perpendicular. Such a pencil will correspond to each pair of points on the two fundamental conics. Conversely if the point in neither plane lie on the fundamental conic, we have a single pair of spheres. We have also a single pair of coincident spheres when one point lies on a fundamental conic and the other does not. When both points lie on the fundamental conics

* The euclidean case was treated first, and at great length, by STUDY in the third part of his *Geometrie der Dynamen*, Leipzig, 1903. The same writer sketched out the non-euclidean cases without proofs in an article *Zur Nichteuklidischen und Liniengeometrie*, Jahresbericht der Deutschen Mathematikervereinigung, vol. 11 (1902). The elliptic case was then studied in detail by the author in his *Dual Projective Geometry of Elliptic and Spherical Space*, Greifswald, 1904, and the hyperbolic by BECK, *Die Strahlenketten im hyperbolischen Raume*, Hannover, 1905. For a briefer account of the non-euclidean cases see the author's *Non-Euclidean Geometry*, already cited, Ch. X.

we have a pencil of spheres through two parallel isotropics the reflections of each other in the origin and in a self-perpendicular plane through the origin.

Before discussing figures generated by sphere-pairs, let us notice that our dual coördinates $(a_0 a_1 a_2)(a_3 a_4 a_5)$ are capable of another interpretation besides that of the system of coaxal complexes. It is, in fact, immediately evident that

$$\frac{a_3}{\sqrt{a_3^2 + a_4^2 + a_5^2}}, \quad \frac{a_4}{\sqrt{a_3^2 + a_4^2 + a_5^2}}, \quad \frac{a_5}{\sqrt{a_3^2\, a_4^2 + + a_5^2}},$$

are the direction cosines of the line of centers of the two spheres. On the other hand if θ_1 and θ_2 be the angles which the two spheres make with the spheres $(i10000)$, $(i01000)$ whose common center is the origin,

$$a_0 : a_1 : a_2 = 1 : i \cos \theta_1 : i \cos \theta_2 .$$

The fundamental relation in dual sphere geometry is touching. If

$$u_0 a_0 + u_1 a_1 + u_2 a_2 = 0, \qquad v_3 a_3 + v_4 a_4 + v_5 a_5 = 0,$$

the two spheres $(a_0 a_1 a_2)$ $(a_3 a_4 a_5)$ touch the two spheres (u) (v).

Suppose that we have a fixed point in one plane, and the points of a line in the other. We have ∞^1 spheres touching a pair of axial spheres, and with their centres either on a line through the origin, or on a sphere whose center is the origin, and with equal radii.

Representing Planes.	S.
Point in one plane not on fundamental conic and line in the other not tangent to the fundamental conic.	One generation of anchor ring whose centre is the origin.
Polar and pole with regard to fundamental conics of foregoing point and line.	Other generation of same ring.

The next most interesting one-parameter family is given by two projective ranges in the two planes

$$a_i = \lambda b_i + \mu c_i, \qquad i = 0, 1, 2, 3, 4, 5. \tag{38}$$

There are three sub-cases.

a) The points of the fundamental conics correspond. We may imagine that these are (b) and (c). Let (r) and (r') be any two spheres of the system.

$$(aa) = 2\lambda\mu(bc), \qquad (a'a') = 2\lambda'\mu'(bc),$$

$$(a \mid a) = 2\lambda\mu(b \mid c), \qquad (a' \mid a') = 2\lambda'\mu'(b \mid c),$$

$$h = \frac{(bc)}{(b \mid c)} = h', \qquad l = l',$$

$$(rr') = (hh' + ll')(aa') + (hl' + lh')(a \mid a'),$$

$$= (h^2 + l^2)(\lambda\mu' + \lambda'\mu)(bc) + 2hl(\lambda\mu' + \lambda'\mu)(b \mid c),$$

$$= 0, \qquad \text{since } h^2 + l^2 = -2l.$$

Projective ranges where the points of fundamental conics correspond.

Pencil of mutually touching spheres which touch two axial spheres.

b) One intersection with a fundamental conic corresponds to one such in the other plane. This is an unsymmetrical case which we pass over.

c) The intersections with the fundamental conics do not correspond.

The one parameter family of spheres all touch a pair of axial spheres and generate a surface which we shall call a C surface. Let us find its order. If

$$\lambda/\mu = z,$$

$$(aa) = z^2(bb) + 2z(bc) + (cc),$$

$$(a \mid a) = z^2(b \mid b) + 2z(b \mid c) + (c \mid c),$$

$$r_0 = \sqrt{[(bb) - (b \mid b)]z^2 + 2[(bc) - (b \mid c)]z + [(cc) - (c \mid c)]}\,(zb_0 + c_0),$$

$$r_3 = \sqrt{[(bb) + (b \mid b)]z^2 + 2[(bc) + (b \mid c)]z + [(cc) + (c \mid c)]}\,(zb_3 + c_3),$$

$$r_0 = \sqrt{f_2(z)}\,(zb_0 + c_0), \qquad \alpha_3 = \sqrt{\phi_2(z)}\,(zb_4 + c_3).$$

If $0 : x_1 : x_2 : x_3 : x_4 : x_5$ be a point of this sphere

$$\sqrt{f_2(z)} \sum_{i}^{1 \ldots 2} (zb_i + c_i)x_i + \sqrt{\phi_2(z)} \sum_{i}^{3 \ldots 5} (zb_i + c_i)x_i = 0.$$

To find the envelop of these spheres we must square this expression getting an expression of the fourth order in z, quadratic in x_i, differentiate with respect to z and eliminate that parameter. This gives an expression of the fourteenth order in x_i.

Theorem 15′. *The surfaces C are of the twenty-eighth order, with the circle at infinity as a curve of the fourteenth order.*

This theorem would be of uncommonly little importance, were it not for a very general and remarkable property of C surfaces. Suppose that we have any analytic congruence of spheres

$$b_i = b_i(uv).$$

Let us find the sphere which touches (b), its adjacent sphere

$$(b) + \frac{\partial (b)}{\partial u} du + \frac{\partial (b)}{\partial v} dv,$$

and the reflections of the two in the origin,

$$a_0 b_0 + a_1 b_1 + a_2 b_2 = a_3 b_3 + a_4 b_4 + a_5 b_5 = 0,$$

$$\sum_i^{0\cdots 2} a_i \left(\frac{\partial b_i}{\partial u} du + \frac{\partial b_i}{\partial v} dv \right) = \sum_i^{3\cdots 5} a_i \left(\frac{\partial b_i}{\partial u} du + \frac{\partial b_i}{\partial v} dv \right) = 0.$$

Let us assume that

$$\left| a_0 \frac{\partial b_1}{\partial u} \frac{\partial b_2}{\partial v} \right| \not\equiv 0, \qquad \left| a_3 \frac{\partial b_4}{\partial u} \frac{\partial b_5}{\partial v} \right| \not\equiv 0.$$

We choose such a (b) (said to be in the general position) that neither of these determinants vanishes. We may then write

$$a_i = du \begin{vmatrix} b_j & b_k \\ \frac{\partial b_j}{\partial u} & \frac{\partial b_k}{\partial u} \end{vmatrix} + dv \begin{vmatrix} b_j & b_k \\ \frac{\partial b_j}{\partial v} & \frac{\partial b_k}{\partial v} \end{vmatrix} \tag{39}$$

where i, j, k are circular permutations of $0, 1, 2$ and $3, 4, 5$. Varying du and dv at will, we have

Theorem 16′. *If an analytic congruence of spheres be given and a point chosen which is neither the vertex of a cone which is the locus of their centres, nor the centre of ∞^1 spheres each passing through the centres of ∞^1 spheres of the congruence of equal radius, then if a sphere of the congruence be chosen, the spheres which touch it and the adjacent spheres of the congruence, as well as the reflections in the fixed point, will generate a C surface.**

An interesting two parameter family is the chain congruence, given by

$$a_i = \lambda b_i + \mu c_i + \nu d_i, \qquad i = 0, 1, 2, \quad \text{or} \quad 3, 4, 5. \tag{40}$$

*This is, of course, the line-sphere transform of the non-euclidean form of Hamilton's famous theorem on the cylindroid. See his paper on *Systems of Rays*, Transactions of the Royal Irish Academy, vol. 15 (1829), and the author's *Non-euclidean Geometry*, chap. X.

We pass over the discussion of the various possible special cases, and take the general case where we can solve three of the equations for λ, μ, ν and substitute in the other three. We may here write

$$a_0 = u_0 a_3, \qquad a_1 = u_1 a_4, \qquad a_2 = u_2 a_5.$$

The spheres which touch two spheres of the congruence and their reflections in the origin will generate the conjugate congruence

$$b_3 = u_0 b_0, \qquad b_4 = u_1 b_1, \qquad b_5 = u_2 b_2.$$

General projectivity between the two planes.	Chain congruence.
Contragredient projectivity.	Conjugate congruence.

When a projective relation is established between two planes, one is thereby established between the ranges on any two corresponding lines in them.

Theorem 17′). *The spheres of a chain congruence which touch one of the conjugate congruence and its reflection in the origin generate a C surface.*

Are there any spheres of our congruence which touch one of the conjugate congruence without touching its reflection? Let us write

$$r_0 = a_0, \qquad r_3 = \frac{h - l}{h + l} a_3, \qquad s_0 = b_0, \qquad s_3 = \frac{h' - l'}{h' + l'} b_3,$$

If $(rs) = 0$,

$$(a_0 b_0 + a_1 b_1 + a_2 b_2)\left[1 + \frac{(h - l)(h' - l')}{(h + l)(h' + l')}\right] = 0.$$

If the first factor vanish (a) touches (b) and also its reflection. If the second vanish

$$hh' + ll' = 0.$$

But there will be a singly infinite set of values for (a) and (b) in the two congruences satisfying that equation. Hence ∞^1 spheres of each congruence touch ∞^1 of the other, but not their reflections.

Theorem 18). *The spheres of a chain congruence which touch a sphere of the conjugate but not its reflection in the origin form one generation of a Dupin cyclide, the other generation belonging to the conjugate congruence.*

Theorem 19). *The spheres which touch pairs of spheres of one generation of a Dupin cyclide and their reflections in a fixed point generate a chain congruence.*

The chain congruence has many other beautiful properties, easily proved from the corresponding properties of the chain congruences of lines by means of the line-sphere transformation.

In closing one remark of a general character may not be out of place. The method of polar reciprocation in a quadric, descriptively considered, is a means of illustrating the principle of duality. Metrically, it shows the perfect correspondence of the concepts of distance and angle in non-euclidean space. The line-sphere transformation, descriptively considered, is one of the most beautiful examples under the general type of contact transformation. Metrically, as we have seen, it brings out numerous relations between the distances of lines and the angles of spheres. Is it not probable that there exists a large class of contact transformations which are not merely descriptive in character, but may be used to establish interesting metrical relations among elements of various sorts? Can we, without a total loss of naturalness, include all contact transformations in such a class?

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NATURAL SYSTEMS OF TRAJECTORIES GENERATING FAMILIES OF LAMÉ*

BY

EDWARD KASNER

As in the author's previous papers in these Transactions,† the term *natural* is employed to designate those quadruply-infinite systems of space curves which can be identified with the totality of extremals of a variation problem of the form

$$\int F(x, y, z)\, ds = \text{minimum} \qquad (ds^2 = dx^2 + dy^2 + dz^2). \tag{1}$$

Such a system has the characteristic property that if ∞^2 of the curves are orthogonal to one surface they are necessarily orthogonal to ∞^1 surfaces. The equation of these ∞^1 surfaces may be written in the form

$$f(x, y, z) = \text{constant} \tag{2}$$

where f satisfies the Hamilton-Jacobi equation

$$f_x^2 + f_y^2 + f_z^2 = F^2 \tag{3}$$

associated with (1).

A set of ∞^1 surfaces of this sort will be termed a set of wave fronts or, more simply, a *wave set*. The natural system being given, every surface determines one of these wave sets. The phraseology employed is of course suggested by the optical interpretation of (1), in which $F(x, y, z)$ represents the index of refraction in a non-homogeneous isotropic medium: the curves of the natural system are then the possible paths of light; and a disturbance starting from any surface will be propagated by means of a set of wave fronts.

The question arises whether wave sets may be families of Lamé, that is, simply infinite families of surfaces which can form part of a triply-orthogonal system. Ordinarily this is not the case. The condition that (2) shall represent a Lamé family is that f satisfy the Darboux-Cayley equation of third order‡

* Presented to the Society, under a different title, December 29. 1909.

† *Natural families of trajectories: conservative fields of force*, vol. 10 (1909), pp. 201–219; *The theorem of Thomson and Tait and natural families of trajectories*, vol. 11 (1910), pp. 121–140.

‡ DARBOUX, *Leçons sur les systèmes orthogonaux*, vol. 1 (1898), p. 23.

$$\begin{vmatrix} H_{xx} & H_{yy} & H_{zz} & H_{yz} & H_{xz} & H_{xy} \\ f_{xx} & f_{yy} & f_{zz} & f_{yz} & f_{xz} & f_{xy} \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2f_x & 0 & 0 & 0 & f_z & f_y \\ 0 & 2f_y & 0 & f_z & 0 & f_x \\ 0 & 0 & 2f_z & f_y & f_x & 0 \end{vmatrix} = 0, \tag{4}$$

where

$$H \equiv (f_x^2 + f_y^2 + f_z^2)^{-\frac{1}{2}}. \tag{5}$$

The problem of the present paper is to find those natural systems for which all the associated wave sets are families of Lamé. The solution together with optical and kinetic formulations follow:

THEOREM I. *The only natural systems whose wave sets are of the Lamé type are those composed of the ∞^4 circles orthogonal to a fixed sphere. The corresponding variation problem* (1) *is of the form*

$$\int \frac{ds}{a_0(x^2+y^2+z^2) + a_1x + a_2y + a_3z + a_4} = \text{minimum}.$$

THEOREM II. *The only isotropic media for which every disturbance spreads out by means of a Lamé family of surfaces are those in which the index of refraction is of the form*

$$[\, a_0(x^2+y^2+z^2) + a_1x + a_2y + a_3z + a_4 \,]^{-1};$$

its value at any point varies inversely as the power of that point with respect to a fixed sphere.

THEOREM III. *The only conservative fields of force such that, when the energy constant is taken equal to zero, every set of surfaces of equal action is of the Lamé type, are those in which the potential at any point varies inversely as the square of the power of the point with respect to a fixed sphere.*

The analytic formulation of our problem is as follows: Determine the function F in such a way that every function f which satisfies the Hamilton-Jacobi equation (3) shall also satisfy the Darboux-Cayley equation (4).

It will be convenient to write the former equation in the form

$$f_x^2 + f_y^2 + f_z^2 = 2W, \tag{6}$$

so that

$$2W \equiv F^2. \tag{7}$$

By differentiation, we have

$$
\begin{aligned}
f_x f_{xx} + f_y f_{yx} + f_z f_{zx} &= W_x, \\
f_x f_{xy} + f_y f_{yy} + f_z f_{zy} &= W_y, \\
f_x f_{xz} + f_y f_{yz} + f_z f_{zz} &= W_z.
\end{aligned} \tag{8}
$$

Hence of the six partial derivatives of the second order which appear in (4) we may eliminate say f_{xx}, f_{yy}, f_{zz}. The final result may be put into the form

$$
\begin{vmatrix}
A & B & C & H_{yz} & H_{zx} & H_{xy} \\
W_x & W_y & W_z & f_{yz} & f_{zx} & f_{xy} \\
f_x & f_y & f_z & 0 & 0 & 0 \\
2W & 0 & 0 & 0 & f_z & f_y \\
0 & 2W & 0 & f_z & 0 & f_x \\
0 & 0 & 2W & f_y & f_x & 0
\end{vmatrix} = 0, \tag{9}
$$

where

$$
\begin{aligned}
A &\equiv f_x H_{xx} + f_y H_{xy} + f_z H_{xz}, \qquad B \equiv f_x H_{yx} + f_y H_{yy} + f_z H_{yz}, \\
C &\equiv f_x H_{zx} + f_y H_{zy} + f_z H_{zz},
\end{aligned} \tag{10}
$$

$$
H \equiv (2W)^{-\frac{1}{2}} \equiv F^{-1}. \tag{11}
$$

Equation (9) may be expanded in the form

$$
\alpha f_{yz} + \beta f_{zx} + \gamma f_{xy} + \delta = 0, \tag{12}
$$

where the coefficients involve only derivatives of first order. Since, in virtue of (8), the quantities f_{yz}, f_{zx}, f_{xy} are numerically arbitrary, it follows that the above coefficients must vanish individually.

It will be sufficient to calculate γ, the minor of f_{xy} in (9). The resulting condition is found to be

$$
f_z\{H_{xy}(f_x^2 - f_y^2) + (H_{yy} - H_{xx}) f_x f_y\} - (f_x^2 + f_y^2)(f_x H_{yz} - f_y H_{zx}) = 0. \tag{13}
$$

This is to hold as a consequence of (6); hence the left hand member of (13), a homogeneous cubic in f_x, f_y, f_z, must be divisible by the non-homogeneous quadratic $f_x^2 + f_y^2 + f_z^2 - 2W$. It follows that (13) must be an identity. Equating the coefficients to zero and adjoining other conditions obtained by cyclic permutation of the letters x, y, z, we find

$$
H_{xx} = H_{yy} = H_{zz}, \qquad H_{yz} = H_{zx} = H_{xy} = 0. \tag{14}
$$

It follows that the function H must be of the form

$$
H = a_0(x^2 + y^2 + z^2) + a_1 x + a_2 y + a_3 z + a_4. \tag{15}
$$

It is easily verified that, for this form of H, equation (9) holds identically and is thus certainly satisfied by all the solutions of (6).

The result of our analytic discussion is that the only cases in which all the solutions of (3) are also solutions of (4) are defined by

$$F = [a_0(x^2 + y^2 + z^2) + a_1 x + a_2 y + a_3 z + a_4]^{-1}. \tag{16}$$

The natural system associated with such a function is easily shown to consist of the ∞^4 circles orthogonal to the fixed sphere S

$$a_0(x^2 + y^2 + z^2) + a_1 x + a_2 y + a_3 z + a_4 = 0. \tag{17}$$

The Lamé families corresponding to the form (15) are well known.* The ∞^1 surfaces of each of these families have for orthogonal trajectories circles orthogonal to S. These circles cut any two of the surfaces and S in four points whose anharmonic ratio is constant.

The ten-parameter group of conformal transformations of space converts natural systems into natural systems and Lamé families into Lamé families; and, therefore, solutions of our problem into solutions. With respect to real transformations there are three types according as the radius of the sphere S is zero, real, or imaginary. The function H may be reduced to one of the forms

$$x^2 + y^2 + z^2, \qquad x^2 + y^2 + z^2 - k^2, \qquad x^2 + y^2 + z^2 + k^2, \tag{18}$$

In the optical interpretation the index of refraction is F or $1/H$. It is thus of one of the forms

$$\frac{1}{r^2}, \qquad \frac{1}{r^2 - k^2}, \qquad \frac{1}{r^2 + k^2}, \tag{19}$$

where r is the distance from the origin and k is a real constant. These three media furnish representations of the parabolic, elliptic, and hyperbolic geometries respectively.

In the kinetic interpretation, the work function W, according to (11), is proportional to H^{-2}, and is thus

$$\frac{1}{r^4}, \qquad \frac{1}{(r^2 - k^2)^2}, \qquad \frac{1}{(r^2 + k^2)^2}. \tag{20}$$

The force is hence central and varies as

$$\frac{1}{r^5}, \qquad \frac{r}{(r^2 - k^2)^3}, \qquad \frac{r}{(r^2 + k^2)^3}. \tag{21}$$

It is to be remembered that the constant of energy must here be taken equal to zero. The dynamical trajectories are then the systems of circles described above. For other values of the energy constant, the trajectories are represented by elliptic functions and the systems do not give rise to Lamé families.

* They were first studied by RIBAUCOUR. Cf. DARBOUX, *loc. cit.*, pp. 53, 56.

It is easily seen that the natural systems obtained are the only ones composed exclusively of circles. This follows from the fact that any family of surfaces whose orthogonal trajectories are circles is necessarily a Lamé family. We thus have

THEOREM IV. *The only natural systems of circles are those formed by the circles orthogonal to a fixed sphere.*

The media (19) are hence the only simply-refracting media in which all light rays are circular.

In conclusion we observe that, since natural systems are characterized by the property of Thomson and Tait,* and Lamé families are characterized by a certain property of Ribaucour,† our first theorem may be restated in the following purely geometric form:

THEOREM V. *If a system of ∞^4 curves in space is to be such that, for an arbitrarily selected surface Σ: 1) there are ∞^2 curves of the system orthogonal to Σ and these form a normal congruence; 2) the osculating circles of these ∞^2 curves for the points of Σ also form a normal congruence; then the system must be composed of the ∞^4 circles orthogonal to a fixed sphere.*‡

Theorem IV is seen to be included in this result as a special case.

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* These Transactions, vol. 11 (1910), p. 121.

† DARBOUX, *loc. cit.*, p. 78.

‡ It is sufficient, in fact, to require merely that the osculating circles shall form a normal congruence. That the system is then natural follows from Theorem II, these Transactions, vol. 11 (1910), p. 130.

A FUNDAMENTAL SYSTEM OF INVARIANTS OF THE GENERAL MODULAR LINEAR GROUP WITH A SOLUTION OF THE FORM PROBLEM*

BY

LEONARD EUGENE DICKSON

1. We shall determine m functions which form a fundamental system of invariants for the group G_m of all linear homogeneous transformations on m variables with coefficients in the Galois field of order p^n. In the so-called form problem for the group G_m, we seek all sets of values of the m variables for which the m fundamental absolute invariants take assigned values. It is shown in § 8 that all sets of solutions are linear combinations of the roots of an equation involving only the powers p^{nm}, $p^{n(m-1)}$, $\cdots$, p^n, 1 of a single variable. This fundamental equation has properties analogous to those of a linear differential equation of the m-th order. In §§ 10–16 we determine the degrees of the irreducible factors of the fundamental equation and, in particular, the smallest field in which it is completely solvable. We obtain a wide generalization of the theory of the equation $\xi^{p^{nm}} - \xi = 0$, which forms the basis of the theory of finite fields. The function defined by the left member of the fundamental equation includes the type of substitution quantics in one variable the theory of which is equivalent to, but preceded historically, the theory of linear modular substitutions on m variables. We here find that the latter theory necessitates a return to the earlier quantics in one variable. Finally, in §§ 17–22, we consider the interpretation of certain invariants.

It follows from the theorem concerning the product of two determinants that a transformation T of G_m replaces the function

$$[e_1, \cdots, e_m] = \begin{vmatrix} x_1^{p^{e_1 n}} & x_2^{p^{e_1 n}} & \cdots & x_m^{p^{e_1 n}} \\ x_1^{p^{e_2 n}} & x_2^{p^{e_2 n}} & \cdots & x_m^{p^{e_2 n}} \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{p^{e_m n}} & x_2^{p^{e_m n}} & \cdots & x_m^{p^{e_m n}} \end{vmatrix}$$

by one which equals $|T|$ times the initial function. By § 2, each of these functions has the factor

$$L_m = [m-1, m-2, \cdots, 1, 0].$$

* Presented to the Society, September 6, 1910.

Certain of the quotients will be given a special notation:

$$Q_{ms} = [m, m-1, \cdots, s+1, s-1, \cdots, 1, 0]/L_m.$$

They are absolute invariants of G_m. We shall prove the

Theorem. *The m invariants L_m, Q_{m1}, $\cdots$, $Q_{m\,m-1}$ are independent and form a fundamental system of invariants of the group G_m.*

2. Consider the product P of all the linear functions

$$a_1x_1 + a_2x_2 + \cdots + a_mx_m$$

in which the a_i are elements not all zero of the $GF[p^n]$ and such that, of the coefficients a_i not zero, the one with smallest subscript is unity:

$$P = \prod_{k=1}^{m} \prod_{a} (x_k + a_{k+1}x_{k+1} + \cdots + a_mx_m),$$

where the inner product extends over the $p^{n(m-k)}$ sets $a_{k+1}, \cdots, a_m$ of $m-k$ elements of the field. Hence the term

$$\prod_{k=1}^{m} x_k^{p^{n(m-k)}} = x_1^{p^{n(m-1)}} \cdots x_{m-1}^{p^n} x_m$$

occurs once and but once in the expansion of P and has the coefficient unity. This term is the product of the elements in the main diagonal of the determinant L_m. Since L_m is invariant * under G_m and has the factor x_1, it follows that L_m is identical † with the product P.

Similarly, $[e_1, \cdots, e_m]$ has the factor L_m.

Theory of Ternary Invariants.

3. In the adjoint determinant of the nine first minors of

$$L_3 = (x^{p^{2n}} y^{p^n} z),$$

consider the three determinants formed of the elements in the first and third columns:

$$\begin{vmatrix} (y^{p^n}z) & (x^{p^n}y) \\ (y^{p^{2n}}z^{p^n}) & (x^{p^{2n}}y^{p^n}) \end{vmatrix}, \quad \begin{vmatrix} (y^{p^{2n}}z) & (x^{p^{2n}}y) \\ (y^{p^{2n}}z^{p^n}) & (x^{p^{2n}}y^{p^n}) \end{vmatrix}, \quad \begin{vmatrix} (y^{p^n}z) & (x^{p^n}y) \\ (y^{p^{2n}}z) & (x^{p^{2n}}y) \end{vmatrix}.$$

They equal $y^{p^n}L_3$, $y^{p^{2n}}L_3$, yL_3, respectively. Hence $L_3^{p^n-1}$ times the first determinant equals the p^n-th power of the third. Transposing the negative terms

* This fact appears to have been first noted by the writer. See his *Linear Groups*, p. 216.

† This theorem is due to Professor E. H. Moore, Bulletin of the American Mathematical Society, vol. 2 (1896), p. 189. His three proofs differ from the above invariantive proof. His sequence of variables is the reverse of that employed here.

and dividing by $(x^{p^{2n}}y^{p^n})(y^{p^{2n}}z^{p^n})$, we get

$$\frac{(y^{p^n}z)\,L_3^{p^n-1}+(y^{p^{3n}}z^{p^n})}{(y^{p^{2n}}z^{p^n})}=\frac{(x^{p^n}y)\,L_3^{p^n-1}+(x^{p^{3n}}y^{p^n})}{(x^{p^{2n}}y^{p^n})}.$$

The second member is invariant under G_2 (§ 1) and is replaced by the first member by the transformation $x'=z$, $z'=-x$, which extends G_2 to G_3. Hence the second member is an invariant of G_3.

Similarly, from the first and second determinants, we get

$$\frac{(x^{p^{2n}}y)\,L_3^{p^n-1}+(x^{p^{3n}}y^{p^{2n}})}{(x^{p^{2n}}y^{p^n})}=\frac{(y^{p^{2n}}z)\,L_3^{p^n-1}+(y^{p^{3n}}z^{p^{2n}})}{(y^{p^{2n}}z^{p^n})},$$

so that the first member is an invariant of G_3.

If we employ the following integral invariants of G_2,

$$(1)\qquad L_2=(x^{p^n}y),\qquad Q_{21}=\frac{(x^{p^{2n}}y)}{L_2}=\frac{x^{p^{2n}-1}-y^{p^{2n}-1}}{x^{p^n-1}-y^{p^n-1}},$$

we may express our ternary invariants in the form

$$(2)\qquad L_3,\qquad Q_{32}=\left(\frac{L_3}{L_2}\right)^{p^n-1}+Q_{21}^{p^n},\qquad Q_{31}=Q_{21}\left(\frac{L_3}{L_2}\right)^{p^n-1}+L_2^{p^{2n}-p^n},$$

the identification of the last two with the quotients Q_{3i} (§ 1) being made in § 4. The fact that L_3 is divisible by L_2 follows from § 2; the quotient may be obtained from the expansion

$$(3)\qquad L_3=z^{p^{2n}}L_2-z^{p^n}L_2Q_{21}+zL_2^{p^n}.$$

We proceed to the proof that the three invariants (2) form a fundamental system. Let I be any homogeneous ternary invariant and let I_0 be the sum of the terms of I which lack z. Then I_0 and the coefficients of the various powers of z in I are invariants of the binary group G_2 on x, y, and hence* are integral functions of the invariants (1).

Let $cL_2^aQ_{21}^b$ be the term of I_0 in which a is a minimum. Then the term of I_0 of minimum degree in y is cx^ey^a, where

$$e=ap^n+bd,\qquad d=p^{2n}-p^n.$$

Since x^ey^a does not occur elsewhere in I_0, cx^ey^a and therefore also cz^ey^a is a term of I. Hence cy^a is a term of an invariant of G_2 and hence a term of kQ_{21}^α. Thus $a=\alpha d$.

Let $c'L_2^{a'}Q_{21}^{b'}$ be any term of I_0. By the homogeneity of I_0,

$$(a'-a)(p^n+1)+(b'-b)d=0.$$

Hence a' is divisible by p^n. We next show that a' is divisible by p^n-1, so that a' is divisible by $d=p^n(p^n-1)$.

* Dickson, these Transactions, this volume, p. 1.

Apply to I_0 a transformation of determinant ρ, where ρ is a primitive root of the $GF[p^n]$. Since Q_{21} is an absolute invariant and L_2 takes the factor ρ (§ 1), it follows that $\rho^{a'} = \rho^a$, whence $a' \equiv a \pmod{p^n - 1}$. But a is divisible by d. Hence a' is divisible by $p^n - 1$.

Let $b = p^\beta B$, where B is prime to p. Let $f = p^\beta (p^n - 1)$. Then

$$Q_{21} = x^a + x^{a-p^n+1} y^{p^n-1} + \cdots, \qquad Q_{21}^{p^\beta} = x^{ap^\beta} + x^{ap^\beta - f} y^f + \cdots,$$

$$Q_{21}^b = x^{bd} + Bx^{bd-f} y^f + \cdots,$$

$$L_2^a = (x^{p^{2n}} y^{p^n} - x^{p^n} y^{p^{2n}})^{a(p^n-1)} = x^{ap^n} y^a + \alpha x^{ap^n - d} y^{a+d} + \cdots.$$

Suppose that $\beta < n$. Then $f < d$ and

$$L_2^a Q_{21}^b = x^{\mu+f} y^a + Bx^\mu y^{a+f} + \cdots \qquad (\mu = ap^n + bd - f).$$

Neither of these terms occurs in another term $c' L_2^{a'} Q_{21}^{b'}$, for which therefore $a' > a$. In fact, $a' \geqq a + d > a + f$. Hence I contains the term $cBz^\mu y^{a+f}$. But $a + f$ is not a multiple of d and B is not zero in the field. Hence $\beta \geqq n$ and b is of the form $p^n b_1$. Hence by (2),

$$I' = I - c Q_{31}^a Q_{32}^{b_1}$$

is an invariant in which I_0' lacks $cL_2^a Q_{21}^b$.

We can similarly delete one after another of the terms free of z and reach ultimately an invariant I_1 in which there are no terms free of z. Since I_1 has the factor z, it has the factor L_3. Thus $I_1 = L_3^e I_2$, where I_2 is either a constant or else is a function having terms free of z. In the latter case we repeat the above process on I_2. It follows that *any integral invariant of* G_3 *is an integral function of the three invariants* (2).

4. For $m = 3$, the invariants Q_{ms} of § 1 are

$$Q_{32} = (x^{p^{3n}} y^{p^n} z)/L_3, \qquad Q_{31} = (x^{p^{3n}} y^{p^{2n}} z)/L_3.$$

Their degrees $p^{3n} - p^{2n}$ and $p^{3n} - p^n$ equal the degrees of the second and third invariants (2), respectively. Hence by the theorem of § 4 the corresponding invariants differ only by a constant factor. We now prove that the factor is unity in each case.

The terms free of z in the quotient Q_{3i} are given by the quotient of the coefficient of z in the numerator by the coefficient of z in the denominator L_3. For $i = 2, 1$, we get respectively

$$(x^{p^{2n}} y^{p^n})/(x^{p^{2n}} y^{p^n}) = Q_{21}^{p^n}, \qquad (x^{p^{3n}} y^{p^{2n}})/(x^{p^{2n}} y^{p^n}) = L_2^{p^{2n} - p^n}.$$

Hence the relations (2) are proved. For another proof by means of a vanishing determinant of the fifth order, see § 6.

Expressions for the Quotients Q_{ms}.

5. The process which led so naturally to invariants (2) can be readily applied also when $m > 3$. For example, if $m = 4$,

$$\begin{vmatrix} (x^{p^{2n}} y^{p^n} w) & (x^{p^{2n}} y^{p^n} z) \\ (x^{p^{3n}} y^{p^n} w) & (x^{p^{3n}} y^{p^n} z) \end{vmatrix} = L_4 \begin{vmatrix} x^{p^n} & y^{p^n} \\ x & y \end{vmatrix},$$

$$\begin{vmatrix} (x^{p^{2n}} y^{p^n} w) & (x^{p^{2n}} y^{p^n} z) \\ (x^{p^{3n}} y^{p^{2n}} w^{p^n}) & (x^{p^{3n}} y^{p^{2n}} z^{p^n}) \end{vmatrix} = L_4 \begin{vmatrix} x^{p^{2n}} & y^{p^{2n}} \\ x^{p^n} & y^{p^n} \end{vmatrix}.$$

Equating the p^n-th power of the upper to $L_4^{p^n-1}$ times the lower, transposing the negative terms, and dividing by $(x^{p^{3n}} y^{p^{2n}} w^{p^n})(x^{p^{3n}} y^{p^{2n}} z^{p^n})$, we get

$$\frac{(x^{p^{2n}} y^{p^n} z) L_4^{p^n-1} + (x^{p^{4n}} y^{p^{2n}} z^{p^n})}{(x^{p^{3n}} y^{p^{2n}} z^{p^n})} = \frac{(x^{p^{2n}} y^{p^n} w) L_4^{p^n-1} + (x^{p^{4n}} y^{p^{2n}} w^{p^n})}{(x^{p^{3n}} y^{p^{2n}} w^{p^n})}.$$

It follows that the left member is an invariant of G_4 (later identified with Q_{43}). We are led in a similar manner to the invariants

$$\frac{(x^{p^{3n}} y^{p^n} z) L_4^{p^n-1} + (x^{p^{4n}} y^{p^{3n}} z^{p^n})}{(x^{p^{3n}} y^{p^{2n}} z^{p^n})}, \qquad \frac{(x^{p^{3n}} y^{p^{2n}} z) L_4^{p^n-1} + (x^{p^{4n}} y^{p^{3n}} z^{p^{2n}})}{(x^{p^{3n}} y^{p^{2n}} z^{p^n})},$$

which will be identified with Q_{42} and Q_{41}, respectively.

6. For general m, we obtain the desired expressions for Q_{ms} by means of the following determinant, which vanishes identically:

$$D_s = \begin{vmatrix} A & A' \\ O & B \end{vmatrix},$$

where O is a matrix of $m-2$ rows and m columns all of whose elements are zero, A' is derived from A by deleting the last column, and *

$$A = \begin{vmatrix} x_1^{p^{nm}} & \cdots & x_m^{p^{nm}} \\ x_1^{p^{n(m-1)}} & \cdots & x_m^{p^{n(m-1)}} \\ \cdots & \cdots & \cdots \\ x_1^{p^n} & \cdots & x_m^{p^n} \\ x_1 & \cdots & x_m \end{vmatrix}, \qquad B = \begin{vmatrix} x_1^{p^{n(m-1)}} & \cdots & x_{m-1}^{p^{n(m-1)}} \\ \cdots & \cdots & \cdots \\ x_1^{p^{n(s+1)}} & \cdots & x_{m-1}^{p^{n(s+1)}} \\ x_1^{p^{n(s-1)}} & \cdots & x_{m-1}^{p^{n(s-1)}} \\ \cdots & \cdots & \cdots \\ x_1^{p^n} & \cdots & x_{m-1}^{p^n} \end{vmatrix}.$$

By Laplace's development of D_s we get

$$\begin{aligned} &(-1)^{m-2}[m, \cdots, 1][m-1, \cdots, s+1, s-1, \cdots, 1, 0] \\ &\quad + (-1)^s (-1)^{m-s-1}[m, \cdots, s+1, s-1, \cdots, 0][m-1, \cdots, 1] \\ &\quad + (-1)^m [m-1, \cdots, 0][m, \cdots, s+1, s-1, \cdots, 1] = 0. \end{aligned}$$

* If $s = 1$, the exponents in the last row of B are p^{2n}; if $s = m-1$, the exponents in the first row are $p^{n(m-2)}$.

Let the factor $(-1)^m$ be removed. For $1 < s < m-1$, we get

$$L_m^{p^m} \cdot Q_{m-1\,s} L_{m-1} - Q_{ms} L_m \cdot L_{m-1}^{p^m} + L_m (Q_{m-1\,s-1} L_{m-1})^{p^m} = 0,$$

$$Q_{ms} = Q_{m-1\,s} \left(\frac{L_m}{L_{m-1}} \right)^{p^{m-1}} + Q_{m-1\,s-1}^{p^m} \qquad (1 < s < m-1). \tag{4}$$

For $s = 1$ and $s = m-1$, we obtain respectively

$$L_m^{p^m} \cdot Q_{m-1\,1} L_{m-1} - Q_{m1} L_m \cdot L_{m-1}^{p^m} + L_m \cdot L_{m-1}^{p^{2n}} = 0,$$

$$L_m^{p^m} \cdot L_{m-1} - Q_{m\,m-1} L_m \cdot L_{m-1}^{p^m} + L_m (Q_{m-1\,m-2} L_{m-1})^{p^m} = 0,$$

$$Q_{m1} = Q_{m-1\,1} \left(\frac{L_m}{L_{m-1}} \right)^{p^{m-1}} + L_{m-1}^{p^{2n}-p^n}, \qquad Q_{m\,m-1} = \left(\frac{L_m}{L_{m-1}} \right)^{p^{m-1}} + Q_{m-1\,m-2}^{p^m}. \tag{5}$$

As a check, we observe that the terms free of x_m, namely the final terms in (4) and (5), equal the quotient of the coefficients of z in the numerator and denominator of Q_{mi}. We note that

$$p^{nm} - p^{ns} = \text{degree of } Q_{ms}, \qquad p^{n(m-1)} + \cdots + p^n + 1 = \text{degree of } L_m. \tag{6}$$

Expanding L_m according to the last column and introducing the Q_{ms} of § 1, we get

$$L_m = x_m L_{m-1}^{p^m} + L_{m-1} \sum_{s=1}^{m-2} (-1)^s x_m^{p^{ns}} Q_{m-1\,s} + (-1)^{m-1} x_m^{p^{n(m-1)}} L_{m-1}. \tag{7}$$

Hence (4) and (5) may be given an integral form.

Fundamental System of Invariants of G_m.

7. **Theorem.** *The functions L_μ, $Q_{\mu 1}, \cdots, Q_{\mu\,\mu-1}$ are independent and form a fundamental system of invariants for G_μ.*

We assume that the theorem is true for $\mu \leqq m$, where $m \geqq 2$, and prove that it is true for $\mu = m+1$.

Let I be any homogeneous integral invariant of G_{m+1}. The coefficients of the various powers of x_{m+1} in I are invariants of G_m and hence by hypothesis are integral functions of L_m, $Q_{m1}, \cdots, Q_{mm-1}$. In particular, the sum I_0 of the terms of I free of x_{m+1} is an aggregate of terms

$$t' = c' L_m^{a'} Q_{m1}^{b'_1} \cdots Q_{mm-1}^{b'_{m-1}} \qquad (c' \neq 0).$$

Consider the set of terms t' in which a' has the minimum value a, the subset in which b'_1 has the minimum value b_1, etc. In the resulting unique term

$$t = c L_m^a Q_{m1}^{b_1} \cdots Q_{mm-1}^{b_{m-1}},$$

the term of minimum degree in x_m is, by (4), (5), (7), $x_m^{a_1} t_1$, where

$$t_1 = c L_{m-1}^{a_1} \prod_{s=2}^{m-1} Q_{m-1\,s-1}^{p^n b_s} \qquad (a_1 = a p^n + b_1 d).$$

Similarly, the term of t_1 of minimum degree in x_{m-1} is $x_{m-1}^{\alpha_1} t_2$,

$$t_2 = cL_{m-2}^{\alpha_1} \prod_{s=3}^{m-1} Q_{m-2\,s-2}^{p^{2n}b_s} \qquad (a_2 = a_1 p^n + p^\nu b_2 d).$$

Proceeding in this manner, we see that t contains a term

$$\tau = cx_m^{a} x_{m-1}^{\alpha_1} x_{m-2}^{\alpha_2} \cdots x_{m-i}^{\alpha_i} \cdots x_1^{\alpha_{m-1}} \quad (a_i = a_{i-1} p^n + p^{n(i-1)} b_i d),$$

where $a_0 = a$. Evidently τ occurs but once in the product t. Further, τ does not occur in a product t' distinct from t. For, if so, $a' = a$ and hence (by a_1) $b_1' = b_1$, then (by a_2) $b_2' = b_2$, etc., so that $t' \equiv t$. Hence I has the isolated term τ and therefore also

$$x_{m+1}^{\alpha_1} \tau_1, \qquad \tau_1 \equiv cx_{m-1}^{\alpha} x_{m-2}^{\alpha_2} \cdots x_1^{\alpha_{m-1}}.$$

Hence τ_1 is a term of an invariant of G_m. The latter invariant has the term

$$x_m^{\alpha_2} \tau_2, \qquad \tau_2 \equiv cx_{m-2}^{\alpha} x_{m-3}^{\alpha_3} \cdots x_1^{\alpha_{m-1}}.$$

Hence τ_2 is a term of an invariant of G_{m-1}. Proceeding in this manner, we see that $\tau_m = cx_1^{\alpha}$ is a term of an invariant of G_2. Hence cx_1^{α} is a term of kQ_{21}^{a}, whence $a = \alpha d$.

By (6_1) the degree of Q_{ms} is a multiple of p^n. Since a is a multiple of p^n, it follows that t is of degree a multiple of p^n. Since this is therefore true of t, and since the degree of L_m is prime to p, by (6_2), it follows that a' is a multiple of p^n. As in § 3, a' is a multiple of $p^n - 1$. Hence *in every term t' of I_0, a' is a multiple of $d = p^n(p^n - 1)$.*

By (4), (5), (7), we have

$$Q_{m1} = L_{m-1}' + x_m^{r} L_{m-1}^{r^2} Q_{m-1\,1} + \cdots, \qquad Q_{ms} = Q_{m-1\,s-1}^{p^n} + x_m^{r} L_{m-1}^{r^2} Q_{m-1\,s} + \cdots \quad (s > 1),$$

the final Q being suppressed if $s = m - 1$. In these series the exponents of x_m differ by multiples of $r = p^n - 1$. Let $b_s = p^{\beta_s} B_s$, where B_s is prime to p. To obtain the power p^β of a sum in a field having modulus p, we have only to multiply every exponent by p^β. Hence we get

$$(8) \qquad L_m^{a} = (L_m^{p^n})^{\alpha r} = x_m^{\alpha} L_{m-1}^{\alpha p^n} - \alpha r x_m^{\alpha+d} L_{m-1}^{\alpha p^n - d} Q_{m-1\,1}^{p^n} + \cdots,$$

$$(9) \qquad Q_{m1}^{b_1} = L_{m-1}^{db_1} + B_1 x_m^{rp^{\beta_1}} L_{m-1}^{e} Q_{m-1\,1}^{p^{\beta_1}} + \cdots \qquad (e = db_1 - rp^{\beta_1}),$$

$$(10) \qquad Q_{ms}^{b_s} = Q_{m-1\,s-1}^{p^n b_s} + B_s x_m^{rp^{\beta_s}} L_{m-1}^{r^2 p^{\beta_s}} Q_{m-1\,s-1}^{e_s} Q_{m-1\,s}^{p^{\beta_s}} + \cdots \quad (e_s = b_s p^n - p^{n+\beta_s}),$$

where in the last series $s > 1$ and the term $Q_{m-1\,s}$ is to be suppressed if $s = m - 1$. Hence t/c contains the terms

$$T_1 = B_1 x_m^{a+rp^{\beta_1}} L_{m-1}^{e+\alpha p^n} Q_{m-1\,1}^{p^{\beta_1}} \prod_{s=2}^{m-1} Q_{m-1\,s-1}^{p^n \beta_s},$$

$$T_\sigma = B_\sigma x_m^{a+rp^{\beta_\sigma}} L_{m-1}^{h_\sigma} Q_{m-1\,\sigma-1}^{e_\sigma} Q_{m-1\,\sigma}^{p^{\beta_\sigma}} \prod Q_{m-1\,s-1}^{p^n b_s} \qquad (h_\sigma = ap^n + db_1 + r^2 p^{\beta_\sigma}),$$

where $\sigma > 1$ and $Q_{m-1\,\sigma}$ is to be suppressed if $\sigma = m-1$, and where in the final product s has the values $2, \cdots, \sigma-1, \sigma+1, \cdots, m-1$.

First, let $\beta_1 < n$. Then $rp^{\beta_1} < d$. The product t contains but one term with the same set of exponents as T_1. For, if we employ a term of (9) after the second, the exponent of x_m exceeds that in T_1; if we employ the second term in (9), we must use the first terms in (8) and (10) and hence get T_1 itself; if we employ the first term of (9), we must use the first term of (8), and obtain as the exponent of L_{m-1} in the product of the two

$$ap^n + db_1 > e + ap^n.$$

Suppose that T_1 is a term of a product t' distinct from t. If $a' > a$, then $a' \geqq a + d$, since a' and a are multiples of d. Thus the minimum exponent a' of x_m in t' would exceed the exponent of x_m in T_1. Hence $a' = a$. Hence by (6) and the homogeneity of our invariant,

$$\sum_{s=1}^{m-1} (b_s' - b_s)(p^{nm} - p^{ns}) = 0. \tag{11}$$

Hence $(b_1' - b_1)p^n$ is a multiple of p^{2n}, so that

$$b_1' \equiv a_1 \pmod{p^n} \qquad \text{when} \qquad a' = a. \tag{12}$$

Thus in $b_1' = p^{\beta_1'} B_1'$, we have $\beta_1' = \beta_1$. Hence T_1 cannot occur in terms of t' other than

$$x_m^{\alpha} L_{m-1}^{ap^n} (L_{m-1}^{tb_1'} + B_1' x_m^{rp^{\beta_1}} L_{m-1}^{e'} Q_{m-1\,1}^{p^{\beta_1}}) \prod_{s=2}^{m-1} Q_{ms}^{b_s'}. \tag{13}$$

If we employ the second term in the parenthesis, we must take the term of each Q_{ms} free of x_m. Then $b_1' = b_1$, from the exponents of L_{m-1}, and $b_s' = b_s$ $(s = 2, \cdots, m-1)$, from the exponents of $Q_{m-1\,s-1}$. But $t' \neq t$. If we employ the first term in the parenthesis in (13), we obtain as the exponent of L_{m-1} in the product of the first two factors

$$ap^n + db_1' > e + ap^n,$$

since $b_1' \geqq b_1$ when $a' = a$. Hence the assumption is false.

We have now shown that T_1 occurs as an isolated term of the invariant. But the exponent of x_m is not a multiple of d and the coefficient B_1 is not zero in the field. Hence this case $\beta < n$ is excluded. Thus b_1 is a multiple of p^n.

Of the numbers $b_1, \cdots, b_{m-1}$ not multiples of p^n, let b_σ be the one with smallest subscript. Then $\sigma > 1$. A term of t with the same set of exponents as T_σ can be obtained only by taking the first terms of (8), (9), (10), for $s < \sigma$. If we use the second term of (10) for $s = \sigma$, we must take the first term of (10) for $s > \sigma$ and then obtain T_σ. If we use the first term of (10) for $s = \sigma$, the exponent of $Q_{m-1\,\sigma-1}$ in the product is $p^n b_\sigma$, which exceeds its exponent e_σ in T_σ.

Next, if T_σ occurs in t', distinct from t, then $a' = a$. From (12) it now follows that b_1' is a multiple of p^n. Analogous to (9),

$$Q_{m1}^{b_1'} = L_{m-1}^{db_1'} + x_m^d K. \tag{14}$$

Hence we must take the first terms of (8) and (14). In the product of these two, the exponent of L_{m-1} is $ap^n + db_1' > h_\sigma$ if $b_1' \geqq b_1 + p^n$. Hence* must $b_1' = b_1$. If $\sigma > 2$, b_2 is by hypothesis a multiple of p^n. Then by (11), b_2' is a multiple of p^n. Hence we must take the first term $Q_{m-11}^{p^n b_2'}$ of $Q_{m2}^{b_2'}$. Since Q_{m-11} does not occur in the expansion of Q_{ms} for $s > 2$, and occurs in T_σ with the exponent $p^n b_2$, we conclude that $b_2' = b_2$. In this manner we may show that we must take the first term of $Q_{ms}^{b_s'}$ $(s < \sigma)$ and that $b_s' = b_s$ $(s = 2, \cdots, \sigma - 1)$. Then by (11), $b_\sigma' \equiv b_\sigma \pmod{p^n}$, whence $\beta_\sigma' = \beta_\sigma$. If we employ the second term in $Q_{m\sigma}^{b_\sigma'}$, we must use the first term in $Q_{ms}^{b_s'}$ $(s > \sigma)$ and we obtain the term T_σ if and only if $b_s' = b_s$ $(s = \sigma, \cdots, m-1)$, as shown by comparing the exponents of Q_{m-1s} $(s \geqq \sigma)$. But then $t' \equiv t$. If we employ the first term in $Q_{m\sigma}^{b_\sigma'}$, the total exponent of $Q_{m-1\sigma-1}$ in t' is $p^n b_\sigma'$ which exceeds its exponent e_σ in T_σ since $b_\sigma' \geqq b_\sigma$, in view of our definition of t.

We have now shown that τ_σ occurs as an isolated term of the invariant. But the exponent of x_m is not a multiple of d and the coefficient B_σ is not zero in the field. Hence our assumption on b_σ is false, so that $b_1, \cdots, b_{m-1}$ are all multiples of p^n. Set $b_s = p^n c_s$. Then

$$I' = I - c\, Q_{m+11}^{\sigma} \prod_{s=1}^{m-1} Q_{m+1s+1}^{c_s}$$

is an invariant of G_{m+1} in which I_0' lacks t. As at the end of § 3, it follows that I is an integral function of L_{m+1}, Q_{m+1i} $(i = 1, \cdots, m)$.

It remains to prove that the latter invariants are independent.† Any rational integral relation between them can be given the form

$$A L_{m+1} + B(Q_{m+11}, \cdots, Q_{m+1m}) = 0.$$

Let $x_{m+1} = 0$. Then by (4) and (5) with m replaced by $m+1$, we get

$$B(L_{m1}', Q_{m1}^{p^n}, \cdots, Q_{mm-1}^{p^n}) = 0.$$

But L_m and the Q_{ms} are independent by hypothesis. Hence $B \equiv 0$. Thus the initial relation has the factor L_{m+1}. Since the relation cannot reduce to $L_{m+1}^\lambda \equiv 0$, it may be given the form $A' L_{m+1} + B' = 0$. As before $B' \equiv 0$. A repetition of this argument shows that no relation exists between the L_{m+1}, Q_{m+1i}.

As a basis for our induction, we note that there is no relation $A L_2 + c Q_{21}^{\epsilon} = 0$

* For $m = 3$, (11) gives $b_2' = b_2$, whence $t' \equiv t$.

† Another proof follows from the existence of solutions of the form problem (§ 8), whatever values be assigned to the fundamental invariants.

between the invariants of G_2. For, by setting $y = 0$, we get $cx^{\alpha\delta} = 0$, whence $c = 0$. Proceeding similarly with $A = A'L_2 + c'Q_{21}' = 0$, we prove that $A \equiv 0$.

The Form Problem.

8. In discussing the solution of a set of equations with coefficients in a finite field having modulus p, it is convenient to introduce the infinite field F_p composed of all the roots of all rational integral equations with integral coefficients taken modulo p. Then F_p, like the field of all complex numbers, has the property that any algebraic equation of degree k with coefficients in the field has k roots in the field.

In the form problem for the group G_m, we seek the sets of values of the m variables x_i for which the m fundamental absolute invariants L_m^r, Q_{ms} $(s = 1, \cdots, m-1)$ take assigned values λ, q_s in F_p. Here $r = p^n - 1$. If l is a particular r-th root of λ, the problem * consists in the solution of

$$L_m(x_i) = l, \qquad Q_{ms}(x_i) = q_s \qquad (s = 1, \cdots, m-1). \tag{15}$$

Let $x_1, \cdots, x_m$ be a set of solutions of (15). Since the determinant L_{m+1} vanishes when x_{m+1} equals one of the x_i $(i \leqq m)$, it follows from (7), with m replaced by $m+1$, that $x_1, \cdots, x_m$ are roots of

$$l\xi^{p^{nm}} + \sum_{s=1}^{m-1} (-1)^{m-s} l q_s \xi^{p^{ns}} + (-1)^m l^{p^n} \xi = 0. \tag{16}$$

Suppose for the present that $l \neq 0$. The preceding equation gives

$$\xi^{p^{nm}} + \sum_{s=1}^{m-1} (-1)^{m-s} q_s \xi^{p^{ns}} + (-1)^m \lambda \xi = 0. \tag{17}$$

This equation has no double root and hence has p^{nm} distinct roots in F_p. If ξ_1 and ξ_2 are roots, then are also $\xi_1 + \xi_2$, $c_1\xi_1$, where c_1 is an element of the $GF[p^n]$. Indeed, in that field,

$$(\xi_1 + \xi_2)^{p^{ns}} = \xi_1^{p^{ns}} + \xi_2^{p^{ns}}, \qquad (c_1\xi_1)^{p^{ns}} = c_1\xi_1^{p^{ns}}.$$

Hence there exist m solutions $\xi_1, \cdots, \xi_m$ of (17), linearly independent with respect to the $GF[p^n]$, while ξ is a solution if and only if

$$\xi = c_1\xi_1 + \cdots + c_m\xi_m \qquad (c\text{'s in } GF[p^n]). \tag{18}$$

Since the x_i are solutions, we have

$$x_i = c_{i1}\xi_1 + \cdots + c_{im}\xi_m \qquad (i = 1, \cdots, m), \tag{19}$$

in which the c_{ij} are elements of non-vanishing determinant of the $GF[p^n]$. Indeed, by § 1,

$$l = L_m(x_i) = |c_{ij}| \cdot L_m(\xi_i). \tag{20}$$

* This problem is the form problem for the subgroup G_m' of all the transformations of determinant unity.

To show conversely that any such set of values (19) satisfy equations (15), let $\xi_1, \cdots, \xi_m$ be any set of roots of (17) linearly independent with respect to the $GF[p^n]$. Then by § 2, $L_m(\xi_i) \neq 0$. Define the x_i by (19), where $|c_{ij}| \neq 0$. Then $l \neq 0$ by (20). By § 2, the p^{nm} expressions (18) are the roots of

$$L_{m+1}(\xi_1, \cdots, \xi_m, \xi) = 0,$$

in which the coefficient of $\xi^{p^{nm}}$ is $\pm L_m(\xi_i) \neq 0$, and hence are the roots of

$$\xi^{p^{nm}} + \sum_{s=1}^{m-1} (-1)^{m-s} Q_{ms}(\xi_i) \xi^{p^{ns}} + (-1)^m [L_m(\xi_i)]^r \xi = 0.$$

Since this equation and (17) have in common the p^{nm} distinct roots (18), they are identical. In view of the absolute invariance of L_m^r and Q_{ms}, we conclude that the expressions (19) satisfy equations (15), in which $l^r = \lambda$.

Theorem. *For $\lambda \neq 0, x_1, \cdots, x_m$ is a set of solutions of*

$$L_m^{p^{n-1}} = \lambda, \qquad Q_{ms} = q_s \qquad (s = 1, \cdots, m-1) \tag{21}$$

if and only if $x_i = c_{i1}\xi_1 + \cdots + c_{im}\xi_m$ $(i = 1, \cdots, m)$, where the c_{ij} are elements of non-vanishing determinant of the $GF[p^n]$, and $\xi_1, \cdots, \xi_m$ is any set of roots of equation (17) *linearly independent with respect to the $GF[p^n]$.*

To obtain the sets of solutions of (15), we restrict the c_{ij} to be of determinant unity and hence, by (20), the ξ_i to be linearly independent roots of (17) for which $L_m(\xi_i) = l$.

Next, let $\lambda = 0$, $q_1 \neq 0$. If the minors of the elements of the first row of L_m all vanished, there would exist (§ 2) a linear relation between each set of $m-1$ of the x's. Applying a linear transformation, we would obtain $x_m = x_{m-1} = 0$. By (7) the quotient L_m/L_{m-1} would vanish and, by (5_1), $Q_{m1} = 0$, in contradiction with $q_1 \neq 0$. Hence the above minors are not all zero. After permuting the variables we may set $L_{m-1} \neq 0$. Then, by (4) and (5),

$$L_{m-1}^{p^n(p^n-1)} = q_1, \qquad Q_{m-1\,s-1}^{p^n} = q_s \qquad (s = 2, \cdots, m-1).$$

As above, $x_1, \cdots, x_{m-1}$ are linear functions of a set of linearly independent roots $\xi_1, \cdots, \xi_{m-1}$ of

$$\xi^{p^{n(m-1)}} + \sum_{\sigma=2}^{m-1} (-1)^{m-\sigma} q_\sigma^{1/p^n} \xi^{p^{n(\sigma-1)}} + (-1)^{m-1} q_1^{1/p^n} \xi = 0,$$

given by (7) upon replacing s by $\sigma - 1$. Raising this equation to the power p^n we obtain (17) for $\lambda = 0$. Each root of the latter is therefore of multiplicity exactly p^n. In view of $L_m = 0$, there exists (§ 2) a linear relation between $x_1, \cdots, x_m$, with coefficients in the $GF[p^n]$. Since $x_1, \cdots, x_{m-1}$ are linearly independent with respect to this field, x_m is a linear function of $x_1, \cdots, x_{m-1}$ and hence of $\xi_1, \cdots, \xi_{m-1}$, with coefficients in this field. Returning to the initial order of the variables, we conclude that, if $\lambda = 0$, $q_1 \neq 0$, $x_1, \cdots, x_m$ are

linear functions of a set $m-1$ linearly independent roots of (17), the matrix of the coefficients being of rank $m-1$.

Next, let $\lambda = q_1 = 0$, $q_2 \neq 0$. Then the minors of the elements of the first row of L_m all vanish. For, if $L_{m-1} \neq 0$, for example, (5_1) would give $Q_{m1} \neq 0$, contrary to $q_1 = 0$. After applying a linear transformation T, we may set $x_m = x_{m-1} = 0$. Then by (7) the quotient L_m / L_{m-1} is zero. Hence by (4) and (5_2),

$$Q^{p^n}_{m-1\,s-1} = q_s \qquad (s = 2, \cdots, m-1).$$

By (7), with m replaced by $m-1$, the quotient L_{m-1}/L_{m-2} vanishes when $x_{m-1} = 0$. Hence by (4) and (5), with m replaced by $m-1$,

$$L^{p^{2n}(p^n-1)}_{m-2} = q_2, \qquad Q^{p^{2n}}_{m-2\,s-2} = q_s \qquad (s = 3, \cdots, m-1).$$

If, for $i \leqq m-2$, we multiply the elements of the i-th column of L_{m-1} by the adjoint minors of the corresponding elements of the last column, we see (compare (7) with m replaced by $m-1$) that $x_1, \cdots, x_{m-2}$ are roots of

$$0 = \xi L^{p^n}_{m-2} + L_{m-2} \sum_{s=1}^{m-3} (-1)^s \xi^{p^{ns}} Q_{m-2\,s} + (-1)^{m-2} \xi^{p^{n(m-2)}} L_{m-2}.$$

Divide by L_{m-2} and raise the resulting equation to the power p^{2n}. We obtain equation (17), since $\lambda = q_1 = 0$. Each root of the latter is now of multiplicity p^{2n}. As above, $x_1, \cdots, x_{m-2}$ are linearly independent linear functions of $m-2$ linearly independent roots $\xi_1, \cdots, \xi_{m-2}$ of (17). Applying the inverse of T, we conclude that the initial values $x_1, \cdots, x_m$ are linear functions of $\xi_1, \cdots, \xi_{m-2}$, the matrix of the coefficients being of rank $m-2$.

Proceeding in a similar manner, we obtain the

Theorem. *If $\lambda = q_1 = \cdots = q_{t-1} = 0$, $q_t \neq 0$, then $x_1, \cdots, x_m$ is a set of solutions of (21) if and only if the x_i are linear functions of $\xi_1, \cdots, \xi_{m-t}$ with coefficients in the $GF[p^n]$ the rank of whose matrix is $m-t$, while $\xi_1, \cdots, \xi_{m-t}$ is any set of roots of (17) linearly independent with respect to the $GF[p^n]$, every root of (17) being a linear function of these $m-t$ roots. If all the invariants are zero, each x_i is zero.*

9. The number of matrices (c_{ij}) of rank $m-t$ with m rows and $m-t$ columns, each c_{ij} being an element of the $GF[p^n]$, is

$$(p^{nm}-1)(p^{nm}-p^n)\cdots(p^{nm}-p^{n(m-t-1)}). \tag{22}$$

Hence this gives the number of distinct sets of solutions of the form problem when $\lambda = q_1 = \cdots = q_{t-1} = 0$, $q_t \neq 0$.

From the above discussion follows the

Theorem. *The determinant L_m is of rank $m-t$ if and only if*

$$L_m = 0, \quad Q_{m1} = 0, \quad \cdots, \quad Q_{m\,t-1} = 0, \quad Q_{mt} \neq 0. \tag{23}$$

These are necessary and sufficient invariantive conditions that the variables x_i shall satisfy exactly t linearly independent linear relations in the $GF[p^n]$.

Solution of the Fundamental Equation.

10. The complete solution of the form problem has been reduced to the solution of the fundamental equation (17). If $\lambda = 0$, the latter equation is the p^n-th power of an equation of like type. Hence it suffices to discuss the solution of equations of type (17) with $\lambda \neq 0$.

We may restrict attention to the case in which the coefficients of (17) belong to the $GF[p^n]$, since in the contrary case the equation is equivalent to one of like form with coefficients in the $GF[p^n]$, but of higher degree. The nature of the proof will be indicated for $m = 2$. Then (17) becomes

$$\xi^{p^{2n}} = q\xi^{p^n} - \lambda\xi. \tag{24}$$

Let q be a root of an equation $Q^2 - aQ + b = 0$, irreducible in the $GF[p^n]$. Its second root is q^{p^n}, so that

$$q^{p^n} + q = a, \qquad q^{p^{n+1}} = b, \qquad q^{p^{2n}} = q.$$

From the p^n-th power of (24) we eliminate $\xi^{p^{2n}}$ and get

$$\xi^{p^{3n}} = (b - \lambda^{p^n})\xi^{p^n} - q^{p^n}\lambda\xi.$$

From the p^n-th power of the latter we eliminate $q\xi^{p^n}$ by (24) and get

$$\xi^{p^{4n}} = \beta\xi^{p^{2n}} - \lambda^{p^{n+1}}\xi, \qquad \beta = b - \lambda^{p^{2n}} - \lambda^{p^n}. \tag{25}$$

If λ belongs to the $GF[p^n]$, the required equation is thus

$$\xi^{p^{4n}} - (b - 2\lambda)\xi^{p^{2n}} + \lambda^2\xi = 0. \tag{26}$$

If λ is a root of an equation $L^2 - rL + s = 0$, irreducible in the $GF[p^n]$, the required equation is obviously

$$\xi^{p^{4n}} - (b - r)\xi^{p^{2n}} + s\xi = 0. \tag{27}$$

If λ is a root of an equation $\lambda^3 - c\lambda^2 + d\lambda - e = 0$, irreducible in the $GF[p^n]$, we make repeated use of the relations

$$\lambda + \lambda^{p^n} + \lambda^{p^{2n}} = c, \qquad \lambda^{1+p^n+p^{2n}} = e,$$

raise (25) to the powers p^{2n}, p^{4n}, $\cdots$, p^{8n}, and find that

$$\xi^{p^{12n}} - \{b(b-c)^2 - 2e\}\xi^{p^{6n}} + e^2\xi = 0. \tag{28}$$

11. We therefore consider the fundamental equation (17) with coefficients in the $GF[p^n]$ and $\lambda \neq 0$. There is no multiple root. Let r be a root $\neq 0$.

If $r^{p^n} = xr$, where x belongs to the $GF[p^n]$ then

$$r^{p^{2n}} = xr^{p^n} = x^2 r, \qquad r^{p^{jn}} = x^j r.$$

Hence, by (17), x must satisfy the *characteristic equation*

$$\Delta(x) \equiv x^m + \sum_{s=1}^{m-1} (-1)^{m-s} q_s x^s + (-1)^m \lambda = 0. \tag{29}$$

Thus each root in the $GF[p^n]$ of $\Delta(x) = 0$ furnishes a factor $\xi^{p^n} - x\xi$ of (17). Let this binomial have a factor $f(\xi)$, of degree d, irreducible in the $GF[p^n]$. Its roots are

$$r,\ r^{p^n} = xr,\ r^{p^{2n}} = x^2 r,\ \cdots,\ r^{p^{(d-1)n}} = x^{d-1} r,$$

while x belongs to the exponent d. Since x is in the field, d is a divisor of $p^n - 1$. It follows that

$$f(\xi) = \xi^d - \delta \qquad (\delta = r^d).$$

Theorem. *The irreducible factors of $\xi^{p^n-1} - x$ are all binomial and of equal degree, namely, the exponent to which x belongs.*

Discussion of the Fundamental Equation for $m = 2$.

12. For the present, let $m = 2$. Since $2p^n - 1 < p^{2n}$, equation (24) has an irreducible factor $F(\xi)$, of degree $D > 1$, not of the preceding type $f(\xi)$, and hence has a root r such that r^{p^m}/r is not an element of the $GF[p^n]$. Since, therefore, r and r^{p^n} are linearly independent with respect to that field, we conclude from (18) that every root of (24) is of the form $c_1 r + c_2 r^{p^n}$, where c_1 and c_2 are elements of the $GF[p^n]$. Hence (24) has all its roots in the $GF[p^{nD}]$, but not all in a smaller field.

Theorem. *For $m = 2$, every irreducible factor of the fundamental equation is of degree a divisor of D; each irreducible factor not of the above binomial type is of degree D.*

13. We proceed to determine this integer D which is such that equation (24) is completely solvable in the $GF[p^{nD}]$, but not in the $GF[p^{nl}]$, for $l < D$. By raising (24) to the powers $p^n, p^{2n}, \cdots$, we may express $\xi^{p^{nt}}$ as a linear function l_t of ξ^{p^n} and ξ. We seek the least value D of t for which $l_t \equiv \xi$. Now the coefficients of l_t are the elements of the first line in S^{t-1}, where

$$S = \begin{pmatrix} q & -\lambda \\ 1 & 0 \end{pmatrix}, \qquad \Delta(x) = x^2 - qx + \lambda.$$

The condition for $l_{D+1} = \xi^{p^n}$ is therefore $S^D = 1$. Hence D is the period of the transformation S. According as the characteristic equation $\Delta(x) = 0$ has distinct roots x_1 and x_2 or equal* roots $x = \frac{1}{2}q = \lambda^{\frac{1}{2}}$, the canonical form for S is

$$\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \qquad \begin{pmatrix} x & x \\ 0 & x \end{pmatrix}.$$

In the first case, the period of S is the least common multiple of the exponents

* We then employ the new variables $V_1 = v_1,\ V_2 = v_1 - xv_2$.

to which the roots x_1 and x_2 belong. In the second case, the period is p times the exponent to which the double root belongs.

Theorem. *For $m = 2$, the fundamental equation is completely solvable in the $GF[p^{nD}]$, but in no lower field, where D is the least common multiple of the exponents to which belong distinct roots of the characteristic equation, or p times the exponent to which its double root belongs.*

Illustrations of the preceding results are afforded by the following examples in which are given all the irreducible factors other than ξ of (24):

$p^n = 2$. $\xi^3 + 1 \equiv (\xi + 1)(\xi^2 + \xi + 1)$, $\xi^3 - \xi + 1$ irreducible.

$p^n = 3$. $\xi^3 - 1 \equiv (\xi + 1)(\xi - 1)(\xi^2 + 1)(\xi^2 + \xi - 1)(\xi^2 - \xi - 1)$,

$\xi^8 \pm \xi^2 - 1$ irreducible, $\xi^8 + \xi^2 + 1 \equiv (\xi+1)(\xi-1)(\xi^3-\xi+1)(\xi^3-\xi-1)$,

$\xi^3 + 1 \equiv (\xi^4+\xi^2-1)(\xi^4-\xi^2-1)$, $\xi^8 - \xi^2 + 1 \equiv (\xi^2+1)(\xi^6 - \xi^4 + \xi^2 + 1)$.

Theory of the General Fundamental Equation.

14. We remove the restriction that $m = 2$ and prove the

Theorem. *If the characteristic function* (29) *reduces in the $GF[p^n]$,*

$$(30)\quad \Delta(x) = \phi(x)\psi(x), \quad \phi(x) = \sum_{i=0}^{\alpha} a_i x^i, \quad \psi(x) = \sum_{j=0}^{m-\alpha} b_j x^j, \quad a_\alpha = b_{m-\alpha} = 1,$$

the fundamental equation (17) *is transformed into*

$$(31)\qquad \Psi(\eta) \equiv \sum_{j=0}^{m-\alpha} b_j \eta^{p^{nj}} = 0$$

by the substitution

$$(32)\qquad \eta = \Phi(\xi) \equiv \sum_{i=0}^{\alpha} a_i \xi^{p^{ni}}.$$

In other words, the fundamental equation factors into*

$$\prod_{k=1}^{p^{n(m-\alpha)}} [\Phi(\xi) - \eta_k] = 0 \qquad [\eta_k \text{ roots of } \Psi(\eta) = 0].$$

For proof, we note that the result of the elimination of η is

$$\sum a_i b_j \xi^{p^{n(i+j)}} \qquad (i = 0, \cdots, \alpha;\ j = 0, \cdots, m-\alpha),$$

which is identical with (17), since by (30) the corresponding sum $\sum a_i b_j x^{i+j}$ is identical with (29).

Since equation (31) has the root $\eta = 0$, we obtain

Corollary I. The fundamental equation has the factor $\Phi(\xi)$ if and only if the characteristic equation has the factor $\phi(x)$ in the $GF[p^n]$.

Corollary II. A root of the fundamental equation satisfies $\Phi(\xi) = 0$, but no similar equation of lower degree, if and only if $\phi(x)$ is a factor of $\Delta(x)$.

* Hence it has the symbolic expression $\psi(\phi)$, as in the theory of the reducibility of linear differential equations.

15. For $k \leqq m$ let D_k denote the number of the non-vanishing roots of the fundamental equation each of which satisfies an equation $\Phi_k(\xi) = 0$, but no equation $\Phi_l(\xi) = 0$, $l < k$. By corollary II, $\phi_k(x)$ must be a factor of $\Delta(x)$. Suppose first that the latter has no multiple factors. Denote by $N_1, N_2, N_3, \cdots$ the number of its irreducible factors of degree $1, 2, 3, \cdots$. Then

$$m = N_1 + 2N_2 + 3N_3 + \cdots.$$

Let $[i] = p^{ni} - 1$. We proceed to show that

$$D_k = \Sigma \binom{N_1}{n_1}\binom{N_2}{n_2}\cdots\binom{N_k}{n_k}[1]^{n_1}[2]^{n_2}\cdots[k]^{n_k}, \tag{33}$$

the sum extending over all sets of positive integers n_i for which

$$k = n_1 + 2n_2 + 3n_3 + \cdots + kn_k, \qquad n_i \leqq N_i.$$

Let a particular factor ϕ_k of $\Delta(x)$ contain $n_1, n_2, n_3, \cdots$ irreducible factors of degree $1, 2, 3, \cdots$. Of the $[k]$ roots of the corresponding equation $\Phi_k(\xi) = 0$, we wish to exclude those which satisfy $\Phi_l(\xi) = 0$, $l < k$. Let ϕ_l have $m_1, m_2, m_3, \cdots$ irreducible factors of degree $1, 2, 3, \cdots$. We assume that (33) holds when k is replaced by a smaller value l. Then the number of roots to be excluded is

$$E = \sum_{l=1}^{k-1} \Sigma \binom{n_1}{m_1}\binom{n_2}{m_2}\cdots\binom{n_l}{m_l}[1]^{m_1}[2]^{m_2}\cdots[l]^{m_l},$$

the inner sum extending over all sets of positive integers m_i for which

$$l = m_1 + 2m_2 + \cdots + lm_l, \qquad m_i \leqq n_i.$$

To prove (33), it remains to show that

$$[k] - E = [1]^{n_1}[2]^{n_2}\cdots[k]^{n_k}.$$

This follows from *

$$1 + E + [1]^{n_1}\cdots[k]^{n_k} = \sum_{l=0}^{k} \Sigma \binom{n_1}{m_1}\cdots\binom{n_l}{m_l}[1]^{m_1}\cdots[l]^{m_l}$$

$$= \left\{\sum_{m_1=0}^{n_1}\binom{n_1}{m_1}[1]^{m_1}\right\}\cdots\left\{\sum_{m_k=0}^{n_k}\binom{n_k}{m_k}[k]^{m_k}\right\} = \{1+[1]\}^{n_1}\cdots\{1+[k]\}^{n_k} = p^{nk}.$$

Hence if $\Delta(x)$ has no multiple root, the number of the roots of the fundamental equation which satisfy no similar equation of lower degree is

$$D_m = [1]^{N_1}[2]^{N_2}\cdots[m]^{N_m}.$$

If the characteristic equation has multiple roots,

$$\Delta(x) = \prod_i F_{di}^{e_i},$$

* Note that, for the added term, $l = k$, whence $m_i = n_i$.

where the F's are distinct irreducible functions, but not necessarily of distinct degrees d_i, the preceding result is to be replaced by

$$D_m = p^{ne} \prod_i [d_i], \qquad e = \sum_i d_i(e_i - 1). \tag{34}$$

From $D_m > 1$ we infer that the fundamental equation has a root r such that $r^{p^{ni}}$ $(i = 0, 1, \cdots, m-1)$ are linearly independent with respect to the $GF[p^n]$. Hence, by (18), every root is of the form

$$\xi = \sum_{i=0}^{m-1} c_i r^{p^{ni}} \qquad (c\text{'s in } GF[p^n]).$$

Let D be the degree of the equation, irreducible in the $GF[p^n]$, which has the root r and hence also the roots $r^{p^{ni}}$. For $i = D$, the latter equals r. By the linear independence, $D \geqq m$.

Theorem. *Every irreducible factor in the $GF[p^n]$ of the fundamental equation is of degree a divisor of D, where D is not less than m and is the common degree of all the irreducible factors which do not divide a similar equation of lower degree. The fundamental equation is completely solvable in the $GF[p^{nD}]$, but not in a smaller field.*

16. We proceed to determine D. By the powers p^n, p^{2n}, $\cdots$ of (17),

$$\xi^{p^{nt}} = \sum_{i=0}^{m-i} a_i \xi^{p^{ni}} \equiv l_t(\xi) \qquad (t \geqq m).$$

We seek the least value D of t for which $l_t \equiv \xi$. Now, the coefficients of $\xi^{p^{n(m-1)}}, \cdots, \xi^{p^n}, \xi$ in l_t are the elements of the first row of S^{t-m+1}, where

$$S = \begin{Vmatrix} q_{m-1} & -q_{m-2} & q_{m-3} & \cdots & (-1)^{m-2} q_1 & (-1)^{m-1}\lambda \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{Vmatrix}.$$

If $l_t = \xi$, then $l_{t+1} = \xi^{p^n}, \cdots, l_{t+m-1} = \xi^{p^{n(m-1)}}$, and hence $S^t = 1$. Thus D is the period of the transformation of S. The minor of the element $\pm\lambda$ in the matrix S equals unity. Hence the characteristic determinant $(-1)^m \Delta(x)$ of S is its single invariant-factor. Hence,* if

$$(-1)^m \Delta(x) = F_k^{\kappa} F_l^{\lambda} \cdots,$$

where $F_k, F_l, \cdots$ are distinct irreducible functions in the $GF[p^n]$, the first having the roots $K_1, \cdots, K_k$, the second the roots $L_1, \cdots, L_l$, the transformation S has the canonical form

$$\eta'_{i1} = K_i \eta_{i1},\ \eta'_{i2} = K_i(\eta_{i1} + \eta_{i2}),\ \eta'_{i3} = K_i(\eta_{i2} + \eta_{i3}),\ \cdots,\ \eta'_{i\kappa} = K_i(\eta_{i\kappa-1} + \eta_{i\kappa}),$$
$$\zeta'_{i1} = L_i \zeta_{i1},\ \zeta'_{i2} = L_i(\zeta_{i1} + \zeta_{i2}),\ \zeta'_{i3} = L_i(\zeta_{i2} + \zeta_{i3}),\ \cdots,\ \zeta'_{i\lambda} = L_i(\zeta_{i\lambda-1} + \zeta_{i\lambda}),\ \cdots$$

* These Transactions, vol. 3 (1902), p. 291.

where $i = 1, \cdots, k$ in the first line, $i = 1, \cdots, l$ in the second, $\cdots$. The transformation S therefore has the maximum number of variables in each chain. Let κ be greatest of the exponents $\kappa, \lambda, \cdots$. Then the longest chain contains κ variables. Determine h so that $p^{h-1} < \kappa \leqq p^h$. Let d be the least common multiple of the exponents to which belong the roots $K_1, L_1, \cdots$. Then * $D = dp^h$.

Theorem. *The smallest field in which the fundamental equation is completely solvable is the* $GF[p^{nD}]$, $D = dp^h$, *where* d *is the least common multiple of the exponents to which belong the roots of the characteristic equation* $\Delta(x) = 0$, *while* p^h *is the least power of* p *which is equal to or greater than the maximum multiplicity of a root of* $\Delta(x) = 0$.

In the case of the classical equation $\xi^{p^m} - \xi = 0$, we have $\Delta(x) = x^m - 1$, so that $D = m$.

In case $\Delta(x)$ is a primitive irreducible function, we have $D = p^{nm} - 1$. Thus the fundamental equation is the product of ξ and an irreducible equation.

The Interpretation of certain Invariants, § 17–22.

17. Since the determinant $[2m - 2, 2m - 4, \cdots, 4, 2, 0]$ is the product of the distinct † linear functions of m variables in the $GF[p^{2n}]$, its quotient by L_m is the product J_m of all distinct quadratic forms in the $GF[p^{2n}]$ on m variables which can be transformed into irreducible binary forms. Indeed, each linear factor of J_m is of the form $l_1 + \rho l_2$, where l_1 and l_2 are linear forms in the $GF[p^n]$, $l_2 \neq 0$, and ρ is a root of a quadratic equation irreducible in that field. If ρ' is the second root, $l_1 + \rho' l_2$ is a factor of J_m. The product of the two factors is a quadratic form in the $GF[p^n]$ which $x_1' = l_1$, $x_2' = l_2$ transforms into an irreducible binary form.

For $m = 2$, J_2 is the invariant Q_{21} of the fundamental system.

For $m = 3$, we have the following expression for J_3:

$$J_3 = Q_{31} Q_{32}^{p^n} - L_3^{p^n(p^n-1)}. \tag{35}$$

For proof, we expand the identically vanishing determinant

$$\begin{vmatrix} a_4 & b_4 & c_4 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 & a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 & a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 & a_0 & b_0 & c_0 \\ 0 & 0 & 0 & a_2 & b_2 & c_2 \end{vmatrix}$$

* Jordan, *Traité des Substitutions*, p. 127. Jordan's q is the present $h - 1$.

† Here and below we shall say that two functions are distinct if their ratio is not a constant.

by Laplace's method according to the minors of the first three columns and get

$$(36) \qquad (a_1b_3c_2)(a_2b_1c_0) - (a_1b_2c_1)(a_3b_2c_0) + (a_1b_2c_0)(a_3b_2c_1) \equiv 0.$$

Taking $a_i = x^{p^{ni}}$, $b_i = y^{p^{ni}}$, $c_i = z^{p^{ni}}$, and dividing by $L_3^{p^n+1}$, we get (35). Since (35) is an irreducible function of its arguments, we conclude that any irreducible binary quadratic form is equivalent to a constant multiple of any other irreducible binary quadratic form within G_3 and hence within G_2.

Invariants relating to Cubic Forms.

18. The product of the distinct linear functions of m variables in the $GF[p^{3n}]$, no one of which is a constant times a linear function in the $GF[p^n]$, equals

$$P_m = [3m-3, 3m-6, \cdots, 6, 3, 0]/L_m.$$

Each linear factor of P_m is of the form

$$\gamma = \sum_{i=1}^{m} \gamma_i x_i, \qquad \gamma_i \equiv c_{i0} + c_{i1}\rho + c_{i2}\rho^2,$$

where ρ is a root of a fixed irreducible cubic R in the $GF[p^n]$, and where the ratios of the γ_i do not all belong to the latter field.

If $m = 2$, the factors are $x_1 - \sigma x_2$, where σ is in the $GF[p^{3n}]$, but not in the $GF[p^n]$. Hence P_2 is the product of the distinct irreducible binary cubic forms in the $GF[p^n]$. See (42) below.

If $m \geqq 3$, the rank of the matrix (c_{ij}) is 3 or 2. In the first case $\sum c_{i0}x_i$, $\sum c_{i1}x_i$, $\sum c_{i2}x_i$ are linearly independent, and there exists a transformation of G_m which replaces γ by $x_1 + \rho x_2 + \rho^2 x_3$. The functions obtained from the latter by replacing ρ by either of the remaining roots of the cubic R are factors of P_m. Hence P_m contains as a factor the product K_m of all distinct m-ary cubic forms equivalent to a non-vanishing ternary cubic form.* If the rank is 2, the linear factor is equivalent to $x_1 - \rho x_2$. Hence $P_m = K_m C_m$, where C_m is the product of all distinct m-ary cubic forms equivalent to an irreducible binary form. Now, there are

$$(37) \qquad N = (p^{nm}-1)(p^{nm}-p^n)(p^{nm}-p^{2n})$$

sets of elements c_{ij} in the $GF[p^n]$ such that matrix (c_{ij}) is of rank 3. But two linear functions whose ratio is one of the $p^{3n}-1$ elements $\neq 0$ of the $GF[p^{3n}]$ give the same factor. Hence the degree of K_m is

$$(38) \qquad k_m = N/(p^{3n}-1).$$

The degree c_m of C_m is therefore $p_m - k_m$, where

$$(39) \qquad p_m = p^{3n(m-1)} - p^{n(m-1)} + p^{3n(m-2)} - p^{n(m-2)} + \cdots + p^{3n} - p^n.$$

* Dickson, Bulletin of the American Mathematical Society, vol. 14 (1908), p. 161.

19. We proceed to the evaluation of the invariant C_3 of degree

$$c_3 = (p^{3n} - p^n)(p^{2n} + p^n + 1) = p^n(p^{3n} - 1)(p^n + 1).$$

To this end we construct an integral function of the fundamental invariants (2) of G_3 which vanishes identically in y, z for $x = \rho y$, where

$$\rho^{p^{3n}} = \rho, \qquad \rho^{p^n} \neq \rho. \tag{40}$$

For $x = \rho y$, (1) gives

$$L_2 = (\rho^{p^n} - \rho) y^{p^{n+1}}, \qquad Q_{21} - k y^{p^{2n} - p^n}, \qquad k \equiv (\rho^{p^{2n}} - \rho)/(\rho^{p^n} - \rho).$$

In view of (40), we have

$$k^{p^n} = (\rho - \rho^{p^n})/(\rho^{p^{2n}} - \rho^{p^n}), \qquad k^{p^{n+1}} = (\rho - \rho^{p^{2n}})/(\rho^{p^{2n}} - \rho^{p^n}),$$

$$(\rho^{p^n} - \rho)^{p^{2n} - p^n} = (\rho^{p^{3n}} - \rho^{p^{2n}})/(\rho^{p^{2n}} - \rho^{p^n}) = k^{p^{n+1}},$$

$$Q_{21}^{p^{n+1}} = L_2^{p^{2n} - p^n}. \tag{41}$$

Since $p_2 = p^{3n} - p^n$ by (39), we conclude that *

$$P_2 = Q_{21}^{p^{n+1}} - L_2^{p^{2n} - p^n}. \tag{42}$$

The desired invariantive relation $C_3 = 0$ is obtained by the elimination of Q_{21} and L_2 from (41), (2_2) and (2_3). We have

$$Q_{31} = Q_{21} Q_{32}, \qquad L_2^{p^{2n} - p^n} = Q_{21}^{p^{n+1}} = (Q_{31}/Q_{32})^{p^{n+1}},$$

$$L_3^{p^{2n} - p^n} \left(\frac{Q_{32}}{Q_{31}} \right)^{p^{n+1}} = \left(\frac{L_3}{L_2} \right)^{p^{2n} - p^n} = (Q_{32} - Q_{21}^{p^n})^{p^n} = Q_{32}^{p^n} - \left(\frac{Q_{31}}{Q_{32}} \right)^{p^{2n}},$$

$$C_3 = L_3^{p^{2n} - p^n} Q_{32}^e - (Q_{31} Q_{32}^{p^n})^{p^{n+1}} + Q_{31}^e \qquad (e = p^{2n} + p^n + 1). \tag{43}$$

We proceed to express the invariant P_3 in terms of the fundamental invariants. With the notation of § 1, we obtain from (36),

$$[623][310] - [631][230] + [630][231] = 0,$$

$$[532][210] - [521][320] + [520][321] = 0,$$

$$[431][120] - [412][310] + [410][312] = 0.$$

We raise the second to the power p^{2n}, the third to the power p^{2n}, and get

$$-[410]^{p^{2n}}[310] + [520]^{p^n}[320] - [630][210]^{p^n} = 0,$$

$$[310]^{p^{3n}}[210]^{p^n} - [410]^{p^{2n}}[320]^{p^n} + [520]^{p^n}[210]^{p^{2n}} = 0,$$

$$-[320]^{p^{3n}}[210]^{p^{2n}} + [310]^{p^{3n}}[310]^{p^{2n}} - [410]^{p^{2n}}[210]^{p^{3n}} = 0.$$

We eliminate $[520]^{p^n}$ and $[410]^{p^{2n}}$ linearly, divide by a power of $[210]$, and get

$$P_3 = [Q_{32}^{p^{3n} + p^{2n}} - Q_{31}^{p^{2n}}][Q_{31}^{p^{n+1}} - Q_{32} L_3^{p^{2n} - p^n}] - Q_{31} Q_{32}^{p^{2n}} L_3^{p^{3n} - p^{2n}}. \tag{44}$$

*Since Q_{21} is an absolute invariant of G_2 and L_2^r, $r = p^n - 1$, is the least power of L_2 giving an absolute invariant, any invariant factor of (42) must be of degree a multiple of $r(p^n + 1)$ and $p^{2n} - p^n$ and hence a multiple of $rp^n(p^n + 1)$. Thus P_2 is an irreducible invariant. Hence any irreducible binary cubic is equivalent within G_2 to a multiple of any other.

We readily verify the identity

$$Q_{32}^{p^{2n}+p^n} P_3 + Q_{31} C_3^{p^n} - Q_{31}^{p^{2n}} C_3 + Q_{32}^{p^{3n}+p^{2n}} C_3 \equiv 0. \tag{45}$$

Hence the product of all non-vanishing ternary cubics is

$$K_3 = -Q_{32}^{p^{3n}-p^n} + (Q_{31}^{p^{2n}} - Q_{31} C_3^{p^n-1})/Q_{32}^{p^{2n}+p^n}. \tag{46}$$

The last expression equals an integral function. By (43)

$$K_3 = -Q_{32}^{p^{3n}-p^n} - \sum_{i=1}^{p^n-1} (Q_{31}^{p^n+1} - Q_{32} L_3^{p^{2n}-p^n})^i Q_{32}^{(p^{2n}+p^n)(i-1)} Q_{31}^{p^{3n}-ei}. \tag{46'}$$

Invariants relating to Quadratic Forms.

20. We next determine the invariantive expression for the product Q of all distinct ternary quadratic forms of non-vanishing discriminant* in the $GF[p^n]$. An integral function has the factor $y^2 - xz$ if and only if it vanishes identically in x and t when we set

$$y = tx, \qquad z = t^2 x.$$

For these values,

$$L_2 = x^{p^n+1}(t - t^{p^n}), \qquad Q_{21} = x^{rp^n}(t - t^{p^{2n}})/(t - t^{p^n}), \qquad r \equiv p^n - 1,$$

$$L_3 = x^e (t - t^{p^n})(t - t^{p^{2n}})(t^{p^n} - t^{p^{2n}}), \qquad e \equiv p^{2n} + p^n + 1.$$

By eliminating t, we get the two relations†

$$x^e L_3 = Q_{21} L_2^{p^n+2}, \qquad x^{p^{2n}} - x^{p^n} Q_{21} + x L_2^r = 0. \tag{47}$$

By means of the rth power of the former, we may express $x^{p^{2n}}$ as a multiple of x. Hence (47_2), its p^n-th power, and its p^{2n}-th power yield three linear relations between $x^{p^{2n}}$, x^{p^n}, x, the determinant of whose coefficients is a power of L_2 times

$$\begin{gathered} A Q_{21}^{p^{2n}-1} L_2^{rp^n} + B L_2^{r(p^{2n}-1)} + Q_{21}^{p^{2n}} L_3^r L_2^{p^{3n}-2p^n+1} - Q_{21}^{p^{2n}+2p^n} L_3^{rp^n} = 0, \\ A = L_3^{rp^n} Q_{21}^{p^n} + L_2^{p^{3n}-p^n}, \qquad B = L_3^{p^{2n}-1} + Q_{21}^{p^n} L_3^{rp^n} L_2^r. \end{gathered} \tag{48}$$

We eliminate Q_{21} from A by means of (2_3), from B by means of (2_2):

$$A = Q_{31}^{p^n} L_2^{rp^n}, \qquad B = Q_{32} L_3^{rp^n} L_2^r.$$

Similarly, the last two terms of (48) equal

$$\begin{aligned} Q_{21}^{p^{2n}} \{ &(L_3^r/L_2^r - Q_{32})(Q_{31}^{p^n} L_2^{rp^n} - L_2^{p^{3n}-p^n}) + L_3^r L_2^{p^{3n}-2p^n+1} \} \\ &= Q_{21}^{p^{2n}} \{ L_3^r Q_{31}^{p^n} L_2^{r^2} + L_2^{rp^n} C \}, \qquad C \equiv Q_{32} L_2^{rp^{2n}} - Q_{32} Q_{31}^{p^n}. \end{aligned}$$

In view of (2_3) the first term of the latter equals

$$Q_{21}^{p^{2n}-1}(Q_{31}^{p^n+1} L_2^{rp^n} - Q_{31}^{p^n} L_2^{2rp^n}).$$

The last term cancels the first term of (48). By (2_3),

$$C = -Q_{32} Q_{21}^{p^n} L_3^{rp^n}/L_2^{rp^n}.$$

*Semi-discriminant S_3 if $p = 2$, these Transactions, vol. 10 (1909), p. 134.

†The product of the second by L_2 is an identity in x, y, in view of (3).

Hence the new form of (48) is

$$L_2^{rp^{2n}} Q_{32} L_3^{rp^n} + Q_{21}^{p^{2n}-1} Q_{31}^{p^n+1} L_2^{rp^n} - Q_{32} Q_{21}^{p^{2n}+p^n} L_3^{rp^n} = 0.$$

The first and third terms equal $Q_{32} L_3^{rp^n} E^{p^n}$, where

$$E \equiv L_2^{rp^n} - Q_{21}^{p^n+1} = Q_{31} + Q_{32} L_2^{p^{2n}-1} / L_3^r - Q_{31} Q_{32} L_2^r / L_3^r,$$

in view of the product of $Q_{21}^{p^n}$ from (2_2) by Q_{21} from (2_3). Hence

$$(49)\quad Q_{21}^{p^{2n}-1} Q_{31}^{p^n+1} L_2^{rp^n} + L_3^{rp^n} Q_{31}^{p^n} Q_{32} + Q_{32}^{p^n+1} L_2^{p^{3n}-p^n} - Q_{31}^{p^n} Q_{32}^{p^n+1} L_2^{rp^n} = 0.$$

By eliminating Q_{21} between (2_2) and (2_3), we get

$$(50)\qquad L_2^{p^{3n}} - L_2^{p^{2n}} Q_{31}^{p^n} + L_2^{1} L_3^{rp^n} Q_{32} - L_2 L_3^{p^{2n}-1} = 0.$$

Multiply (49) by $L_2^{p^n}$ and eliminate $L_2^{p^{2n}}$ by (50). Then multiply by Q_{21}/L_2^r, replace Q_{21} by its value from (2_3), and $Q_{21}^{p^{2n}}$ by the p^n-th power of the value of $Q_{21}^{p^n}$ from (2_3). We get

$$(51)\quad \begin{aligned}(Q_{31} - L_2^{rp^n})(Q_{31}^{p^n} Q_{32} L_2^{p^n} L_3^{rp^n} - L_2^{p^n} L_3^{rp^n} Q_{32}^{p^n+2} + L_2 L_3^{p^{2n}-1} Q_{32}^{p^n+1}) \\ + L_3^r L_2^{p^{2n}-p^n+1} Q_{31}^{p^n+1} Q_{32}^{p^n} - L_2 L_3^{p^{2n}-1} Q_{31}^{p^n+1} = 0.\end{aligned}$$

Multiply (50) by $Q_{31} Q_{32}^{p^n+1} - Q_{31}^{p^n+1}$, add the result to (51); then divide by $L_2^{rp^n}$. We get

$$(52)\quad \begin{aligned}L_2^{p^{3n}-p^{2n}+p^n}(Q_{31} Q_{32}^{p^n+1} - Q_{31}^{p^n+1}) + L_2^{p^n}[L_3^{rp^n} Q_{32}^{p^n+2} - L_3^{rp^n} Q_{31}^{p^n} Q_{32} \\ - (Q_{31} Q_{32})^{p^n+1} + Q_{31}^{2p^n+1}] + L_2(L_3^r Q_{31}^{p^n+1} Q_{32}^{p^n} - L_3^{p^{2n}-1} Q_{32}^{p^n+1}) = 0.\end{aligned}$$

From the p^n-th power of (51) we eliminate $L_2^{p^{3n}}$ by (50) and obtain an equation involving the same three powers of L_2 as in (52). Eliminating the highest power of L_2 and dividing the resulting relation by $L_3^{rp^n}$, we obtain $FG = 0$, where

$$(53)\qquad F = L_2^{p^n}(Q_{31}^{p^n} - Q_{32}^{p^n+1}) + L_2 L_3^{p^n-1} Q_{32}^{p^n},$$

$$(54)\quad \begin{aligned}G = L_3^{p^{3n}-p^n} Q_{32}^e + L_3^{p^{3n}-p^{2n}}(Q_{31}^{p^{2n}+1} Q_{32}^{p^n+1} - Q_{31} Q_{32}^{e+p^n} - Q_{31}^e) \\ - L_3^{p^{2n}-p^n} Q_{31}^{p^{2n}+p^n} Q_{32}^{p^{2n}+1} + Q_{31}^{e+p^n} Q_{32}^{p^{2n}}\end{aligned}$$

The factor F is extraneous, since it does not vanish for $x \neq 0$, t in the $GF[p^{2n}]$ but not in the $GF[p^n]$. Indeed, we then have

$$L_2 \neq 0,\ Q_{21} = 0,\ L_3 = 0,\ Q_{32} = 0,\ Q_{31} = L_2^{p^{2n}-p^n},\ F = L_2^{p^{3n}-p^{2n}+p^n}.$$

Hence the desired invariant Q is a factor of G. Now

$$(55)\quad G = J_3(Q_{31}^{p^{2n}+p^n} Q_{32}^{p^{2n}+1} - L_3^{p^{3n}-p^{2n}} Q_{32}^e) + J_3^{p^n}(Q_{31}^e - Q_{31}^{p^{2n}+1} Q_{32}^{p^n+1}),$$

where J_3 is defined by (35). *The invariant Q equals G/J_3.* This follows from the facts that any ternary quadratic form T of non-vanishing discriminant (semi-discriminant, if $p = 2$) is equivalent (*Linear Groups*, p. 158, p. 197)

under the total group G_3 to $c(y^2 - xz)$, and that the number of the forms T, no two with a constant ratio, is $N = p^{2n}(p^{3n} - 1)$, while the degree of Q is $2N$.

We may, however, give a direct proof that $Q = G/J_3$ and deduce the preceding facts as corollaries. This proof depends upon the fact that G/J_3 is not the product of two integral invariants of G_3. Since G/J_3 contains a term free of L_3, a factor f must contain such a term and hence be an absolute invariant of G_3. Hence the exponent of L_3 in each term of f is a multiple of $p^n - 1$. But the degrees of Q_{31} and Q_{32} are multiples of p^n. Hence the exponents of L_3 are multiples of $p^n(p^n - 1)$. Thus f is an integral function of J_3. By (35) and (55), we get

$$(55') \quad G/J_3 = Q_{31}^{p^n} Q_{32}^{p^{2n}+1} D^{p^n} + Q_{32}^{e} J_3^{p^n} + J_3^{p^n-1} Q_{31}^{p^{2n}+1} D, \qquad D \equiv Q_{31}^{p^n} - Q_{32}^{p^n+1}.$$

If this were reducible in J_3, then

$$(56) \qquad \sigma^{p^n} + \sigma Q_{31}^{p^{2n}} / Q_{32}^{p^{2n}+1} + Q_{32}^{p^n}$$

would be reducible in $\sigma = Q_{31} D/J_3$. But values of x, y, z may be found for which Q_{31} and Q_{32} take any assigned values (§ 8). Since the extraction of the p^n-th root is here uniquely possible, the coefficients of (56) may be given any assigned values. By § 13, values may be assigned such that (56) is the product of a linear and an irreducible factor of degree $p^n - 1$. Hence (55′) is either irreducible in its arguments or has a factor f_1 linear in J_3. Suppose the latter to be the case. The coefficient of J_3 is either 1 or Q_{32}^{e} in view of the part Q_{31}^{e} of the final term of (55′). If this coefficient is 1, the remaining terms of f_1 are of the same degree as J_3, so that

$$f_1 = J_3 - c Q_{31} Q_{32}^{p^n}.$$

But this is obviously not a factor of (55′). Next, for the factors

$$f_1 = Q_{32}^{e} J_3 + \lambda, \qquad J_3^{p^n-1} - c Q_{31} Q_{32}^{p^n} J_3^{p^n-2} + \cdots,$$

a comparison of the coefficient of $J_3^{p^n-1}$ of the product with that in (55′) gives

$$\lambda = c Q_{31} Q_{32}^{e+p^n} + Q_{31}^{p^{2n}+1} D.$$

Since λ contains the term Q_{31}^{e}, while e exceeds the exponent of Q_{31} in each part of the first term of (55′), f_1 is not a factor. Hence G/J_3 is not the product of two integral invariants of G_3 and thus equals Q.

It may happen that Q is the product of integral invariants of the subgroup G_3' of transformations of determinant unity, namely, that Q is a reducible in the arguments L_3, Q_{31}, Q_{32}. Since Q has a term involving only Q_{31} and Q_{32}, whose degrees are $\rho(p^n + 1)$ and ρp^n, where $\rho = p^n(p^n - 1)$, any factor f of Q is of degree a multiple of ρ. Thus the exponent a of L_3 in any term of f is such that ae is a multiple of ρ. The greatest common divisor of $e = p^{2n} + p^n + 1$ and ρ is 3 or 1 according as p^n is of the form $3l + 1$ or not. If $p^n \neq 3l + 1$,

the exponent a of L_3 is a multiple of ρ, and f is a function of J_3; but Q was shown to be irreducible in J_3. Hence *if* $p^n \neq 3l+1$, *any ternary quadratic form of non-vanishing discriminant is equivalent under* G_3' *to* $c(y^2 - xz)$.

For $p^n = 3l+1$, Q is the product of three integral functions of $L_3^{\rho/3}$. This is in agreement with the fact that the types are now

$$c(y^2 - kxz) \qquad (k = 1, \epsilon, \epsilon^2),$$

where ϵ is a fixed not-cube (for example, a primitive root), while no one of the three types is equivalent under G_3' to a constant multiple of another type.

21. Theorem. *If* α *is a primitive root of the* $GF[p^n]$, $p^n > 2$, *and* β *is any element, the function*

$$\Sigma = \sigma^{p^n} - \alpha\sigma + \beta$$

is the product of a linear and an irreducible function of degree $p^n - 1$.

Let ξ and η be two distinct roots of $\Sigma = 0$. Let $z = \xi - \eta$. Then

$$z^{p^n} = \alpha z, \qquad z^{p^{2n}} = \alpha z^{p^n} = \alpha^2 z, \quad \cdots, \qquad z^{p^{kn}} = \alpha^k z.$$

Since α belongs to the exponent $p^n - 1$, z belongs to the $GF[p^{kn}]$ if $k = p^n - 1$, but not if k is smaller.

Suppose that Σ has a factor f of degree $d (d < p^n)$, irreducible in the $GF[p^n]$. The roots of $f = 0$ are ξ, ξ^{p^n}, $\xi^{p^{2n}}$, $\cdots$ and belong to the $GF[p^{dn}]$. The difference z of two of these roots is not zero and belongs to the latter field. Hence by the earlier result, $d = p^n - 1$.

For $p^n > 2$, we have $\alpha \neq 1$. But Σ vanishes if $\sigma = \beta/(\alpha - 1)$, an element of the $GF[p^n]$. Hence there is a linear factor.

For $p^n = 2$, the theorem holds for $\beta = 0$, but fails if $\beta = 1$.

22. We have determined the product Q of the distinct ternary quadratic forms not equivalent to a binary form, and the product J_3 of those equivalent to an irreducible binary form. A quadratic form equivalent to a reducible binary form is the product of two distinct linear forms; hence the product of all such ternary forms is $L_3^{p^{2n}+p^n}$. Finally, the product of the distinct quadratic forms equivalent to a unary form is L_3^2. Since $QJ_3 = G$, we conclude that the product of all distinct ternary quadratic forms is $GL_3^{p^{2n}+p^{n+2}}$. The degree of the latter product is $2(p^{6n}-1)/(p^n-1)$, as should be the case. The invariant G, given by (54), may be expressed as the following determinant:

$$(57) \qquad G = \begin{vmatrix} L_3^{p^{3n}-p^{2n}} & Q_{31} & 1 \\ L_3^{p^{3n}-p^{2n}} Q_{32}^{p^n+1} & L_3^{p^{2n}-p^n} Q_{32} & Q_{31}^{p^n} \\ Q_{31}^{p^{2n}+p^n} Q_{32}^{p^{2n}} & Q_{31}^{p^{2n}+1} & Q_{32}^{p^{2n}+p^n} \end{vmatrix}.$$

THE UNIVERSITY OF CHICAGO,
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LINEAR DIFFERENCE EQUATIONS AND THEIR ANALYTIC SOLUTIONS*

BY

R. D. CARMICHAEL

Introduction.

The first existence theorem for difference equations in which analytic solutions were treated was obtained by GUICHARD † who, in a paper published in 1887, proved that if $\phi(x)$ is any entire function whatever there exists another entire function $f(x)$ satisfying the relation

$$f(x+1)-f(x)=\phi(x).$$

Demonstrations of this theorem have also been given by APPELL ‡ and HURWITZ. § HURWITZ further showed how to solve the equation

$$\phi(x)f(x+1)-\chi(x)f(x)=\psi(x),$$

where $\phi(x)$, $\chi(x)$, $\psi(x)$ are meromorphic functions.

MELLIN || has considered the nature of the solution of

$$f(x+1)-r(x)f(x)=s(x),$$

where $r(x)$ and $s(x)$ are rational functions.

BARNES ¶ has contributed an interesting study of the equation

$$\phi(x)f(x+\omega)-\chi(x)f(x)=\psi(x),$$

where ω is any constant different from zero and $\phi(x)$, $\chi(x)$, $\psi(x)$ are entire functions.

WATSON ** has considered the equation of the second order

$$A(x)f(x+1)-B(x)f(x)+C(x)f(x-1)=0,$$

* Presented to the Society September 6, 1910.

† Annales de l'École Normale supérieure, ser. 3, vol. 4 (1887), p. 361.

‡ Journal de Mathématiques, ser. 4, vol. 7 (1891), p. 157, especially chapter I, pp. 159–176.

§ Acta Mathematica, vol. 20 (1897), p. 285.

|| Acta Mathematica, vol. 15 (1891), p. 317.

¶ Proceedings of the London Mathematical Society, ser. 2, vol. 2 (1905), p. 438.

** Proceedings of the London Mathematical Society, ser. 2, vol. 8 (1910), p. 125.

with some restrictions on $A(x)$, $B(x)$, $C(x)$, the object of his investigation being to obtain a solution which is of use in applications to the theory of linear differential equations. Two analytic solutions are obtained, but the question of their independence is left unanswered.

Finally, GALBRUN* has employed the Laplace transformation which carries a linear difference equation with polynomial coefficients into a differential equation, and by means of this transformation has obtained important existence theorems for the case in which the difference equation has polynomial coefficients.

The primary object of the present paper is to prove that there exists a fundamental system of simple analytic solutions of the system of n linear homogeneous difference equations

$$(A) \qquad G_i(x+1) = x^{\alpha} \sum_{j=1}^{n} a_{ij}(x) G_j(x) \qquad (i=1, \cdots, n),$$

the known quantities entering into the equations being defined as follows:

1) α is a constant;

2) the functions $a_{ij}(x)$ are single-valued and

$$a_{ij}(x) = a_{ij} + a'_{ij}x^{-1} + a''_{ij}x^{-2} \cdots \qquad (i, j = 1, \cdots, n) \qquad |x| > R';$$

3) the constants a_{ij} are such that the roots of the equation

$$|a_{ij} - \delta_{ij}\rho| = 0, \qquad \delta_{ij} = \begin{cases} 0 \text{ if } i \neq j, \\ 1 \text{ if } i = j, \end{cases}$$

are in absolute value different from each other and from zero.

In § 1 a form of the method of successive approximation is developed in connection with a single equation of the first order and by means of it a study is made of the nature of a simple particular solution of the equation.

The methods of § 1 are readily extended in § 2 in such a way that they lead to a similar solution of the normal system

$$(B) \qquad f_i(x+1) - a_i\left(1 + \frac{1}{x}\right)^{m_i} f_i(x) = \sum_{j=1}^{n} \phi_{ij}(x) f_j(x) \qquad (i=1, \cdots, n),$$

where

$$0 < |a_1| < |a_2| < \cdots < |a_n|$$

and

$$\phi_{ij}(x) = \phi''_{ij}x^{-2} + \phi'''_{ij}x^{-3} + \cdots \qquad (i, j = 1, \cdots, n) \qquad |x| \geqq R > 1.$$

A method of induction in § 3 leads to the determination of a fundamental system of solutions of (B) and to the development of important properties of the functions of which it is composed.

In § 4 it is shown that system (A) can be reduced to the normal form (B)

*Comptes rendus, vol. 148 (1909), p. 905; vol. 149 (1909), p. 1046; vol. 150 (1910), p. 206; vol. 151 (1910), p. 1114.

and thus the general existence theorem for (A) is obtained. Finally, an application of this result is made in deriving the general existence theorem for a single equation of the nth order.

I desire to express my cordial thanks to Professor G. D. Birkhoff for valuable suggestions and encouragement in the preparation of the paper.

§ 1. *The method of successive approximation for a single homogeneous equation of the first order.*

We shall begin by considering a single homogeneous equation of the first order in the normal form *

$$f(x+1) - a\left(1+\frac{1}{x}\right)^m f(x) = \phi(x)f(x), \qquad a \neq 0, \tag{1}$$

or

$$f(x+1) = \psi(x)f(x), \qquad \psi(x) \equiv a\left(1+\frac{1}{x}\right)^m + \phi(x),$$

where

$$\phi(x) = \phi'' x^{-2} + \phi''' x^{-3} + \cdots, \qquad \text{for } |x| \geqq R > 1. \tag{2}$$

We assume that $\psi(x)$ is single-valued and analytic at all points of the complex plane where it is defined, and that that determination of $(1+1/x)^m$ is taken which approaches 1 as x approaches infinity. The sequence of equations

$$\begin{aligned}
&f^{(1)}(x+1) - a\left(1+\frac{1}{x}\right)^m f^{(1)}(x) = 0,\\
&f^{(2)}(x+1) - a\left(1+\frac{1}{x}\right)^m f^{(2)}(x) = \phi(x)f^{(1)}(x),\\
&f^{(3)}(x+1) - a\left(1+\frac{1}{x}\right)^m f^{(3)}(x) = \phi(x)f^{(2)}(x),\\
&\qquad\qquad \cdots\cdots\cdots\cdots\cdots,
\end{aligned} \tag{3}$$

will serve to define a sequence of approximation functions $f^{(1)}(x), f^{(2)}(x), \cdots$. A particular solution of the first equation is evidently $\alpha^x x^m$. All the other equations of the set have the general non-homogeneous form

$$g(x+1) - a\left(1+\frac{1}{x}\right)^m g(x) = \eta(x). \tag{4}$$

* It is easy to reduce the general first order equation

$$G(x+1) = x^\alpha(\theta + \theta' x^{-1} + \theta'' x^{-2} + \cdots)G(x), \qquad \theta \neq 0,$$

to this form. Put

$$G(x) = \{\Gamma(x)\}^\alpha f(x).$$

Then

$$f(x+1) = (\theta + \theta' x^{-1} + \theta'' x^{-2} + \cdots)f(x);$$

and for $|x|$ sufficiently large this evidently may be put in the form (1).

One readily obtains a formal solution of this equation as follows: Let

$$g(x) = a^x x^m \bar{g}(x),$$

where $a^x x^m$ has any determination fixed by making a cut along the negative axis of reals in the x-plane. Substituting in (4) this value of $g(x)$ and dividing by $a^{x+1}(x+1)^m$ one has

$$\bar{g}(x+1) - \bar{g}(x) = \frac{\eta(x)}{a^{x+1}(x+1)^m},$$

an equation which has the formal solution

$$\bar{g}(x) = -\frac{\eta(x)}{a^{x+1}(x+1)^m} - \frac{\eta(x+1)}{a^{x+2}(x+2)^m} - \frac{\eta(x+2)}{a^{x+3}(x+3)^m} - \cdots.$$

Hence

$$g(x) = -\frac{\eta(x)}{a\left(1+\frac{1}{x}\right)^m} - \frac{\eta(x+1)}{a^2\left(1+\frac{2}{x}\right)^m} - \frac{\eta(x+2)}{a^3\left(1+\frac{3}{x}\right)^m} - \cdots. \tag{5}$$

Now

$$a^r\left(1+\frac{r}{x}\right)^m = \lambda(x)\lambda(x+1)\cdots\lambda(x+r-1),$$

where

$$\lambda(x) = a\left(1+\frac{1}{x}\right)^m,$$

and therefore (5) may be written in the form *

$$g(x) = -\sum_{i=0}^{\infty} \frac{\eta(x+i)}{\lambda(x)\lambda(x+1)\cdots\lambda(x+i)}. \tag{6}$$

This yields not merely a formal but an actual solution for any values of x for which it converges; and if $\eta(x)$ is analytic, then (by Weierstrass's theorem) it yields an analytic solution throughout any two-dimensional continuum where it converges uniformly.

By the aid of (6) we obtain a sequence of formal solutions $f^{(i)}(x)$ satisfying (3) identically. Employing the notation

$$S_x(\chi) = -\sum_{i=0}^{\infty} \frac{\chi(x+i)}{\lambda(x)\lambda(x+1)\cdots\lambda(x+i)}$$

and using for each equation that particular formal solution which is obtained by adding the solution $a^x x^m$ of the homogeneous equation to the solution derived from (6), we have

$$f^{(1)}(x) = a^x x^m,$$

$$f^{(2)}(x) = a^x x^m + S_x(\phi a^x x^m),$$

$$f^{(3)}(x) = a^x x^m + S_x\{\phi a^x x^m + \phi S_x(\phi a^x x^m)\},$$

. . . . . .

*Cf. MELLIN, Acta Mathematica, vol. 15 (1891), p. 319.

It is easy to see that this leads to a formal solution $f_1(x)$ of (1), defined by the series

$$f_1(x) = a^x x^m + S_x(\phi a^x x^m) + S_x\{\phi S_x(\phi a^x x^m)\} + S_x[\phi S_x\{\phi S_x(\phi a^x x^m)\}] + \cdots. \tag{7}$$

That this series formally satisfies (1) we may verify by forming the difference $f_1(x+1) - a(1+1/x)^m f_1(x)$ and effecting a reduction of the resulting series by means of the general formula

$$S_{x+1}(\chi) - a\left(1+\frac{1}{x}\right)^m S_x(\chi) = \chi.$$

Consequently whenever all the series S_x in the terms of (7) and also the series (7) itself converge, this series represents a solution of (1).

In studying the nature of the formal solution (7) (and in the work of the succeeding sections also) it will be convenient to have names assigned to certain regions of the plane about which we shall have to speak often. Let R be the

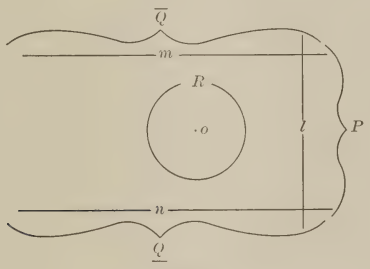

FIG. 1.

circle $|x| = R$. Let l be a line cutting the positive axis of reals at right angles and not cutting R. The part of the finite plane to the right of and including l will be called a P-region. Let m and n be parallel to the axis of reals, m lying above and n below R. The part of the finite plane lying above and including m will be named a $\bar{Q}$-region; that lying below and including n, a $\underline{Q}$-region; these two together, a Q-region. A region formed by the combination of the P- and Q-regions will be called a D-region. The part of the finite plane not included within D will be called the E-region corresponding to D. It is scarcely necessary to remark that $\bar{Q}$ and $\underline{Q}$ have each a part in common with P.

In what follows immediately we are to prove the existence of $f_1(x)$ and develop its properties in particular regions P_1, Q_1, D_1, E_1; for convenience all these regions are defined at once. For any positive constant c, less than $|a|$, there

exists an R' such that

$$\left| a\left(1+\frac{1}{x}\right)^m \right| > c, \qquad \text{for } |x| > R'.$$

From the nature of the function $\phi(x)$ it follows that there exists an M such that

$$|\phi(x)| < \frac{Mc}{|x|^2}, \qquad \text{for } |x| \geqq R.$$

If one writes

$$x = u \pm v\sqrt{-1}$$

where v is positive and both u and v are real, it is now clear that one may choose particular regions P, Q, D, E—call them P_1, Q_1, D_1, E_1 respectively—such that

$$|\phi(x)| < \frac{Mc}{|x|^2}, \qquad \left| a\left(1+\frac{1}{x}\right)^m \right| > c, \qquad \text{for } x \text{ in } D_1; \tag{8}$$

$$u > M+1, \qquad \text{for } x \text{ in } P_1; \tag{9}$$

$$\frac{A}{v} < \frac{M\pi}{v} + \frac{2M}{v^2} < 1, \qquad \text{for } x \text{ in } Q_1,\ A \text{ being a constant}; \tag{10}$$

$$a\left(1+\frac{1}{x}\right)^m + \phi(x) \neq 0, \qquad \text{for } x \text{ in } D_1. \tag{11}$$

Let us consider the function

$$\bar{S}_x(\phi a^x x^m) \equiv \sum_{i=0}^{\infty} \left| \frac{\phi(x+i)a^{x+i}(x+i)^m}{\lambda(x)\lambda(x+1)\cdots\lambda(x+i)} \right| = \sum_{i=0}^{\infty} \left| \frac{\phi(x+i)a^x x^m}{\lambda(x+i)} \right|. \tag{12}$$

Employing inequalities (8) one has the relation *

$$\sum_{i=0}^{\infty} \left| \frac{\phi(x+i)a^x x^m}{\lambda(x+i)} \right| \ll \sum_{i=0}^{\infty} \frac{M|a^x x^m|}{(u+i)^2+v^2} \tag{13}$$

subsisting throughout D_1. The series of the last member of (13) is evidently uniformly convergent when x lies in any closed region in D_1. The same then is true of the series in the first member, and hence also of the series in (12). Therefore $\bar{S}_x(\phi a^x x^m)$ is a uniformly convergent series in the same closed region and hence, in accordance with a well-known theorem of Weierstrass, represents in this region an analytic function of x. From this it follows that the first and second terms of the series in (7) represent functions which are analytic in D_1.

In order to show that the remaining terms are analytic in the same region it is necessary to know a bound to the magnitude of $|\bar{S}_x(\phi a^x x^m)|$ as x varies. In carrying out the needful reckoning it is convenient to treat the regions P_1 and Q_1 separately.

* The symbol $\ll$ is to be read *is less term by term than.*

First, let us consider the region P_1. Of the two series

$$\sum_{i=0}^{\infty} \frac{M}{(u+i)^2+v^2}, \qquad \sum_{i=0}^{\infty} \frac{M}{(u+i)^2}, \quad \text{for } x \text{ in } P_1,$$

the terms of the first in order are equal to or less than the corresponding terms of the second. But the latter series is less than the integral

$$\int_{u-1}^{\infty} \frac{M}{z^2}\, dz = \frac{M}{u-1};$$

and therefore

$$\bar{S}_x(\phi a^x x^m) < |a^x x^m| \frac{M}{u-1}. \tag{14}$$

Hence

$$|S_x(\phi a^x x^m)| < |a^x x^m| \frac{M}{u-1}.$$

Using this result and making an argument similar to that associated with inequality (13) one may show that $S_x\{\phi S_x(\phi a^x x^m)\}$ is analytic throughout P_1. Again, in view of (14) we have

$$\bar{S}_x\{\phi \bar{S}_x(\phi a^x x^m)\} < \bar{S}_x\left\{\phi |a^x x^m| \frac{M}{u-1}\right\} < \frac{M}{u-1}\bar{S}_x(\phi a^x x^m) < |a^x x^m| \frac{M^2}{(u-1)^2}.$$

Similarly it may be proved that $S_x[\phi S_x\{\phi S_x(\phi a^x x^m)\}]$ is analytic in P_1; an upper bound for the value of $\bar{S}_x[\phi \bar{S}_x\{\phi \bar{S}_x(\phi a^x x^m)\}]$ may be obtained; and so on. Hence in series (7) every term is analytic throughout P_1; moreover the following inequality has been established for x in P_1:

$$\bar{S}_x(\phi a^x x^m) + \bar{S}_x\{\phi \bar{S}_x(\phi a^x x^m)\} + \bar{S}_x[\phi \bar{S}_x\{\phi \bar{S}_x(\phi a^x x^m)\}] + \cdots \ll \frac{M|a^x x^m|}{u-1} + \frac{M^2|a^x x^m|}{(u-1)^2} + \frac{M^3|a^x x^m|}{(u-1)^3} + \cdots. \tag{15}$$

From (9) it follows that the second member of (15) is absolutely and uniformly convergent throughout any closed region in P_1; and therefore the same is true of the first member. Hence series (7) is uniformly and absolutely convergent throughout this region. But its terms are analytic functions of x; and therefore, by Weierstrass's theorem, its sum $f_1(x)$ is an analytic function of x throughout P_1. Evidently, then, $f(x) = f_1(x)$ is a solution of (1).

A comparison of (7) and (15) shows that the following inequalities are valid for x in P_1:

$$|a^x x^m|\left\{1 - \frac{M}{u-1} - \frac{M^2}{(u-1)^2} - \cdots\right\} < |f_1(x)| < |a^x x^m|\left\{1 + \frac{M}{u-1} + \frac{M^2}{(u-1)^2} + \cdots\right\}.$$

Hence we have

$$|a^x x^m|\frac{u-2M-1}{u-M-1}<|f_1(x)|<|a^x x^m|\frac{u-1}{u-M-1}, \tag{16}$$

a set of fundamental inequalities bounding the increase or decrease of $f_1(x)$ throughout P_1.

Again using relation (13) as a point of departure, we obtain similar results valid in Q_1. In this region v is different from zero. It is evident that

$$\sum_{i=0}^{\infty}\frac{M}{(u+i)^2+v^2}<\sum_{i=-\infty}^{\infty}\frac{M}{(u+i)^2+v^2}\leqq 2\left\{M\int_0^{\infty}\frac{dz}{z^2+v^2}+\frac{M}{v^2}\right\}=\frac{M\pi}{v}+\frac{2M}{v^2}<\frac{A}{v},$$

A being a properly chosen constant. Comparing this result with (12) and (13) one has

$$|S_x(\phi a^x x^m)|\leqq \bar{S}_x(\phi a^x x^m)<|a^x x^m|\frac{A}{v}. \tag{17}$$

On account of (10) and (18) it is clear that one may employ an argument similar to that associated with (13) to show that $S_x\{\phi S_x(\phi a^x x^m)\}$ is an analytic function of x in Q_1. Again, on account of (17) one has

$$\bar{S}_x\{\phi\bar{S}_x(\phi a^x x^m)\}<\bar{S}_x\left\{\phi|a^x x^m|\frac{A}{v}\right\}=\frac{A}{v}\bar{S}_x(\phi a^x x^m)<|a^x x^m|\frac{A^2}{v^2}. \tag{18}$$

In like manner it may be shown that the function $S_x[\phi S_x\{\phi S_x(\phi a^x x^m)\}]$ is analytic throughout Q_1; also, an upper bound for the value of

$$\bar{S}_x[\phi\bar{S}_x\{\phi\bar{S}_x(\phi a^x x^m)\}]$$

may be obtained; and so on. Hence every term in series (7) is analytic in Q_1. Moreover, for this region the following relation exists:

$$\bar{S}_x(\phi a^x x^m)+\bar{S}_x\{\phi\bar{S}_x(\phi a^x x^m)\}+\bar{S}_x[\phi\bar{S}_x\{\phi\bar{S}_x(\phi a^x x^m)\}]+\cdots \ll |a^x x^m|\frac{A}{v}+|a^x x^m|\frac{A^2}{v^2}+|a^x x^m|\frac{A^3}{v^3}+\cdots. \tag{19}$$

By a method similar to that employed in the previous case it is now easy to show that $f_1(x)$ is analytic in Q_1, and that throughout this region it is bounded as in the inequalities

$$|a^x x^m|\frac{v-2A}{v-A}<|f_1(x)|<|a^x x^m|\frac{v}{v-A}. \tag{20}$$

It remains to determine the nature of $f_1(x)$ in the region E_1. For this study series (7) does not furnish a convenient means. We proceed as follows: If we solve (1) for $f(x)$ and in the result write $x-1$ for x we have

$$f(x-1)=\bar{\psi}(x-1)f(x) \tag{21}$$

where

$$\bar{\psi}(x) = \{\psi(x)\}^{-1}. \tag{22}$$

By means of (21), the solution $f_1(x)$ which is known to exist in P_1 can be extended across the plane to the left; and in this way the nature of $f_1(x)$ in E_1 is determined.

Consider the set of points $[S]$ which is formed of the set $[\alpha]$ and of the sets $[\alpha - i]$, $i = 1, 2, 3, \cdots$, where $[\alpha]$ is the set of singularities of $\bar{\psi}(x)$. A point $\alpha - i$ is said to be *congruent to* α *on the left;* in a similar way *congruence on the right* is defined. It is easy to see that $f_1(x)$ is analytic at every point x' not belonging to $[S]$. For, by (21),

$$f_1(x') = \bar{\psi}(x')\bar{\psi}(x'+1)\cdots\bar{\psi}(x'+\nu-1)f_1(x'+\nu).$$

ν can be so chosen that $x'+\nu$ lies in P_1. For such a value of ν, $f_1(x)$ is analytic at $x'+\nu$; also, $\bar{\psi}(x)$ is analytic at each of the points x', $x'+1$, $\cdots$, $x'+\nu-1$. Therefore $f_1(x)$ is analytic at x'. Combining this with previous results, we see that $f_1(x)$ is analytic throughout the finite plane except at the singularities $[\alpha]$ of $\bar{\psi}(x)$ and the points congruent to them on the left; at these points its character depends upon the nature of the particular $\bar{\psi}(x)$ involved.

Let each point of $[S]$ be inclosed by a small circle of radius ϵ having the point as its center. Denote by $E_{1\epsilon}$ the whole of that part of E_1 no point of which is within one of these small circles.

Inequalities bounding the increase or decrease of $f_1(x)$ for x in $E_{1\epsilon}$ may be found as follows: Consider any positive constants a_- and a_+ satisfying the relations

$$a_- < |a| < a_+.$$

Since

$$\lim_{x=\infty}\left\{a\left(1+\frac{1}{x}\right)^m + \phi(x)\right\} = a,$$

it is clear that r may be chosen so that

$$a_+ > \left|a\left(1+\frac{1}{x}\right)^m + \phi(x)\right| > a_-, \quad \text{for } |x| > r,$$

and also so that the circle $|x| = r$ cuts the boundary of E_1 twice on the left of the origin and at no other point. Since $f_1(x)$ is everywhere analytic in $E_{1\epsilon}$ it follows that there exist constants $\bar{N}_1$ and $\bar{N}_2$ such that

$$\bar{N}_1 \leqq |f_1(x'')| < \bar{N}_2, \qquad \text{for } x'' \text{ in } E_{1\epsilon} \text{ and } r \leqq |x''| \leqq r+1.$$

But (21) yields the equation

$$f_1(x''-\mu) = \bar{\psi}(x''-\mu)\bar{\psi}(x''-\mu+1)\cdots\psi(x''-1)f_1(x'').$$

Hence we have

$$\overline{N}_1 a_+^{-\mu} \leqq |f(x''-\mu)| < \overline{N}_2 a_-^{-\mu}, \qquad \text{for } x'' \text{ in } E_{1\epsilon} \text{ and } r \leqq |x''| \leqq r+1.$$

As μ varies over the range $\mu = 0, 1, 2, \cdots$, $x''-\mu$ runs over all points in $E_{1\epsilon}$ and outside of the circle $|x| = r$. Also, $f_1(x)$ is finite in the part of $E_{1\epsilon}$ which is in the circle $|x| = r$. Hence it is clear that there exist constants N_1 and N_2 such that

$$N_1 |a_+^x| \leqq |f_1(x)| < N_2 |a_-^x|, \quad \text{for } x \text{ in } E_{1\epsilon}, \tag{23}$$

the result sought.

So far we have said nothing of the position of the zeros of $f_1(x)$. By means of equation (1) we shall show that they can occur only at the singularities of $\psi(x)$ and at points congruent to them on the left. We have

$$f_1(x'+\mu) = \psi(x'+\mu-1)\psi(x'+\mu-2)\cdots\psi(x')f_1(x'),$$

where μ is a positive integer. From inequality (16) it follows that for every x' there exists a μ such that $f_1(x'+\mu)$ is different from zero. Hence $f_1(x')$ can be zero only when one at least of the factors $\psi(x'), \psi(x'+1), \cdots, \psi(x'+\mu-1)$ is infinite: and therefore the zeros of $f_1(x)$ must all be contained in the set of points composed of the singularities of $\psi(x)$ and the points congruent to them on the left.

The principal results contained in the section may be enunciated as follows:

The difference equation

$$f(x+1) = \psi(x)f(x) \text{ or } f(x-1) = \bar{\psi}(x-1)f(x), \qquad \bar{\psi}(x) \equiv \{\psi(x)\}^{-1},$$

in which $\psi(x)$ is single-valued and

$$\psi(x) = a\left(1+\frac{1}{x}\right)^m + \phi(x), \qquad \phi(x) = \phi'' x^{-2} + \phi''' x^{-3} + \cdots, \text{ for } |x| \geqq R > 1,$$

has a particular solution $f_1(x)$ which is analytic throughout the finite plane except at the singularities $[\alpha]$ of $\bar{\psi}(x)$ and the points congruent to them on the left, and which can vanish only at the singularities $[\beta]$ of $\psi(x)$ and at points congruent to them on the left. If D_1 is defined by relations (8) to (11), then throughout any closed region in D_1 this solution is representable as a uniformly converging series in the form

$$f_1(x) = a^x x^m + S_x(\phi a^x x^m) + S_x\{\phi S_x(\phi a^x x^m)\} + S_x[\phi S_x\{\phi S_x(\phi a^x x^m)\}] + \cdots.$$

The increase or decrease of $f_1(x)$ for x in the corresponding regions P_1 and Q_1 is bounded as in the inequalities

$$|a^x x^m| \frac{u-2M-1}{u-M-1} < |f_1(x)| < |a^x x^m| \frac{u-1}{u-M-1}, \quad \text{for } x \text{ in } P_1;$$

$$|a^x x^m| \frac{v-2A}{v-A} < |f_1(x)| < |a^x x^m| \frac{v}{v-A}, \quad \text{for } x \text{ in } Q_1.$$

Also, for any positive constants a_- and a_+ satisfying the relations

$$a_- < |a| < a_+$$

there exist other constants N_1, N_2 such that

$$N_1|a_+^x| \leqq |f_1(x)| < N_2|a_-^x|, \quad \text{for } x \text{ in } E_{1\epsilon}.$$

REMARK. About each statement in the conclusion of this theorem it should be observed that, insofar as the statement refers to the region D_1 or any part of it, it is independent of the nature of $\phi(x)$ in the region exterior to D_1. More explicitly: *If $\phi(x)$ is any function which is analytic in some D-region and if also*

$$|\phi(x)| < \frac{Mc}{|x|^2}, \quad \text{for } x \text{ in } D,$$

then there exist regions P_1, Q_1, D_1 having as before the properties (8) *to* (10), *and there is a particular solution $f_1(x)$ of the difference equation having in D_1 the same properties as the solution $f_1(x)$ of the equation in the theorem.* These conclusions follow from the preceding discussion without change of argument.

§ 2. *The method of successive approximation for a normal system of n equations. Properties of the solution thus obtained.*

Let us consider the system of n linear difference equations of the first order in the normal form

$$f_i(x+1) - a_i\left(1+\frac{1}{x}\right)^{m_i} f_i(x) = \sum_{j=1}^{n} \phi_{ij}(x) f_j(x) \qquad (i=1, \cdots, n), \tag{24}$$

or

$$f_i(x+1) = \sum_{j=1}^{n} \psi_{ij}(x) f_j(x), \qquad \psi_{ij}(x) \equiv \delta_{ij} a_i\left(1+\frac{1}{x}\right)^{m_i} + \phi_{ij}(x),$$

where

$$0 < |a_1| < |a_2| < \cdots < |a_n| \tag{25}$$

and

$$\phi_{ij}(x) = \phi_{ij}'' x^{-2} + \phi_{ij}''' x^{-3} + \cdots \qquad (i, j=1, \cdots, n), \quad \text{for} \quad |x| \geqq R > 1. \tag{26}$$

We assume that $\psi_{ij}(x)$ is single-valued. If we solve (24) for $f_i(x)$ in terms of $f_i(x+1)$, $i=1, \cdots, n$, and in the result write $x-1$ for x, we obtain

$$f_i(x-1) = \sum_{j=1}^{n} \bar{\psi}_{ij}(x-1) f_j(x) \qquad (i=1, \cdots, n), \tag{27}$$

$\bar{\psi}_{ij}(x)$ having the form

$$\bar{\psi}_{ij}(x) = \frac{\alpha_{ij}(x)}{\Delta} \tag{28}$$

where Δ is the determinant

$$\Delta \equiv \left| \delta_{ij} a_i\left(1+\frac{1}{x}\right)^{m_i} + \phi_{ij}(x) \right|$$

and $\alpha_{ij}(x)$ is the cofactor of the element in the jth column and the ith row of Δ.

The object of this section is to obtain a particular solution of (24) and to study its properties throughout the plane.

On account of (25) and (26) it is clear that there exist particular regions P_1, Q_1, D_1, E_1 and constants c, M and A such that the following inequalities are true:

$$(30)\quad |\phi_{ij}(x)| < \frac{Mc}{n|x|^2}, \qquad \left|a_i\left(1+\frac{1}{x}\right)^{m_i}\right| > c \qquad (i,j=1,\cdots,n), \quad \text{for } x \text{ in } D_1;$$

$$(31)\quad \left|a_1\left(1+\frac{1}{x}\right)^{m_1}\right| < \left|a_i\left(1+\frac{1}{x}\right)^{m_i}\right| \qquad (i=2,\cdots,n), \quad \text{for } x \text{ in } D_1;$$

$$(32)\quad u > M+1, \quad \text{for } x \text{ in } P_1, \quad \text{where } x = u \pm v\sqrt{-1};$$

$$(33)\quad \frac{A}{v} < \frac{M\pi}{v} + \frac{2M}{v^2} < 1, \quad \text{for } x \text{ in } Q_1;$$

$$(34)\quad \Delta \neq 0, \quad \text{for } |x| \geqq \bar{R} \geqq R.$$

In the study of our system of equations these regions are treated separately.

It will be convenient to employ the notation

$$S_{xi}(\chi) \equiv -\sum_{\nu=0}^{\infty} \frac{\chi(x+\nu)}{\lambda_i(x)\lambda_i(x+1)\cdots\lambda_i(x+\nu)}, \qquad \lambda_i(x) \equiv a_i\left(1+\frac{1}{x}\right)^{m_i};$$

also we write

$$u_i \equiv C_i a_i^x x^{m_i},$$

where C_i is any periodic function of x of period 1 and analytic throughout D_1.

The form of the method of successive approximation employed in the preceding section is readily generalized so as to lead to a particular solution of (24). Consider the set of systems each of n linear equations, all except the first being non-homogeneous:

$$f_i^{(1)}(x+1) - a_i\left(1+\frac{1}{x}\right)^{m_i} f_i^{(1)}(x) = 0,$$

$$f_i^{(2)}(x+1) - a_i\left(1+\frac{1}{x}\right)^{m_i} f_i^{(2)}(x) = \sum_{j=1}^{n} \phi_{ij}(x) f_j^{(1)}(x), \qquad (i=1,\cdots,n).$$

$$f_i^{(3)}(x+1) - a_i\left(1+\frac{1}{x}\right)^{m_i} f_i^{(3)}(x) = \sum_{j=1}^{n} \phi_{ij}(x) f_j^{(2)}(x),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

Proceeding as in the foregoing section, one writes out in order formal solutions

of these systems, obtaining thus the following set of approximation functions:

$$f_i^{(1)}(x) = u_i.$$

$$f_i^{(2)}(x) = u_i + S_{xi}\left(\sum_{j=1}^{n}\phi_{ij}u_j\right),$$

$$f_i^{(3)}(x) = u_i + S_{xi}\left(\sum_{j=1}^{n}\phi_{ij}u_j\right) + S_{xi}\left\{\sum_{j=1}^{n}\phi_{ij}S_{xj}\left(\sum_{k=1}^{n}\phi_{jk}u_k\right)\right\}, \qquad (i=1,\cdots,n).$$

. . . . . .

It is easy to verify by actual substitution that a solution of system (24) is obtained in the limit functions (if they exist) of the sequences $f_i^{(1)}, f_i^{(2)}, f_i^{(3)}, \cdots, i=1,\cdots,n$; that is, in

$$f_i(x) = u_i + S_{xi}\left(\sum_{j=1}^{n}\phi_{ij}u_j\right) + S_{xi}\left\{\sum_{j=1}^{n}\phi_{ij}S_{xj}\left(\sum_{k=1}^{n}\phi_{jk}u_k\right)\right\} + S_{xi}\left[\sum_{j=1}^{n}\phi_{ij}S_{xj}\left\{\sum_{k=1}^{n}\phi_{jk}S_{xk}\left(\sum_{l=1}^{n}\phi_{kl}u_l\right)\right\}\right] + \cdots \qquad (i=1,\cdots,n). \tag{35}$$

This solution is in general illusory on account of the divergence of certain series contained in the terms of the expansions formally representing the f's. In fact, let a_r and a_s be any two a's such that $|a_r| > |a_s|$. Put

$$C_i = 0, \ i \neq r; \qquad C_r = 1.$$

For $i = s$ the second term in the series in (35) becomes

$$S_{xs}(\phi_{sr}a_r^x x^{m_r}) \equiv -\sum_{\nu=0}^{\infty}\frac{\phi_{sr}(x+\nu)\,a_r^{x+\nu}(x+\nu)^{m_r}}{\lambda_s(x)\cdots\lambda_s(x+\nu)} = -\sum_{\nu=0}^{\infty}\frac{\phi_{sr}(x+\nu)\,a_r^{x+\nu}(x+\nu)^{m_r}}{a_s^{x+\nu}(x+\nu)^{m_s}}, \tag{36}$$

a series which is evidently divergent at least for some values of $\phi_{sr}(x)$ as for instance $\phi_{sr}(x) \equiv 1/x^2$.

On the other hand, there is at least one particular convergent solution, arising from the following special choice of values of the periodic functions C_i:

$$C_i = 0, \qquad i \neq 1; \qquad C_1 = 1.$$

If we call this particular solution $f_{i1}(x)$, we have formally from (35)

$$f_{i1}(x) = \delta_{i1}a_1^x x^{m_1} + S_{xi}(\phi_{i1}a_1^x x^{m_1}) + S_{xi}\left\{\sum_{j=1}^{n}\phi_{ij}S_{xj}(\phi_{j1}a_1^x x^{m_1})\right\} + S_{xi}\left[\sum_{j=1}^{n}\phi_{ij}S_{xj}\left\{\sum_{k=1}^{n}\phi_{jk}S_{xk}(\phi_{k1}a_1^x x^{m_1})\right\}\right] + \cdots \quad (i=1,\cdots,n). \tag{37}$$

That all the infinite series contained in (37) are actually convergent in D_1 is readily shown. Consider the series

$$\delta_{i1}a_1^x x^{m_1} + \bar{S}_{xi}\left(\frac{Mc}{nx^2}a_1^x x^{m_1}\right) + \bar{S}_{xi}\left\{\frac{Mc}{nx^2}\sum_{j=1}^{n}\bar{S}_{xj}\left(\frac{Mc}{nx^2}a_1^x x^{m_1}\right)\right\} + \bar{S}_{xi}\left[\frac{Mc}{nx^2}\sum_{j=1}^{n}\bar{S}_{xj}\left\{\frac{Mc}{nx^2}\sum_{k=1}^{n}\bar{S}_{xk}\left(\frac{Mc}{nx^2}a_1^x x^{m_1}\right)\right\}\right] + \cdots, \tag{38}$$

where $\bar{S}_{xi}$ denotes what S_{xi} becomes when the terms of the series which it indicates are replaced by their absolute values. On account of the first inequality (30) it is evident that all the infinite series in (37) are convergent provided that all those in (38) are likewise convergent; for if these expansions are written out in detail, the former, term by term, is less in absolute value than the latter.

Compare (38) with the following series

$$a_1^x x^{m_1} + \bar{S}_{x1}\left(\frac{Mc}{x^2}a_1^x x^{m_1}\right) + \bar{S}_{x1}\left\{\frac{Mc}{x^2}\bar{S}_{x1}\left(\frac{Mc}{x^2}a_1^x x^{m_1}\right)\right\} + \bar{S}_{x1}\left[\frac{Mc}{x^2}\bar{S}_{x1}\left\{\frac{Mc}{x^2}\bar{S}_{x1}\left(\frac{Mc}{x^2}a_1^x x^{m_1}\right)\right\}\right] + \cdots. \tag{39}$$

On account of inequality (31) it is clear that the convergence of (39) carries with it the convergence of (38); this is at once apparent from a term by term comparison of the two series, since (39) is obtained from (38) by replacing each denominator by one at least as great. But (39) (except for its first term) is of the same form as the series in the first members of (15) and (19). Hence, just as in the case of the latter series, one proves that throughout the region D_1 the successive terms of (38) represent analytic functions and finally that the series itself represents an analytic function.

We shall now find an upper bound for the values of the $n-1$ functions $f_{i1}(x)$, $i = 2, \cdots, n$, as x varies within P_1; and, for similar variation of x, both upper and lower bounds for the values of $f_{11}(x)$. We have evidently

$$f_{i1}(x) < \bar{S}_{x1}\left(\frac{Mc}{x^2}a_1^x x^{m_1}\right) + \bar{S}_{x1}\left\{\frac{Mc}{x^2}\bar{S}_{x1}\left(\frac{Mc}{x^2}a_1^x x^{m_1}\right)\right\} + \cdots$$

$$< |a_1^x x^{m_1}|\left\{\frac{M}{u-1} + \frac{M^2}{(u-1)^2} + \cdots\right\} \qquad (i = 2, \cdots, n),$$

the latter inequality being deduced by aid of (15). Hence it is clear that

$$|f_{i1}(x)| < |a_1^x x^{m_1}|\frac{M}{u-M-1} \qquad (i = 2, \cdots, n). \tag{40}$$

Similarly, one has

$$|f_{11}(x)| < |a_1^x x^{m_1}|\left(1 + \frac{M}{u-M-1}\right) = |a_1^x x^{m_1}|\frac{u-1}{u-M-1};$$

and also,

$$|f_{11}(x)| > |a_1^x x^{m_1}| \left(1 - \frac{M}{u-M-1}\right) = |a_1^x x^{m_1}| \frac{u-2M-1}{u-M-1}.$$

Combining these two inequalities we get

$$|a_1^x x^{m_1}| \frac{u-2M-1}{u-M-1} < |f_{11}(x)| < |a_1^x x^{m_1}| \frac{u-1}{u-M-1}. \tag{41}$$

The desired results for the region P_1 are contained in inequalities (40) and (41).

Starting from (19) and working in a similar manner one obtains the following inequalities valid in Q_1:

$$|f_{i1}(x)| < |a_1^x x^{m_1}| \frac{A}{v-A} \qquad (i=2, \cdots, n); \tag{42}$$

$$|a_1^x x^{m_1}| \frac{v-2A}{v-A} < |f_{11}(x)| < |a_1^x x^{m_1}| \frac{v}{v-A}. \tag{43}$$

By means of (27) we are able to determine the nature of $f_{i1}(x)$, $i=1, \cdots, n$, in the region E_1; for we have

$$f_{i1}(x-1) = \sum_{j=1}^{n} \bar{\psi}_{ij}(x-1) f_{j1}(x). \tag{44}$$

Now the functions $f_{i1}(x)$ are analytic throughout P_1. This solution may therefore be extended across the plane to the left by means of (44); it is evident from the nature of the process that the singularities of $f_{i1}(x)$ for x in E_1 are at the singularities of $\bar{\psi}_{ij}(x)$, $i, j=1, \cdots, n$, and the points congruent to them on the left. The nature of the singularities which the functions $f_{i1}(x)$ may have depends upon the particular functions $\bar{\psi}_{ij}(x)$ entering into a given particular system of equations. In special cases it may happen that some or all of the functions have no singularities at the points indicated.

Inequalities bounding the increase or decrease of $|f_{i1}(x)|$ for x in E_1 are not as complete as those for x in P_1 and Q_1. We shall however obtain bounds for $|f_{i1}(x)|$ when x is in $E_{1\epsilon}$, $E_{1\epsilon}$ being defined as follows: Let $[S]$ be the set of points composed of all the singularities of $\bar{\psi}_{ij}(x)$, $i, j=1, \cdots, n$, and the points congruent to them on the left. Enclose each point of $[S]$ by a circle of radius ϵ having the point for center. The part of E_1 no point of which is interior to one of these circles is the region $E_{1\epsilon}$.

Referring to the definition of $\bar{\psi}_{ij}(x)$ in equation (28), it is easy to see that

$$\lim_{x=\infty} \alpha_{ij}(x) = \begin{cases} 0 \text{ if } j \neq i, \\ a_i^{-1} \prod_{s=1}^{n} a_s \text{ if } j = i, \end{cases}$$

and that

$$\underset{x=\infty}{L}\,\Delta = \prod_{s=1}^{n} a_s .$$

Hence we have

$$\underset{x=\infty}{L}\,\bar{\psi}_{ij}(x) = \begin{cases} 0 \text{ if } j \neq i, \\ a_i^{-1} \text{ if } j = i. \end{cases}$$

From this it follows that for every positive constant a such that $a < |a_1|$ there exists a positive constant r for which

$$\sum_{j=1}^{n} |\bar{\psi}_{ij}(x)| < a^{-1} \qquad (i = 1, \cdots, n) \qquad |x| \geqq r.$$

It will be assumed that r is so taken that the circle $|x| = r$ cuts the boundary of E_1 in two points on the left of the origin and in no other point.

Since each of the functions $f_{i1}(x)$ is analytic throughout $E_{1\epsilon}$ it follows that a constant $\bar{N}$ exists such that

$$|f_{i1}(x'')| < \bar{N} \text{ for } x'' \text{ in } E_{1\epsilon} \text{ and } r \leqq |x''| \leqq r + 1.$$

But from (27) we have

$$f_{i1}(x'' - \mu) = \sum_{j_1=1}^{n} \bar{\psi}_{ij_1}(x'' - \mu) \sum_{j_2=1}^{n} \bar{\psi}_{j_1 j_2}(x'' - \mu + 1) \cdots \sum_{j_\mu=1}^{n} \bar{\psi}_{j_{\mu-1} j_\mu}(x'' - 1) f_{j_\mu 1}(x'').$$

Hence we get

$$|f_{i1}(x'' - \mu)| < \bar{N} a^{-\mu}.$$

As μ varies over the range $\mu = 0, 1, 2, \cdots$, $x'' - \mu$ runs over all points in $E_{1\epsilon}$ and outside of the circle $|x| = r$. Also, $|f_{i1}(x)|$ is finite in the part of $E_{1\epsilon}$ which is in the circle $|x| = r$. Hence it is clear that there exists a constant N, such that

$$|f_{i1}(x)| < N |a^x| \qquad (i = 1, \cdots, n) \qquad x \text{ in } E_{1\epsilon}.$$

The principal results of the section may be stated as follows:

The system of difference equations

$$f_i(x+1) = \sum_{j=1}^{n} \psi_{ij}(x) f_j(x) \quad \text{or} \quad f_i(x-1) = \sum_{j=1}^{n} \bar{\psi}_{ij}(x-1) f_j(x) \quad (i = 1, \cdots, n),$$

in which $\psi_{ij}(x)$ is single-valued and

$$\psi_{ij}(x) = \delta_{ij} a_i \left(1 + \frac{1}{x}\right)^{m_i} + \phi_{ij}(x), \qquad \phi_{ij}(x) = \phi''_{ij} x^{-2} + \phi'''_{ij} x^{-3} + \cdots$$

for $|x| \geqq R > 1$,

$$\delta_{ij} = 0 \text{ when } i \neq j; \qquad \delta_{ii} = 1,$$

has a particular solution, $f_{i1}(x)$, each function of which is analytic throughout the finite plane except at the singularities $[\alpha]$ of $\bar{\psi}_{ij}(x)$ and the points con-

gruent to them on the left. If D_1 is defined by relations (30) *to* (34), *then throughout any closed region in D_1 this solution is representable as a uniformly converging series in the form*

$$f_{i1}(x) = \delta_{i1}\alpha_1^x x^{m_1} + S_{xi}(\phi_{i1}\alpha_1^x x^{m_1}) + S_{xi}\left\{\sum_{j=1}^{n}\phi_{ij}\,S_{xj}(\phi_{j1}\alpha_1^x x^{m_1})\right\}$$

$$+ S_{xi}\left[\sum_{j=1}^{n}\phi_{ij}\,S_{xj}\left\{\sum_{k=1}^{n}\phi_{jk}S_{xk}(\phi_{k1}\alpha_1^x x^{m_1})\right\}\right] + \cdots \quad (i=1,\cdots,n).$$

The increase or decrease of these functions for x in the corresponding regions P_1 and Q_1 is bounded as in the inequalities

$$|\alpha_1^x x^{m_1}|\frac{u-2M-1}{u-M-1} < |f_{11}(x)| < |\alpha_1^x x^{m_1}|\frac{u-1}{u-M-1}, \quad \textit{for } x \textit{ in } P_1;$$

$$|\alpha_1^x x^{m_1}|\frac{v-A}{v-2A} < |f_{11}(x)| < |\alpha_1^x x^{m_1}|\frac{v}{v-A}, \quad \textit{for } x \textit{ in } Q_1;$$

$$|f_{i1}(x)| < |\alpha_1^x x^{m_1}|\frac{M}{u-M-1} \qquad (i=2,\cdots,n), \quad \textit{for } x \textit{ in } P_1;$$

$$|f_{i1}(x)| < |\alpha_1^x x^{m_1}|\frac{A}{v-A} \qquad (i=2,\cdots,n), \quad \textit{for } x \textit{ in } Q_1.$$

Moreover for any positive constant a such that $a < |a_1|$ there exists another constant N such that

$$|f_{i1}(x)| < N|a^x| \qquad (i=1,\cdots,n), \quad \textit{for } x \textit{ in } E_{1\epsilon}.$$

REMARK. It is important for later use to point out the fact that, in so far as any statement in the conclusion of the theorem refers to the region D_1 or any part of it, it is independent of the nature of $\phi_{ij}(x)$, $i,j=1,\cdots,n$, in the region exterior to D_1. Hence, *if the functions $\phi_{ij}(x)$ are analytic in some D-region and satisfy the inequalities*

$$(45) \qquad |\phi_{ij}(x)| < \frac{Mc}{n|x|^2} \qquad (i,j=1,\cdots,n), \quad \textit{for } x \textit{ in } D,$$

then there exists a region D_1 having properties as in the theorem, and the system of equations has a particular solution $f_{i1}(x)$ having in D_1 the properties of $f_{i1}(x)$ of the theorem.

§ 3. *Existence of n independent solutions of the normal system.*

For the system of equations considered in the preceding section we shall now obtain n particular solutions which are independent and therefore constitute a fundamental system of solutions.

Since by (37) it is seen that $f_{11}(x)$ is not identically zero, we may make the substitution

$$g_1(x) = f_1(x), \tag{46}$$

$$g_i(x) = f_i(x) - \frac{f_{i1}(x)}{f_{11}(x)} f_1(x) \qquad (i = 2, \cdots, n), \tag{47}$$

of which the inverse is

$$f_1(x) = g_1(x), \qquad f_i(x) = \frac{f_{i1}(x)}{f_{11}(x)} g_1(x) + g_i(x) \quad (i = 2, \cdots, n), \tag{48}$$

where $f_{i1}(x)$, $i = 1, \cdots, n$, is the solution of (24) obtained in the preceding section. Then we have

$$\begin{aligned} g_i(x+1) &= f_i(x+1) - \frac{f_{i1}(x+1)}{f_{11}(x+1)} f_1(x+1) \\ &= \sum_{j=1}^{n} \left\{ \psi_{ij}(x) - \frac{f_{i1}(x+1)}{f_{11}(x+1)} \psi_{1j}(x) \right\} f_j(x) \end{aligned} \qquad (i = 2, \cdots, n).$$

Substituting in this equation the value of $f_j(x)$ obtained from (48) we have

$$\begin{aligned} g_i(x+1) &= \sum_{j=2}^{n} \left\{ \psi_{ij}(x) - \frac{f_{i1}(x+1)}{f_{11}(x+1)} \psi_{1j}(x) \right\} g_j(x) \\ &\quad + f_1(x) \sum_{j=1}^{n} \left\{ \psi_{ij}(x) - \frac{f_{i1}(x+1)}{f_{11}(x+1)} \psi_{1j}(x) \right\} \frac{f_{j1}(x)}{f_{11}(x)} \end{aligned} \qquad (i = 2, \cdots, n). \tag{49}$$

The last line of the expression may be written in the form

$$\frac{g_1(x)}{f_{11}(x)} \left\{ \sum_{j=1}^{n} \psi_{ij}(x) f_{j1}(x) - \frac{f_{i1}(x+1)}{f_{11}(x+1)} \sum_{j=1}^{n} \psi_{1j}(x) f_{j1}(x) \right\}.$$

Since $f_{j1}(x)$, $j = 1, \cdots, n$, is a solution of (24), this reduces to

$$\frac{g_1(x)}{f_{11}(x)} \left\{ f_{i1}(x+1) - \frac{f_{i1}(x+1)}{f_{11}(x+1)} f_{11}(x+1) \right\},$$

an expression which is identically zero. Hence in writing (49) the last line may be omitted. Thus we have

$$g_i(x+1) = \sum_{j=2}^{n} \left\{ \psi_{ij}(x) - \frac{f_{i1}(x+1)}{f_{11}(x+1)} \psi_{1j}(x) \right\} g_j(x) \quad (i = 2, \cdots, n). \tag{50}$$

Substituting the values of $f_i(x)$, $i = 1, \cdots, n$, obtained from (48) in the first equation of system (24), we have

$$g_1(x+1) = \sum_{j=1}^{n} \frac{f_{j1}(x)}{f_{11}(x)} \psi_{1j}(x) g_1(x) + \sum_{j=2}^{n} \psi_{1j}(x) g_j(x). \tag{51}$$

Replacing in (51) and (50) the functions $\psi_{ij}(x)$ by their original values, we

obtain the transformed system of equations:

$$(52)\quad g_1(x+1)-a_1\left(1+\frac{1}{x}\right)^{m_1}g_1(x)=\sum_{j=1}^{n}\frac{f_{j1}(x)}{f_{11}(x)}\phi_{1j}(x)g_1(x)+\sum_{j=2}^{n}\phi_{1j}(x)g_j(x),$$

$$(53)\quad g_i(x+1)-a_i\left(1+\frac{1}{x}\right)^{m_i}g_i(x)=\sum_{j=2}^{n}\left\{\phi_{ij}(x)-\frac{f_{i1}(x+1)}{f_{11}(x+1)}\phi_{1j}(x)\right\}g_j(x)$$
$$(i=2,\cdots,n).$$

This system has the simple solution

$$g_1(x)=f_{11}(x),\qquad g_i(x)=0\qquad(i=2,\cdots,n).$$

By substitution of this solution in (48) one reproduces of course the known solution of (24). We shall prove next that system (52), (53) has a second solution, and by means of it find a second solution of (24).

It will be noticed that the $n-1$ equations (53) involve only the $n-1$ functions $g_i(x)$, $i=2,\cdots,n$; and therefore (53) is a system of $n-1$ equations in these unknowns. It satisfies the hypotheses of the theorem indicated in the remark at the close of § 2; that is, there exist a constant M' ($\geqq M$) and a D-region such that each of the coefficient functions

$$\phi_{ij}(x)-\frac{f_{i1}(x+1)}{f_{11}(x+1)}\phi_{1j}(x)\qquad(i,j=2,\cdots,n),$$

is analytic in D and the inequality

$$\left|\phi_{ij}(x)-\frac{f_{i1}(x+1)}{f_{11}(x+1)}\phi_{1j}(x)\right|<\frac{M'c}{(n-1)|x|^2}$$

is satisfied by every x in D. The statement referring to the analytic nature of the coefficients is evidently true; for in a suitable D-region $f_{11}(x)$ does not vanish. It is clear also that there exists a constant M' such that the above inequality is true provided that there is a D-region in which $f_{i1}(x+1)/f_{11}(x+1)$ is in absolute value equal to or less than 1. That this is true follows at once from inequalities (40) to (43).

From the above considerations and the theorem indicated in the remark referred to may be drawn several conclusions which are important for the succeeding argument:

1) There exist particular regions P_2, Q_2, D_2 and constants c and M_1 ($\geqq M'$) such that inequalities analogous to (30) to (34) hold and such that also

$$(54)\quad u>2M_1+1,\qquad A_1/v<\tfrac{1}{2},\qquad |f_{i1}(x)/f_{11}(x)|\leqq 1,\quad \text{for } x \text{ in } D_2.$$

2) The system of equations (53) has a particular solution, $g_{i2}(x)$, $i=2,\cdots,n$, each function of which is analytic throughout D_2.

3) The increase or decrease of the functions which constitute this solution is

bounded as follows:

$$
\begin{aligned}
&|a_2^x x^{m_2}|\,\frac{v-2A_1}{v-A_1}<|g_{22}(x)|<|a_2^x x^{m_2}|\,\frac{v}{v-A_1},\\
&|g_{i2}(x)|<|a_2^x x^{m_2}|\,\frac{A_1}{v-A_1}\quad (i=3,\ \cdots,\ n),
\end{aligned}
\tag{55}
$$

for x in Q_2;

$$
\begin{aligned}
&|a_2^x x^{m_2}|\,\frac{u-2M_1-1}{u-M_1-1}<|g_{22}(x)|<|a_2^x x^{m_2}|\,\frac{u-1}{u-M_1-1},\\
&|g_{i2}(x)|<|a_2^x x^{m_2}|\,\frac{M_1}{u-M_1-1}\quad (i=3,\ \cdots,\ n),
\end{aligned}
\tag{56}
$$

for x in P_2.

These $n-1$ functions are a part of a particular solution of the system (52), (53). To find the other function $g_{12}(x)$ of this solution we have the non-homogeneous equation

$$
\begin{aligned}
&g_{12}(x+1)-a_1\left(1+\frac{1}{x}\right)^{m_1}g_{12}(x)\\
&\qquad=\sum_{j=1}^{n}\frac{f_{j1}(x)}{f_{11}(x)}\phi_{1j}(x)g_{12}(x)+\sum_{j=2}^{n}\phi_{1j}(x)g_{j2}(x).
\end{aligned}
\tag{57}
$$

We obtain first a solution of the reduced equation

$$
g(x+1)-a_1\left(1+\frac{1}{x}\right)^{m_1}g(x)=\sum_{j=1}^{n}\frac{f_{j1}(x)}{f_{11}(x)}\phi_{1j}(x)g(x),
\tag{58}
$$

an equation which satisfies the hypotheses of the theorem indicated in the remark at the close of § 1, as is shown by relations (54) and those indicated in the lines preceding (54). Accordingly it has a particular solution $\bar{g}(x)$ analytic and not zero throughout D_2 and approximately characterized by the inequalities

$$
\begin{aligned}
&|a_1^x x^{m_1}|\,\frac{v-2A_1}{v-A_1}<|\bar{g}(x)|<|a_1^x x^{m_1}|\,\frac{v}{v-A_1},\quad \text{for } x \text{ in } Q_2;\\
&|a_1^x x^{m_1}|\,\frac{u-2M_1-1}{u-M_1-1}<|\bar{g}(x)|<|a_1^x x^{m_1}|\,\frac{u-1}{u-M_1-1},\quad \text{for } x \text{ in } P_2.
\end{aligned}
\tag{59}
$$

Substituting in (57)

$$
g_{12}(x)=h(x)\bar{g}(x)
\tag{60}
$$

and making use of the fact that $\bar{g}(x)$ satisfies (58), one obtains by easy reduction

$$
h(x+1)-h(x)=\sum_{j=2}^{n}\frac{\phi_{1j}(x)g_{j2}(x)}{\bar{g}(x+1)}=\eta(x).
\tag{61}
$$

It is clear that $\eta(x)$ is analytic throughout D_2.

Bounds of increase or decrease of $\eta(x)$ may be readily obtained. From (55),

(56), and (59) we have

$$(62)\qquad \begin{aligned} |\eta(x)| &< \left(\frac{a_2}{a_1}\right)^x x^{m_2-m_1-2}\,\frac{Mv}{v-2A_1}, \quad \text{for } x \text{ in } Q_2;\\ |\eta(x)| &< \left(\frac{a_2}{a_1}\right)^x x^{m_2-m_1-2}\,\frac{M(u-1)}{u-2M_1-1}, \quad \text{for } x \text{ in } P_2. \end{aligned}$$

From the first of these inequalities it is clear that the series

$$(63)\qquad s(x)=\eta(x-1)+\eta(x-2)+\eta(x-3)+\cdots$$

is uniformly convergent throughout any closed region in Q_2; and hence the function $s(x)$ is analytic in Q_2. We have also

$$(64)\qquad s(x+1)-s(x)=\eta(x),$$

so that $h(x)=s(x)$ is a solution of (61) which is analytic in Q_2. But its behavior near the positive real axis is complicated; and the solution of (52), (53), and hence of (24), to which it leads is composed of functions which are complicated in their character in the same region. We seek therefore another solution, one which will avoid this difficulty near the real axis; it will of course differ from $s(x)$ by a periodic function.

A solution of (61) in the form of a contour integral, the path of integration going to infinity along two parallel lines, will be found to serve our purpose. We form the integral

$$(65)\qquad I(x)=\int_L \frac{\eta(z)\,dz}{1-e^{2\pi i(x-z)}},$$

where the path of integration L is yet to be chosen. Since $\eta(x)$ is analytic throughout D_2, it is evident that all the infinities which the integrand has in D_2 are at the points $x \pm r$, where r is zero or a positive integer. Let AB be the straight line containing all these points. Let the path of integration L or $CKRH$ lie entirely in D_2 and be formed in the following manner: It consists

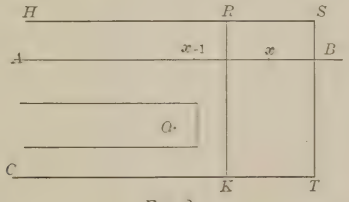

FIG. 2.

of three parts: a part RH lying entirely above AB and extending to infinity in a negative direction parallel to the axis of reals; a part KC lying below AB and

extending to infinity in a negative direction parallel to the axis of reals; a finite part KR which crosses AB once between $x-1$ and x and at no other point. Moreover the path $CKRH$ is such that E_2 lies entirely to the left of it. From (62) it is evident that the integral along such a path has a meaning. It is clear that for a particular x the path of integration may be deformed at will without changing the value of $I(x)$, provided that during the deformation it has always the properties stated in the definition of $CKRH$. Furthermore, $I(x)$ is analytic in D_2.

We show that $h(x) = I(x)$ is a solution of (61). In view of the fact that the integrand considered as a function of x is periodic of period 1 it is clear that we have

$$I(x+1) = \int_{L'} \frac{\eta(z)\,dz}{1-e^{2\pi i(x-z)}}, \tag{66}$$

where L' is the path $CTSH$. Now

$$\int_{L'} - \int_{L} = \int_{KTSRK}.$$

But by Cauchy's theorem the last integral is equal to $\eta(x)$. Hence, subtracting (65) from (66), member by member, we have

$$I(x+1) - I(x) = \eta(x);$$

that is,

$$h(x) = I(x)$$

is a solution of (61). It is this particular solution which we shall take for the value of $h(x)$ in the remaining discussion. It is analytic in D_2, as we have seen.

It is necessary, for later use, to know a bound to the increase or decrease of $|h(x)|$ for x in Q_2 and P_2. We consider the two regions separately, first taking the region Q_2. On account of (64) and the analytic nature of both $h(x)$ and $s(x)$ in Q_2, it is evident that $h(x) - s(x)$ is a periodic function of x of period 1 and analytic throughout Q_2. Now

$$|h(x)| \leqq |h(x) - s(x)| + |s(x)|; \tag{67}$$

accordingly we find upper bounds for $|h(x) - s(x)|$ and $|s(x)|$ separately. Consider the function $h(x) - s(x)$ for $x = u \pm v\sqrt{-1}$, x being in Q_2, where v is fixed and where u runs over any stretch of length 1. For such variation of x, $|h(x) - s(x)|$ has evidently a maximum which is entirely independent of the range for u but depends directly on v. Call this maximum $m(\pm v)$. Then we have

$$|h(x) - s(x)| \leqq m(\pm v), \quad \text{for } x \text{ in } Q_2, \tag{68}$$

where $m(\pm v)$ is a single-valued function of $\pm v$.

Now

$$|s(x)| \leqq \sum_{i=1}^{\infty} |\eta(x-i)| < \frac{Mv}{v-2A_1} \sum_{i=1}^{\infty} \left| \left(\frac{a_2}{a_1}\right)^{x-i} (x-i)^{m_2-m_1-2} \right|,$$

by (62). Hence we have

$$|s(x)| < \frac{Mv}{v-2A_1} \left| \left(\frac{a_2}{a_1}\right)^{x} x^{m_2-m_1-2} \right| \sum_{i=1}^{\infty} \left| \left(\frac{a_1}{a_2}\right)^{i} \left(1-\frac{i}{x}\right)^{m_2-m_1-2} \right|.$$

Therefore we obtain the result

$$|s(x)| < \frac{B_1}{|x|^2} \left| \left(\frac{a_2}{a_1}\right)^{x} x^{m_2-m_1} \right|, \quad \text{for } x \text{ in } Q_2, \tag{69}$$

where B_1 is a properly chosen constant. Then, finally, from (67), (68), (69) we have

$$|h(x)| < m(\pm v) + \frac{B_1}{|x|^2} \left| \left(\frac{a_2}{a_1}\right)^{x} x^{m_2-m_1} \right|, \quad \text{for } x \text{ in } Q_2. \tag{70}$$

In order to find an upper bound for $|h(x)|$ when x is in P_2 we proceed as follows: Let AB be the line parallel to the axis of reals and containing the

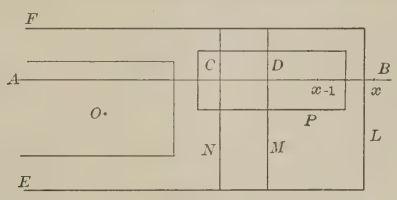

FIG. 3.

point x. Construct the contours ELF, EMF, ENF, P as in the figure, where the stretch CD is equal to $\frac{1}{2}$. Let us denote by $I_L(x)$, $I_M(x)$, $I_N(x)$, $I_P(x)$ respectively the integral

$$\int \frac{\eta(z)\,dz}{1-e^{2\pi i(x-z)}}$$

taken around these contours. For any value of x, either $I_M(x)$ or $I_N(x)$ represents an integral which is convergent, though the path of integration of one of them may pass through an infinity of the integrand. $I_L(x)$ and $I_P(x)$ always have a meaning, if it is understood that the contour P may vary so as not to cross AB at a point congruent to x.

Since $h(x) \equiv I_L(x)$, we clearly have

$$\left.\begin{aligned} |h(x)| &\leqq |I_M(x)| + \bar{I}_P(x), \\ |h(x)| &\leqq |I_N(x)| + \bar{I}_P(x), \end{aligned}\right\} \quad \text{for } x \text{ in } P_2,$$

where $\bar{I}_P(x)$ is the quantity obtained by applying to $I_P(x)$ the calculus of residues and replacing by its absolute value each term of the sum so obtained. Let x vary along AB. For each x take the smaller of the quantities $|I_M(x)|$ and $|I_N(x)|$; call this quantity $\tau(x)$. From the nature of the integrand in $I(x)$, it is evident that $\tau(x)$ has a maximum; we denote this maximum by $\mu(\pm v)$, since it depends on $\pm v$ alone. Thus we obtain

$$|h(x)| \leqq \mu(\pm v) + \bar{I}_P(x), \quad \text{for } x \text{ in } P_2.$$

Now from (62) and the calculus of residues it is clear that when x is in P_2 we have

$$|I_P(x)| < \frac{M(u-1)}{u-2M_1-1} \sum_{i=1}^{\infty} \left| \left(\frac{a_2}{a_1}\right)^{x-i} (x-i)^{m_2-m_1-2} \right|.$$

Hence we obtain

$$|I_P(x)| < \frac{B_2}{|x|^2} \left| \left(\frac{a_2}{a_1}\right)^{x} x^{m_2-m_1} \right|, \quad \text{for } x \text{ in } P_2,$$

where B_2 is a properly chosen constant. Therefore we have

$$(\bar{70}) \qquad |h(x)| < \mu(\pm v) + \frac{B_2}{|x|^2} \left| \left(\frac{a_2}{a_1}\right)^{x} x^{m_2-m_1} \right|, \quad \text{for } x \text{ in } P_2.$$

Since $h(x)$ and $\bar{g}(x)$ are known functions, both analytic throughout D_2, equation (60) gives us a function $g_{12}(x)$ analytic in D_2. From (59), (70), (70) we have

$$(71) \qquad \begin{aligned} |g_{12}(x)| &< |a_1^x x^{m_1}| \frac{v \cdot m(\pm v)}{v - A_1} + |a_2^x x^{m_2}| \frac{B_2}{|x|^2} \cdot \frac{v}{v - A_1}, \quad \text{for } x \text{ in } Q_2; \\ |g_{12}(x)| &< |a_1^x x^{m_1}| \frac{\mu(\pm v)(u-1)}{u - M_1 - 1} + |a_2^x x^{m_2}| \frac{B_2}{|x|^2} \cdot \frac{u-1}{u - M_1 - 1}, \quad \text{for } x \text{ in } P_2. \end{aligned}$$

From (48) we now have a second solution of (24); calling it $f_{i2}(x)$, $i = 1, \cdots, n$, we may write

$$(72) \qquad \begin{aligned} f_{12}(x) &= g_{12}(x), \\ f_{i2}(x) &= \frac{f_{i1}(x)}{f_{11}(x)} g_{12}(x) + g_{i2}(x) \qquad (i = 2, \cdots, n). \end{aligned}$$

Also from (55), (56), (71) we have the following inequalities:

$$(73) \qquad \left\{ \begin{aligned} &|f_{12}(x)| < |a_2^x x^{m_2}| \frac{B_2}{|x|^2} \cdot \frac{v}{v - A_1} + |a_1^x x^{m_1}| \frac{v \cdot m(\pm v)}{v - A_1}, \\ &|f_{22}(x)| < |a_2^x x^{m_2}| \frac{v}{v - A_1} + |f_{12}(x)|, \\ &|f_{22}(x)| > |a_2^x x^{m_2}| \frac{v - 2A_1}{v - A_1} - |f_{12}(x)|, \\ &|f_{i2}(x)| < |a_2^x x^{m_2}| \frac{A_1}{v - A_1} + |f_{12}(x)| \quad (i = 3, \cdots, n), \end{aligned} \right. \qquad \text{for } x \text{ in } Q_2;$$

$$(74)\quad \begin{cases} |f_{12}(x)| < |a_2^x x^{m_2}| \dfrac{B_2}{|x|^2} \cdot \dfrac{u-1}{u-M_1-1} + |a_1^x x^{m_1}| \dfrac{\mu(\pm v)(u-1)}{u-M_1-1}, \\ |f_{22}(x)| < |a_2^x x^{m_2}| \dfrac{u-1}{u-M_1-1} + |f_{12}(x)|, \\ |f_{22}(x)| > |a_2^x x^{m_2}| \dfrac{u-2M_1-1}{u-M_1-1} - |f_{12}(x)|, \\ |f_{i2}(x)| < |a_2^x x^{m_2}| \dfrac{M_1}{u-M_1-1} + |f_{12}(x)| \quad (i=3, \cdots, n), \end{cases} \qquad \text{for } x \text{ in } P_2.$$

It should be observed that the results of this paragraph, inasmuch as they refer only to the region D_2, are again independent of the nature of the functions $\phi_{ij}(x)$ in E_2.

If $n > 2$ we may obtain a third solution of (24) in the following manner: The solution of (24) just obtained depends intimately upon the two smallest a's in the system, as inequalities (73), (74) show. Hence it follows that (53) has a corresponding solution depending in like manner upon the two smallest a's in that system; this solution we shall call $g_{i3}(x)$, $i = 2, \cdots, n$. It is analytic throughout some region D_3 and satisfies the inequalities corresponding to (73):

$$(75)\quad \begin{cases} |g_{23}(x)| < |a_3^x x^{m_3}| \dfrac{B_3}{|x|^2} \cdot \dfrac{v}{v-A_2} + |a_2^x x^{m_2}| \dfrac{v \cdot m_1(\pm v)}{v-A_2}, \\ |g_{33}(x)| < |a_3^x x^{m_3}| \dfrac{v}{v-A_2} + |g_{23}(x)|, \\ |g_{33}(x)| > |a_3^x x^{m_3}| \dfrac{v-2A_2}{v-A_2} - |g_{23}(x)|, \\ |g_{i3}(x)| < |a_3^x x^{m_3}| \dfrac{A_2}{v-A_2} + |g_{23}(x)| \quad (i=4, \cdots, n), \end{cases} \qquad \text{for } x \text{ in } Q_3;$$

where D_3, Q_3, $m_1(\pm v)$, A_2, B_3 are defined in a manner analogous respectively to the definitions of D_2, Q_2, $m(\pm v)$, A_1, B_1. Also there exists of course a set of inequalities similar to (74) for x in P_3; it is not necessary to write them out.

We have thus $n-1$ functions of a third solution of (52), (53). To find the other function, which we will call $g_{13}(x)$, we have an equation corresponding to (57) above. If we write

$$g_{13}(x) = h_1(x)\bar{g}(x),$$

where $\bar{g}(x)$ is defined as before, it is easy to see that $h_1(x)$ is a solution of

$$(76)\qquad h_1(x+1) - h_1(x) = \sum_{j=2}^{n} \frac{\phi_{1j}(x)\, g_{j3}(x)}{\bar{g}(x+1)} \equiv \eta_1(x).$$

From inequalities (59), (75) and those for x in P_3 corresponding to (75) and the fact that $\eta_1(x)$ is analytic throughout D_3 it follows that the method used for solving (61) is applicable to (76) and that a solution may be obtained as before in the form of a contour integral. We thus find a third particular solution of (52), (53). Just as before this leads to a solution of (24). It is evident that this process may be continued until n particular solutions of (24) are obtained and that there exists a D-region in which each of the n^2 functions of these solutions is analytic. We shall call these n solutions

$$f_{1j}(x),\ f_{2j}(x),\ \cdots,\ f_{nj}(x) \qquad (j=1,\cdots,n). \tag{77}$$

It is clear that we may employ (27) to extend each of these solutions across the plane to the left and in this way prove that each function $f_{ij}(x)$ is analytic throughout the finite plane except at the singularities of $\bar{\psi}_{ij}(x)$ and the points congruent to them on the left.

A reference to the discussion (in the preceding section) of the bounds of $|f_{ii}(x)|$ in the region E_1 will bring out the fact that an argument similar to that employed there is applicable to any one of the n particular solutions which we have obtained, and that the result in each case takes the same form as that obtained for the first solution.

Let us now write

$$f_{ij}(x) = a_j^x x^{m_j} U_{ij}(x) \qquad (i,j=1,\cdots,n),$$

where the functions $U_{ij}(x)$ are defined by these equations. Inequalities (40) to (43), (73), (74) and the analogous inequalities for the other solutions enable us to conclude (without further argument) that when x approaches infinity in the positive direction along any line parallel to the real axis we have *

$$\lim U_{ij}(x) = \begin{cases} 0 \text{ if } i \neq j, \\ 1 \text{ if } i = j. \end{cases} \tag{78}$$

This fact will enable us to prove the independence of the solutions (77). The necessary and sufficient condition for such independence † is that the determinant of which $f_{ij}(x)$ is the general element is not identically zero. Now we have

$$|f_{ij}(x)| = a_1^x a_2^x \cdots a_n^x x^{m_1+m_2+\cdots+m_n} |U_{ij}(x)|.$$

From this result and (78) it follows that the determinant of which $f_{ij}(x)$ is the

* This result brings into prominent notice the important question of the asymptotic character of the solutions for every approach to infinity. The full discussion of this subject requires a separate investigation. There are several excellent papers by HORN, FORD, PERRON and others treating the highly important special case in which x runs over a set of points x_0, x_0+1, x_0+2, $\cdots$. See references on pp. 160, 190, 191 of HORN's paper in Crelle's Journal, vol. 138 (1910). To the papers which he lists should be added one by PERRON in Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 19 (1910), pp. 129–137. See also the paper by WATSON referred to in the introduction to the present paper.

† Cf. PINCHERLE and AMALDI, *Le operazione distributive*, p. 218.

general element is not identically zero. Hence the solutions (77) are independent and form therefore a fundamental system of solutions of the system (24); and hence the general solution is

$$f_i(x) = \sum_{j=1}^{n} \omega_j(x) f_{ij}(x) \qquad (i = 1, \cdots, n),$$

where the functions $\omega_j(x)$, $j = 1, \cdots, n$, are arbitrary periodic functions of period 1.

In the theorem below, the inequalities bounding $|f_{ij}(x)|$, $i, j = 1, \cdots, n$, are not given; otherwise the principal results of the section are as follows:

The system of equations

$$f_i(x+1) = \sum_{j=1}^{n} \psi_{ij}(x) f_j(x) \quad \text{or} \quad f_i(x-1) = \sum_{j=1}^{n} \bar{\psi}_{ij}(x-1) f_j(x) \quad (i = 1, \cdots, n),$$

$\psi_{ij}(x)$ being restricted as in the theorem of § 2, *has the fundamental system of solutions*

$$f_{1j}(x),\ f_{2j}(x),\ \cdots,\ f_{nj}(x) \qquad (j = 1, \cdots, n),$$

determined in the preceding pages and having the following properties: Each function $f_{ij}(x)$ is analytic throughout the finite plane except at the singularities of $\bar{\psi}_{ij}(x)$ and the points congruent to them on the left. The general solution of the system of equations is

$$f_i(x) = \sum_{j=1}^{n} \omega_j(x) f_{ij}(x) \qquad (i = 1, \cdots, n),$$

where the functions $\omega_j(x)$, $j = 1, \cdots, n$, are arbitrary periodic functions of period 1. *Further, for x approaching infinity along any line parallel to the positive real axis we have*

$$\lim f_{ij}(x)\, a_j^{-x} x^{-m_j} = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j. \end{cases}$$

§ 4. *Reduction to normal form. General existence theorems.*

Let

$$G_i(x+1) = x^{\alpha} \sum_{j=1}^{n} a_{ij}(x) G_j(x) \qquad (i = 1, \cdots, n), \tag{79}$$

be a system of n first order linear homogeneous difference equations involving n unknown functions $G_1(x), \cdots, G_n(x)$ of the complex variable x, the known quantities entering into the equation being defined as follows:

1) α is a constant;

2) the functions $a_{ij}(x)$ are single-valued and

$$a_{ij}(x) = a_{ij} + a'_{ij} x^{-1} + a''_{ij} x^{-2} + \cdots \qquad (i, j = 1, \cdots, n) \qquad |x| > R;$$

3) the constants a_{ij} are such that the roots of the equation

$$|a_{ij} - \delta_{ij}\rho| = 0, \qquad \delta_{ij} = \begin{cases} 0 \text{ if } i \neq j, \\ 1 \text{ if } i = j, \end{cases} \tag{80}$$

are in absolute value different from each other and from zero.

Equation (80) is the *characteristic equation* associated with (79). For convenience we assume that its roots $a_1, \cdots, a_n$ are arranged so that

$$0 < |a_1| < |a_2| < \cdots < |a_n|. \tag{81}$$

The object of this section is to extend the theorem of § 3 to the system in its more general form and then to apply the result to a single equation of the nth order. The work is carried out by first reducing (79) to the normal form already studied. In this discussion two properties of the Γ-function will be assumed; namely, that it is analytic to the right of the axis of imaginaries, and that it satisfies the functional equation

$$\Gamma(x+1) = x\Gamma(x). \tag{82}$$

It is obvious that α can be reduced to zero by means of the transformation

$$G_i(x) = \{\Gamma(x)\}^{\alpha}\bar{G}_i(x). \tag{83}$$

Effecting this substitution in (79) and reducing by means of (82) we have

$$\bar{G}_i(x+1) = \sum_{j=1}^{n} a_{ij}(x)\bar{G}_j(x) \qquad (i=1, \cdots, n). \tag{84}$$

The first step in the normalization of (84) will be made by means of a linear transformation with constant coefficients. This done, a further reduction (of a different type) will enable us to remove the term in x^{-1} from the resulting coefficient functions $\bar{a}_{ij}(x)$, $i \neq j$.

Let us put

$$\bar{G}_i(x) = \sum_{j=1}^{n} \alpha_{ij} F_j(x) \qquad (i=1, \cdots, n), \tag{85}$$

where the determinant $|\alpha_{ij}|$ is different from zero. Making this substitution in (84) and solving for $F_i(x+1)$ one may write the result in the form

$$F_i(x+1) = \sum_{j=1}^{n} \bar{a}_{ij}(x) F_j(x) \qquad (i=1, \cdots, n), \tag{86}$$

where the functions $\bar{a}_{ij}(x)$, being linear combinations of the functions $a_{ij}(x)$ with constant coefficients, may be expanded in powers of x^{-1} for $|x| > R'$.

In connection with this transformation, let us consider the effect in the linear substitution

$$y_i' = \sum_{j=1}^{n} a_{ij} y_j \qquad (i=1, \cdots, n), \tag{87}$$

of replacing y_i' and y_i by their values as given in the transformation

$$y_i = \sum_{j=1}^{n} \alpha_{ij} \bar{y}_j, \qquad y_i' = \sum_{j=1}^{n} \alpha_{ij} \bar{y}_j' \qquad (i = 1, \cdots, n). \tag{88}$$

Having made these substitutions, solve for $\bar{y}_i'$ and write the result

$$\bar{y}_i' = \sum_{j=1}^{n} \bar{a}_{ij} \bar{y}_j \qquad (i = 1, \cdots, n).$$

Comparing the operations of this paragraph with those of the preceding, it is easy to see that the constant term in the function $\bar{a}_{ij}(x)$ must be identical with $\bar{a}_{ij}$ for each pair of values i and j. But, on account of (81), it follows from the well-known theory * of linear substitutions that a proper determination of α_{ij} in (88) will transform (87) to the simpler system

$$\bar{y}_i' = a_i \bar{y}_i \qquad (i = 1, \cdots, n). \tag{89}$$

Suitable values of α_{ij} may be determined * by solving the following n systems of equations:

$$\sum_{j=1}^{n} a_{ij} \alpha_{j\tau} - a_\tau \alpha_{i\tau} = 0 \qquad (i, \tau = 1, \cdots, n), \tag{90}$$

a separate system being formed for each τ. That each system (90) is consistent follows from the fact that $\rho = a_\tau$ is a solution of the characteristic equation (80). Hence the ratios of $\alpha_{i\tau}$ for fixed τ and $i = 1, \cdots, n$ are determined by (90). Out of the infinity of proportional solutions of each of these n systems of equations we suppose a particular one chosen, and that the quantities α_{ij} have these fixed values throughout the remaining discussion.

If the values of α_{ij} so determined are used in (85), that transformation throws (84) into the simpler system

$$F_i(x+1) = a_i F_i(x) + \sum_{j=1}^{n} b_{ij}(x) F_j(x) \qquad (i = 1, \cdots, n), \tag{91}$$

where

$$b_{ij}(x) = b_{ij} x^{-1} + b'_{ij} x^{-2} + \cdots \qquad (i, j = 1, \cdots, n) \qquad |x| > R'.$$

In order to remove the term in x^{-1} from each of the coefficient functions $b_{ij}(x)$, $i \neq j$, we employ a transformation of the form

$$F_i(x) = f_i(x) + \sum_{j=1}^{n} \frac{\beta_{ij}}{x} f_j(x), \qquad \beta_{ii} = 0, \qquad (i = 1, \cdots, n), \tag{92}$$

where the constants β_{ij}, $i \neq j$, are as yet undetermined. The determinant of this transformation is

$$\Delta \equiv \left| \delta_{ij} + \frac{\beta_{ij}}{x} \right| = 1 + \text{terms in } x^{-2}, x^{-3}, \cdots.$$

* Cf. HORN, *Gewöhnliche Differentialgleichungen*, pp. 72–74.

Substituting these values for $F_i(x)$ in (91) and writing only those terms which do not involve $x^{-2}, x^{-3}, \cdots$, we have, when $|x| > R'$,

$$f_i(x+1) + \sum_{j=1}^{n}\left(\frac{\beta_{ij}}{x+1}\right)f_j(x+1) = a_i f_i(x) + a_i \sum_{j=1}^{n}\frac{\beta_{ij}}{x} f_j(x)$$

$$+ \sum_{j=1}^{n}\left\{\left(\frac{b_{ij}}{x}+\cdots\right)\left[f_j(x) + \sum_{k=1}^{n}\frac{\beta_{ik}}{x}f_k(x)\right]\right\} \tag{93}$$

$$= \left(a_i + \frac{b_{ii}}{x} + \cdots\right)f_i(x) + \sum_{j=1}^{n}{}'\left(\frac{a_i\beta_{ij} + b_{ij}}{x} + \cdots\right)f_j(x),$$

the accent in Σ' denoting that in the summation the term for which $j = i$ is to be omitted.

It is necessary to solve (93) for $f_i(x+1)$. We proceed thus: The cofactor Δ_{ij} of the element in the ith row and jth column of Δ has clearly no constant term, except when i equals j, when there is a constant term 1: the term in $1/x$ for $i \neq j$ is clearly $-\beta_{ji}/x$. For, if we cross out the ith row and jth column of Δ, there remain $n-2$ elements with a constant term 1; and therefore the expansion of this minor can contain only one term in $1/x$, namely that term which contains all the $n-2$ factors 1 and whose remaining factor is not in the same rows or columns as these. Therefore this term is $\pm\beta_{ji}/x$; and, since an even number of interchanges of rows and columns bring the elements β_{ij}/x and β_{ji}/x to the place in the first row second column and second row first column respectively, the negative sign must be chosen. In the cofactor Δ_{ii} there is no term in $1/x$; for every term in the expansion containing one factor $1/x$ necessarily contains another. Hence the cofactors may be written

$$\Delta_{ij} = -\frac{\beta_{ji}}{x} + \text{terms in } x^{-2}, x^{-3}, \cdots, \qquad (i \neq j);$$

$$\Delta_{ii} = 1 + \text{terms in } x^{-2}, x^{-3}, \cdots.$$

Solving (93) for $f_i(x+1)$ we readily obtain

$$\bar{\Delta} f_i(x+1) = \left(a_i + \frac{b_{ii}}{x} + \cdots\right)f_i(x) + \sum_{j=1}^{n}{}'\left\{\frac{(a_i - a_j)\beta_{ij} + b_{ij}}{x} + \cdots\right\}f_j(x) \tag{94}$$

$$(i = 1, \cdots, n),$$

$\bar{\Delta}$ denoting the determinant

$$\bar{\Delta} = \left|\delta_{ij} + \frac{\beta_{ij}}{x+1}\right|.$$

For values of x such that $|x| \geqq R''$, R'' being a properly chosen constant greater than R' and greater than 1, the determinant $\bar{\Delta}$ is everywhere different from zero and is expansible in the form $1 +$ terms in $x^{-2}, x^{-3}, \cdots$. If we divide by

$\bar{\Delta}$ we obtain therefore

$$(95)\quad f_i(x+1)=\left(a_i+\frac{b_{ii}}{x}+\cdots\right)f_i(x)+\sum_{j=1}^{n}{}'\left\{\frac{(a_i-a_j)\beta_{ij}+b_{ij}}{x}+\cdots\right\}f_j(x) \qquad |x|\geqq R'';$$

the terms of order x^{-2}, x^{-3}, $\cdots$ in the coefficients of $f_j(x)$ in the right hand member are of course different from the corresponding terms in (94).

On account of (81) it is always possible to determine β_{ij}, $i\neq j$, so that

$$(a_i-a_j)\beta_{ij}+b_{ij}=0.$$

When the values of β_{ij} so determined are substituted in (95), that system reduces to the form

$$(96)\quad f_i(x+1)=\left(a_i+\frac{b_{ii}}{x}\right)f_i(x)+\sum_{j=1}^{n}\left(\frac{c_{ij}}{x^2}+\cdots\right)f_j(x) \quad (i=1,\cdots,n) \qquad |x|\geqq R''.$$

Now

$$a_i\left(1+\frac{1}{x}\right)^{\frac{b_{ii}}{a_i}}=a_i+\frac{b_{ii}}{x}+\cdots;$$

and hence, changing to more convenient notation and employing relations (81), we obtain the normal system

$$(97)\qquad f_i(x+1)-a_i\left(1+\frac{1}{x}\right)^{m_i}f_i(x)=\sum_{j=1}^{n}\phi_{ij}(x)f_j(x) \qquad (i=1,\cdots,n),$$

where

$$0<|a_1|<|a_2|<\cdots<|a_n|, \qquad m_i=\frac{b_{ii}}{a_i},$$

and

$$\phi_{ij}(x)=\phi''_{ij}x^{-2}+\phi'''_{ij}x^{-3}+\cdots \qquad |x|\geqq R,$$

where $R\geqq R''$. This is the normal form which we have studied in the preceding sections.

Combining the transformations (83), (85), (92) we readily obtain

$$(98)\qquad G_i(x)=\{\Gamma(x)\}^{\alpha}\sum_{k=1}^{n}\alpha_{ik}\left[f_k(x)+\sum_{l=1}^{n}\frac{\beta_{kl}}{x}f_l(x)\right] \qquad (i=1,\cdots,n).$$

It is evident therefore that if we replace $f_i(x)$, $i=1,\cdots,n$, by the particular solution $f_{ij}(x)$, $i=1,\cdots,n$, j fixed, we obtain a particular solution of (79) which we will call $G_{ij}(x)$, $i=1,\cdots,n$. Doing this for $j=1,\cdots,n$, we have n particular solutions

$$G_{1j}(x),\ G_{2j}(x),\ \cdots,\ G_{nj}(x) \qquad (j=1,\cdots,n),$$

where

$$G_{ij}(x) = \{\Gamma(x)\}^{\alpha} \sum_{k=1}^{n} \alpha_{ik} \left[f_{kj}(x) + \sum_{l=1}^{n} \frac{\beta_{kl}}{x} f_{lj}(x) \right] \qquad (i, j = 1, \cdots, n). \tag{99}$$

Let us associate with (79) the corresponding equation

$$G_i(x-1) = \sum_{j=1}^{n} A_{ij}(x-1) G_j(x) \qquad (i = 1, \cdots, n). \tag{100}$$

Now there is a P-region, as we have seen, in which the functions $f_{ij}(x)$ are all analytic. It follows from (99) that the functions $G_{ij}(x)$ likewise are all analytic in the same region. By means of (100) the functions of each of these solutions may be extended across the plane to the left; and in this way it may be shown that each of the n^2 functions $G_{ij}(x)$ is analytic throughout the entire plane except at the singularities of $A_{ij}(x)$ and the points congruent to them on the left.

Let us again write

$$f_{ij}(x) = a_j^r x^{m_j} U_{ij}(x) \qquad (i, j = 1, \cdots, n).$$

If we substitute these values for $f_{ij}(x)$ in (99) we have

$$G_{ij}(x) = \{\Gamma(x)\}^{\alpha} a_j^r x^{m_j} V_{ij}(x), \qquad (i, j = 1, \cdots, n),$$

where

$$V_{ij}(x) = \sum_{k=1} \alpha_{ik} \left[U_{kj}(x) + \sum_{l=1}^{n} \frac{\beta_{kl}}{x} U_{lj}(x) \right].$$

From (78) it follows that when x approaches infinity in the positive direction along any line parallel to the real axis we have

$$\lim V_{ij}(x) = \alpha_{ij}.$$

This result and the fact that the determinant $|\alpha_{ij}|$ is different from zero lead easily to the conclusion that the n particular solutions $G_{ij}(x)$ are independent. The method of proof is that employed for showing the independence of the solutions $f_{ij}(x)$ in the preceding section. Hence the general solution of (79) is

$$G_i(x) = \sum_{j=1}^{n} \omega_j(x) G_{ij}(x) \qquad (i = 1, \cdots, n),$$

where the function $\omega_j(x)$, $j = 1, \cdots, n$, are arbitrary periodic functions of period 1.

The principal results of the section may be stated in the following theorem:

Let there be given the system of equations

$$G_i(x+1) = x^{\alpha} \sum_{j=1}^{n} a_{ij}(x) G_j(x) \qquad (i = 1, \cdots, n),$$

where

1) *α is a constant;*

2) *the functions* $a_{ij}(x)$ *are single-valued and*

$$a_{ij}(x) = a_{ij} + a'_{ij}x^{-1} + a''_{ij}x^{-2} + \cdots \qquad (i, j = 1, \cdots, n) \qquad |x| > R';$$

3) *the constants* a_{ij} *are such that the roots* $\rho = a_i$ *of the equation*

$$|a_{ij} - \delta_{ij}\rho| = 0$$

are in absolute value different from each other and from zero.

Associate with the given system the corresponding one

$$G_i(x-1) = \sum_{j=1}^{n} A_{ij}(x-1)G_j(x) \qquad (i = 1, \cdots, n).$$

Then the given set of equations has the fundamental system of solutions

$$G_{1j}(x),\ G_{2j}(x),\ \cdots,\ G_{nj}(x) \qquad (j = 1, \cdots, n),$$

determined in the preceding pages and having the following properties: Each of the n^2 *functions is analytic throughout the finite plane except at the singularities of* $A_{ij}(x)$ *and the points congruent to them on the left. The general solution of the system is*

$$G_i(x) = \sum_{j=1}^{n} \omega_j(x)G_{ij}(x) \qquad (i = 1, \cdots, n),$$

where the functions $\omega_j(x), j = 1, \cdots, n$, *are arbitrary periodic functions of period* 1. *Further, if* x *approaches infinity in the positive direction along any line parallel to the real axis we have*

$$\lim G_{ij}(x)\{\Gamma(x)\}^{-\alpha} a_j^{-x} x^{-m_j} = \alpha_{ij},$$

where the constants α_{ij} *are determined by equations* (90) *and the constants* m_j *are those introduced in* (97).

Let us now consider the single difference equation of the nth order

$$\begin{aligned} &H(x+n) + (x+n)^{\alpha} b_1(x) H(x+n-1) + (x+n)^{\alpha}(x+n-1)^{\alpha} \\ &\quad \times b_2(x) H(x+n-2) + \cdots + (x+n)^{\alpha} \cdots (x+1)^{\alpha} x^{\alpha} b_n(x) H(x) = 0 \end{aligned} \tag{101}$$

where α is a constant and $b_i(x)$ is single-valued and

$$b_i(x) = b_i + b'_i x^{-1} + b''_i x^{-2} + \cdots \qquad (i = 1, \cdots, n) \qquad |x| > R,$$

and where the constants b_i are such that the roots $a_1, \cdots, a_n$ of the equation

$$\rho^n + b_1\rho^{n-1} + b_2\rho^{n-2} + \cdots + b_{n-1}\rho + b_n = 0 \tag{102}$$

are in absolute value different from each other and from zero.

Making the substitution

$$H(x) = \{\Gamma(x)\}^{\alpha} \bar{H}(x) \tag{103}$$

we have

$$\bar{H}(x+n)+b_1(x)\bar{H}(x+n-1)+\cdots+b_n(x)\bar{H}(x)=0. \tag{104}$$

It is evident that this equation is equivalent to the following system

$$\begin{aligned}&\bar{H}_1(x+1)=\sum_{j=1}^{n}\{-b_j(x)\}\bar{H}_j(x)=0,\\ &\bar{H}_i(x+1)=\bar{H}_{i-1}(x) \qquad (i=2,\cdots,n),\end{aligned} \tag{105}$$

where

$$\bar{H}_n(x)\equiv\bar{H}(x). \tag{106}$$

Now the characteristic equation [compare (80)] associated with (105) is readily reduced to the form (102).

If we solve (105) for $\bar{H}_i(x)$ in terms of $\bar{H}_i(x+1)$, $i=1,\cdots,n$, and in the result write $x-1$ for x, we have

$$\bar{H}_i(x-1)=\bar{H}_{i+1}(x) \qquad (i=1,\cdots,n-1),$$

$$\bar{H}_n(x-1)=-\frac{1}{b_n(x-1)}\bar{H}_1(x)-\sum_{j=1}^{n-1}\frac{b_j(x-1)}{b_n(x-1)}\bar{H}_{j+1}(x).$$

Consequently, from the foregoing theorem, we have the following conclusions: The system (105) has a fundamental system of solutions

$$\bar{H}_{1j}(x),\ \bar{H}_{2j}(x),\ \cdots,\ \bar{H}_{nj}(x) \qquad (j=1,\cdots,n),$$

each function of which is analytic throughout the finite plane except at the singularities of the functions

$$\frac{1}{b_n(x)},\qquad \frac{b_j(x)}{b_n(x)} \qquad (j=1,\cdots,n),$$

and points congruent to them on the left; further, if x approaches infinity in the positive direction along any line parallel to the real axis, we have

$$\lim \bar{H}_{nj}(x)\alpha_j^{-x}x^{-m_j}=\bar{\alpha}_{ij},$$

where $\bar{\alpha}_{ij}$ and m_j are constants corresponding to the α_{ij} and m_j of the above theorem. Since only the ratios of $\bar{\alpha}_{ij}$, j fixed, $i=1,\cdots,n$, are determined, we take $\bar{\alpha}_{nj}=1$, $j=1,\cdots,n$.

Combining (103) and (106) we have

$$H(x)=\{\Gamma(x)\}^a\bar{H}_n(x)$$

Hence we have a fundamental system of solutions $H_j(x)$, $j=1,\cdots,n$, of (101) in the form

$$H_j(x)=\{\Gamma(x)\}^a\bar{H}_{nj}(x) \qquad (j=1,\cdots,n).$$

Making use of this equation and the properties of $\bar{H}_{nj}(x)$ stated above and determining the position of the singularities from the equation itself by extending the solutions across the plane to the left, we obtain the results stated in the conclusion of the following theorem:

Given, the difference equation

$$H(x+n)+(x+n)^{\alpha}b_1(x)H(x+n-1)+\cdots+(x+n)^{\alpha}\cdots(x+1)^{\alpha}x^{\alpha}b_n(x)H(x)=0$$

or

$$\beta_n(x)H(x+n)+\beta_{n-1}(x)H(x+n-1)+\cdots+\beta_1(x)H(x+1)+H(x)=0,$$

where α is a constant and $b_i(x)$ is single-valued and

$$b_i(x)=b_i+b_i'x^{-1}+b_i''x^{-2}+\cdots \qquad (i=1,\cdots,n) \qquad |x|>R,$$

and where the constants b_i are such that the roots $a_1,\cdots,a_n$ of the equation

$$\rho^n+b_1\rho^{n-1}+\cdots+b_{n-1}\rho+b_n=0$$

are in absolute value different from each other and from zero.

This equation has the fundamental system of solutions

$$H_1(x),\ H_2(x),\ \cdots,\ H_n(x)$$

determined above and possessing the following properties: Each of the functions $H_1(x),\cdots,H_n(x)$ is analytic throughout the finite plane except at the singularities of the functions $\beta_i(x)$, $i=1,\cdots,n$, and the points congruent to them on the left. If x approaches infinity in the positive direction along any line parallel to the real axis, we have

$$\lim H_j(x)\{\Gamma(x)\}^{-\alpha}a_j^{-x}x^{-m_j}=1 \qquad (j=1,\cdots,n),$$

where m_j are constants.

The general solution of the equation is

$$H(x)=\omega_1(x)H_1(x)+\cdots+\omega_n(x)H_n(x)$$

where $\omega_1(x),\cdots,\omega_n(x)$ are arbitrary periodic functions of period 1.

It may be observed that a modification of the foregoing method will lead to a different fundamental system of solutions of (79) and likewise of (101). We confine our attention to (79). A few remarks relative to the equation

$$g(x+1)-a\left(1+\frac{1}{x}\right)^m g(x)=\eta(x)$$

will sufficiently indicate the necessary changes. Employing the notation

$$\lambda(x)\equiv a\left(1+\frac{1}{x}\right)^m,$$

we have two formal solutions of the above equation as follows:

$$g(x) = -\sum_{i=0}^{\infty} \frac{\eta(x+i)}{\lambda(x)\lambda(x+1)\cdots\lambda(x+i)},$$

$$g(x) = \eta(x-1) + \sum_{i=1}^{\infty} \lambda(x-1)\lambda(x-2)\cdots\lambda(x-i)\eta(x-i-1).$$

The first of these, as we have seen, leads to a solution of (1). The second may also be employed in a similar way to obtain a solution of the same equation. The extension to a system of n equations is obvious. In this way a second fundamental system of solutions of (79) may be found.

Suppose further that transformation (83) is modified by the use of a function $\bar{\Gamma}(x)$ which satisfies the equation

$$\bar{\Gamma}(x+1) = x\bar{\Gamma}(x)$$

and is analytic to the left of the axis of imaginaries, this function replacing $\Gamma(x)$ in (83). This modification having been made, it will turn out that each function of the fundamental system just found is analytic throughout the finite plane except at the singularities of $a_{ij}(x)$ and the points congruent to them on the right; and their properties may be developed by a method similar throughout to that employed in the paper and with like results.

PRINCETON UNIVERSITY,
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BIORTHOGONAL SYSTEMS OF FUNCTIONS*

BY

ANNA JOHNSON PELL

Introduction.

In boundary value problems of differential equations which are not self-adjoint, biorthogonal systems of functions play the same rôle as the orthogonal systems do in the self-adjoint case. LIOUVILLE† has considered special non-self-adjoint differential equations with real characteristic values of the parameter; BIRKHOFF‡ has proved the existence of the characteristic values (in general complex) for the differential equation of the nth order and obtained the related expansions.

If the integral equation

$$u(s) = \lambda \int_a^b L(s, t)u(t)\,dt$$

with the unsymmetric kernel $L(s, t)$ has solutions $u(s)$, and therefore the integral equation

$$v(s) = \lambda \int_a^b L(t, s)v(t)\,dt$$

solutions $v(s)$, it has been shown by PLEMELJ§ and GOURSAT‖ that the solutions or functions closely related to them form a biorthogonal system. But expansions in terms of these solutions have not been obtained, and no criteria have been given for the existence of real characteristic numbers of an unsymmetric kernel.¶

The object of this paper is the development of a theory of biorthogonal systems of functions independent of their connection with integral or differential equations. In the theory frequent use is made of the theorems by RIESZ, FISCHER, and TOEPLITZ (Lemmas 1, 2, 3, and 4, § 2).

* Presented to the Society (Chicago) April 10, 1909.

† Liouville's Journal, ser. 1, vol. 3 (1838).

‡ Transactions of the American Mathematical Society, vol. 9 (1908).

§ Monatshefte für Mathematik und Physik, vol. 15 (1904).

‖ Annales de Toulouse, 1908.

¶ Since this paper was written J. MARTY has published some results for unsymmetric kernels, Comptes Rendus, February 28, March 7, April 25, and June 6, 1910. See also the note by the author in Bulletin of the American Mathematical Society, July, 1910.

Necessary and sufficient conditions for the existence of the adjoint system $\{v_i\}$ of any system of linearly independent functions $\{u_i\}$ are deduced (theorem 2, § 3). Theorems for biorthogonal systems analogous to those of RIESZ and FISCHER for orthogonal systems are: (*a*) if

$$\sum_{i=1}^{\infty}\left(\int fv_i\right)^2$$

converges for every function f which is integrable and has an integrable square, then for every system of constants $\{c_i\}$ of finite norm there exists a function g such that $c_i = \int gu_i$ (corollary 1, theorem 6); (*b*) if the system $\{u_i\}$ is complete and if certain conditions are imposed on the function g, then (theorem 8)

$$\int fg = \sum_{i=1}^{\infty}\int fv_i \int u_i g.$$

The equivalence of two biorthogonal systems is defined and a classification into types is made (§ 7). With each type satisfying a certain condition there is connected uniquely (§ 6) and in a reversible way (theorem 20, § 8) a single-valued functional transformation* $T(f)$, which transforms every function which is integrable and has an integrable square into a function of the same kind, and is defined by the properties

$$T(a_1f_1 + a_2f_2) = a_1T(f_1) + a_2T(f_2), \tag{1}$$

$$\int [f_1T(f_2) - f_2T(f_1)] = 0, \tag{2}$$

$$\int fT(f) \geqq 0. \tag{3}$$

The class of all orthogonal systems of functions is a special type, for which $T(f)$ is the identical transformation. Other special cases of $T(f)$ are

$$T(f) = f(s) - \frac{p(s)\int_a^b p(s)f(s)\,ds}{\int_a^b [p(s)]^2 ds},$$

$$T(f) = \int_a^b K(s, t)f(t)\,dt.$$

By means of the theorems by RIESZ, FISCHER and TOEPLITZ, it can be shown that there is a one-to-one correspondence between this functional transformation $T(f)$ and positive definite limited quadratic forms in infinitely many variables.

* Linear functional transformations have recently been studied by F. RIESZ, Mathematische Annalen, vol. 69 (1910), p. 449.

§ 1. *Fundamental Notations.*

We consider an arbitrary interval I: $a \leqq s \leqq b$: of the real variable s, and the corresponding square S: $a \leqq s \leqq b$, $a \leqq t \leqq b$: of the real variables s and t, and real functions of these variables. We denote functions of a single variable by f, g, φ, ϕ, etc., omitting the argument. $\mathfrak{F}$ is the class of all functions f such that f and f^2 are integrable in the sense of Lebesgue* on the interval I. As usual, we regard two functions of $\mathfrak{F}$ as equal if they differ only on a set of points of content zero.† We call two such functions "essentially equal."

When there is no ambiguity we omit the variable of integration In all cases the limits of integration are omitted, since I is the only interval of integration considered in this paper.

Constants and functions with the subscripts i, j, k denote sequences of constants and functions.

$\mathfrak{G}$ and $\mathfrak{H}$ denote subclasses of $\mathfrak{F}$ and the elements of these subclasses are denoted by the small letters g and h.

To express that a relation involving the function g holds for every function of the class we write (g) after the relation; for example

$$\int f^2 \geqq 0 \qquad (f).$$

We use a similar notation in the case of several variables; for example

$$\left(\int f_1 f_2\right)^2 \leqq \int f_1^2 \int f_2^2 \qquad (f_1, f_2).$$

Let $\{f_i\}$ denote any sequence of linearly independent functions of the class $\mathfrak{F}$, and $\{\phi_i\}$ the sequence of normalized and orthogonal‡ functions which are obtained linearly from the functions f_i by means of the construction given by E. Schmidt,§

$$\phi_i = \frac{f_i - \sum_{k=1}^{i-1} \phi_k \int \phi_k f_i}{\sqrt{\int \left(f_i - \sum_{k=1}^{i-1} \phi_k \int \phi_k f_i\right)^2}} \qquad (i).$$

Each ϕ_i may be expressed linearly and homogeneously with constant coefficients in terms of $f_1, f_2, \cdots, f_i$, and conversely:

$$(1) \qquad \phi_i = \sum_{k=1}^{i} a_{ik} f_k, \qquad f_i = \sum_{k=1}^{i} b_{ik} \phi_k \qquad (i).$$

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* Lebesgue, *Sur l'intégration*, p. 115.

† Lebesgue, l. c., p. 106.

‡ A function f is normalized if $\int f^2 = 1$. Two functions f_1 and f_2 are orthogonal if $\int f_1 f_2 = 0$.

§ *Zur Theorie der linearen und nichtlinearen Integralgleichungen.* Mathematische Annalen, vol. 63 (1907), p. 442.

The constants a_{ik}, b_{ik} have the following properties:

$$(2)\qquad \begin{aligned} a_{ii} &\neq 0 && (i),\\ a_{ik} &= 0 && (k>i),\\ b_{ii} &\neq 0 && (i),\\ b_{ik} &= 0 && (k>i). \end{aligned}$$

The matrices formed from the coefficients a_{ik} and b_{ik} we denote by $(a_{ik})_f$ and $(b_{ik})_f$ respectively, or simply by $(a)_f$ and $(b)_f$.

If $\{\bar{\phi}_i\}$ are the normalized and orthogonal functions corresponding to $\{\bar{f}_i = \lambda_i f_i\}$ where $\{\lambda_i\}$ are constants different from zero, then $\bar{\phi}_i = \phi_i$.

§ 2. *Preliminary definitions and lemmas.*

Definition. A system of functions of $\mathfrak{F}$ is called *complete* * for the interval I if there exists no function f of $\mathfrak{F}$, essentially different from zero, which is orthogonal to all the functions of the system in the interval I.

Definition. A sequence of constants $\{c_i\}$ is of *finite norm* if the sum of the squares,

$$\sum_{i=1}^{\infty} c_i^2,$$

converges, and it has the norm M if

$$\sum_{i=1}^{\infty} c_i^2 = M.$$

For any system $\{\phi_i\}$ of normalized and orthogonal functions of $\mathfrak{F}$ we have the three following lemmas.†

Lemma 1. The sequence of constants

$$\left\{\int f\phi_i\right\}$$

is of finite norm for every function f of $\mathfrak{F}$.

Lemma 2. If the system $\{\phi_i\}$ is complete,

$$\int f_1 f_2 = \sum_{i=1}^{\infty} \int f_1 \phi_i \int \phi_i f_2 \qquad (f_1, f_2).$$

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* F. Riesz, Comptes Rendus, November, 1906, p. 738.

† E. Schmidt, l. c., p. 439; F. Riesz, Comptes Rendus, March, 1907, p. 615, and April, 1907, p. 734; E. Fischer, Comptes Rendus, May, 1907, p. 1022.

Lemma 3. To every system of constants $\{c_i\}$ of finite norm there corresponds a function f of $\mathfrak{F}$ such that

$$c_i = \int f\phi_i \qquad (i),$$

and this function f is essentially unique if the system $\{\phi_i\}$ is complete.

Definition. A matrix (α_{ik}) is *limited* if

$$\sum_{i=1}^{\infty} \alpha_{ik} x_i$$

converges for every k, and

$$\left\{ \sum_{i=1}^{\infty} \alpha_{ik} x_i \right\}$$

is of finite norm for every system of values of $\{x_i\}$ of finite norm.

Definition. A bilinear form of the infinitely many variables x_i and y_i

$$\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \alpha_{ik} x_i y_k$$

is limited if for some positive constant M

$$\left| \sum_{k=1}^{n} \sum_{i=1}^{n} \alpha_{ik} x_i y_k \right| \leqq M \qquad (n)$$

for all values of $\{x_i\}$ and $\{y_i\}$ of norm $\leqq 1$ (H, IV, p. 176 *).

Lemma 4. The matrix of the coefficients of a limited bilinear form is limited (H, IV, p. 179); conversely, in virtue of a theorem by TOEPLITZ † a bilinear form whose coefficients are the elements of a limited matrix is limited.

Lemma 5. If the matrix (α_{ik}) is limited the matrix (α_{ki}) is limited also.

Lemma 6. The product of two limited matrices is a limited matrix (H, IV, p. 179).

Definition. A matrix (α_{ik}) is orthogonal if

$$\sum_{i=1}^{\infty} \alpha_{ij}\alpha_{ik} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \qquad (j, k),$$

$$\sum_{i=1}^{\infty} \alpha_{ji}\alpha_{ki} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \qquad (j, k).$$

Lemma 7. An orthogonal matrix is limited (H, IV, p. 180). If some of the elements of the orthogonal matrix are replaced by zeros, the resulting matrix is limited.

* The references to the memoirs by HILBERT in the Göttinger Nachrichten, 1904–1906, are denoted by (H, I, II, ···, V).

† E. SCHMIDT, Palermo Rendiconti, vol. 25 (1908), p. 2; HELLINGER and TOEPLITZ, Göttinger Nachrichten, 1906, p. 351, and Mathematische Annalen, vol. 69 (1910), p. 289.

Lemma 8. The value of a limited bilinear form for arguments $\{x_i\}$ and $\{y_i\}$ of norm 1 is obtained by summation either by rows or by columns (*H*, IV, p. 180).

Lemma 9. If α_{ik} are the elements of a limited matrix, and if $\alpha_{ik} = \alpha_{ki}$, then the system of homogeneous linear equations

$$x_i - \lambda \sum_{k=1}^{\infty} \alpha_{ik} x_k = 0 \qquad (i)$$

has a solution $\{x_i\}$ of finite norm when and only when λ is a "characteristic number," that is, a root of a certain associated equation, and the number of linearly independent solutions is equal to the multiplicity of the root (*H*, IV, p. 198).

Definition. Two systems* (u_i) and (v_i) of functions of $\mathfrak{F}$ form a *biorthogonal system* of functions (u_i, v_i) if a one-to-one correspondence can be established between them such that the integral of the product of two corresponding functions is equal to unity and the integral of the product of two non-corresponding functions is equal to zero; i. e.,

$$\int u_i v_j = \begin{cases} 1 & (i=j), \\ 0 & (i \neq j). \end{cases} \qquad (3)$$

Each one of the two systems (u_i), (v_i) is called an *adjoint* system of the other.

The biorthogonal system (u_i, v_i) is *complete as to* u, v, respectively, if the system (u_i), (v_i), respectively, is complete.

For a biorthogonal system (u_i, v_i) the functions u_i and also v_i are linearly independent. Suppose there existed a linear relation

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0.$$

Multiply by $v_i (i = 1, 2, \cdots, n)$ and integrate; from the properties (3) we obtain

$$c_i = 0 \qquad (i = 1, 2, \cdots, n).$$

§ 3. *Denumerability and existence of the adjoint system.*

Theorem 1. *A system (u_i) of functions of $\mathfrak{F}$ having an adjoint system (v_i) is denumerable.*

We follow the method used by RIESZ † to prove that an orthogonal system of functions is denumerable. The proof is based on the following theorem: If for

* Until after theorem 1 the subscripts i and j are used to denote not only sequences but any system of functions.

† Comptes Rendus, November, 1906, p. 738.

every sequence $\{f_i\}$ of linearly independent functions selected from a given system

$$\int (f_{i_1} - f_{i_2})^2 \geqq 1 \qquad (i_1, i_2),$$

then the given system is denumerable.

Without loss of generality we may assume that

$$\int v_i^2 = 1 \qquad (i).$$

Let $\{u_i\}$ be any sequence of distinct functions selected from the system (u_i) and let v_i be the corresponding functions of the adjoint system. By Bessel's inequality we have

$$1 = \Big[\int v_{i_1}(u_{i_1} - u_{i_2})\Big]^2 \leqq \int (u_{i_1} - u_{i_2})^2 \qquad (i_1, i_2).$$

The hypotheses of the theorem are satisfied and the system (u_i) is denumerable. When the system (u_i) is complete it cannot be finite, and is therefore denumerably infinite.

Let $\{\varphi_i\}$ and $\{\psi_i\}$ be the systems of normalized orthogonal functions constructed linearly from the functions $\{u_i\}$ and $\{v_i\}$, respectively, of the biorthogonal system (u_i, v_i). Connected with these systems of functions we have the four matrices $(a)_u, (b)_u, (a)_v, (b)_v$. (§ 1.)

Theorem 2. *The necessary and sufficient condition that a given system of functions* $\{u_i\}$ *of* $\mathfrak{F}$ *have an adjoint system is, that the* a_{ik} *of* $(a)_u$ *be of finite norm for every* k.

Suppose the condition is satisfied; then from lemma 3 follows the existence of a system of functions $\{v_i\}$ of $\mathfrak{F}$ such that

$$a_{ik} = \int \varphi_i v_k \qquad (i, k). \tag{4}$$

The equations connecting $\{u_i\}$ and $\{\varphi_i\}$ are

$$\varphi_i = \sum_{k=1}^{i} a_{ik} u_k \qquad (i) \tag{5}$$

and

$$u_i = \sum_{k=1}^{i} b_{ik} \varphi_k. \tag{6}$$

Multiply the equation (6) by v_j and integrate:

$$\int u_i v_j = \sum_{k=1}^{i} b_{ik} \int \varphi_k v_j.$$

By substitution from the equation (4):

$$\int u_i v_j = \sum_{k=1}^{i} b_{ik} a_{kj}.$$

Since the matrices (b_{ik}) and (a_{kj}) are each the inverse of the other we obtain

$$\int u_i v_j = \begin{cases} 0 & i=j, \\ 1 & i+j. \end{cases}$$

Thus by definition the system $\{v_i\}$ is an adjoint system of $\{u_i\}$.

Suppose conversely that an adjoint system $\{v_i\}$ exists. Multiply the equations (5) by v_k and integrate

$$a_{ik} = \int \varphi_i v_k \qquad (i, k).$$

Applying lemma 1, we see that the condition is necessary.

Example. Let $u_i = s^{i-1}$, $a = -1$, $b = +1$. Since the Legendrian polynomials $P_i(s)$ form an orthogonal system of functions for the interval $(-1, +1)$, and since $P_i(s)$ is linear in $1, s, s^2, \cdots, s^i$, it is clear that the functions φ_i defined by the equations (5) are the Legendrian polynomials multiplied by the proper constants. The value* of $a_{2m,1}$ is given by

$$a_{2m,1} = \sqrt{\frac{4m+1}{2}}\,\frac{1\cdot 3\cdot 5\cdots(2m-1)}{2\cdot 4\cdots 2m}.$$

Hence

$$a_{2m,1}^2 > \frac{4m+1}{2(2m+1)}\cdot\frac{2}{\pi}.$$

And the constants $\{a_{2m,1}\}$ cannot be of finite norm. This is sufficient to show that the system of the powers of s has no adjoint system in the interval $(-1, +1)$.

§ 4. *Limited matrices.*

Lemma 1, § 2, can be extended to biorthogonal systems only when certain conditions indicated in the following theorem are satisfied.

Theorem 3. *If $\{f_i\}$ is any sequence of linearly independent functions of $\mathfrak{F}$, the necessary and sufficient condition that the sequence $\left\{\int f f_i\right\}$ be of finite norm for every function f of $\mathfrak{F}$ is that the matrix $(b)_f$ be limited.*

Multiply the second equation of (1) by any function f of $\mathfrak{F}$ and integrate:

$$\int f f_i = \sum_{k=1}^{i} b_{ik} \int \phi_k f \qquad (i, f).$$

By lemma 1, $\left\{\int \phi_k f\right\}$ is of finite norm for every f of $\mathfrak{F}$, and therefore $\left\{\int f f_i\right\}$ is of finite norm for every f of $\mathfrak{F}$ if the matrix $(b)_f$ is limited.

On the other hand, if

$$\left\{\int f f_i\right\}$$

* BYERLY, *Fourier's Series*, pp. 180 and 70.

is of finite norm for every f, then

$$\left\{\sum_{k=1}^{i} b_{ik}x_k\right\}$$

is of finite norm for every $\{x_i\}$ of finite norm because

$$\int ff_i = \sum_{k=1}^{i} b_{ik}x_k,$$

where f is the function of $\mathfrak{F}$ which by lemma 3 corresponds to $\{x_i\}$.

Remark. We can always multiply the functions $\{f_i\}$ by constants $\lambda_i \neq 0$ so that

$$\left\{\lambda_i \int ff_i\right\} \qquad (f)$$

is of finite norm. This is evident since it follows from Bessel's inequality that

$$\sum_{i=1}^{\infty}\left(\lambda_i \int ff_i\right)^2 \leqq \int f^2 \sum_{i=1}^{\infty}\lambda_i^2 \int f_i^2 \qquad (f).$$

Corollary 1. If $\{f_i\}$ is any sequence of linearly independent functions of $\mathfrak{F}$, the necessary and sufficient condition that the sequence $\left\{\int ff_i\right\}$ be of finite norm for every function f of $\mathfrak{F}$ is that

$$\Sigma\left(\int ff_i\right)^2 \leqq M\int f^2,$$

where M is a constant depending only on the sequence $\{f_i\}$.

This is a consequence of the property of a limited matrix that

$$\sum_i\left(\sum_k a_{ik}x_k\right)^2 \leqq M\Sigma x_i^2,$$

where M does not depend on Σx_i^2 (lemma 4).

Corollary 2. If $\{u_i\}$ and $\{v_i\}$ form a biorthogonal system of functions, the necessary and sufficient condition that $\left\{\int fu_i\right\}$, $\left\{\int fv_i\right\}$, respectively, be of finite norm $\leqq M\int f^2$ for every function f of $\mathfrak{F}$ is that $(b)_u$, $(b)_v$, respectively, be limited.

Theorem 4. *If the biorthogonal system* (u_i, v_i) *is complete as to* u, v, *respectively, the necessary and sufficient condition that* $\left\{\int fv_i\right\}$, $\left\{\int fu_i\right\}$, *respectively, be of finite norm for every function* f *of* $\mathfrak{F}$ *is that the matrices* $(a)_u$, $(a)_v$, *respectively, be limited.*

We prove the theorem for the case that $\{u_i\}$ is complete. From the equations (5) it is clear that the system $\{\varphi_i\}$ is also complete. Applying lemma 2 and the equations (4) we obtain the following equations:

$$\int fv_i = \sum_{k=1}^{\infty} a_{ki}\int \varphi_k f \qquad (i, f). \tag{7}$$

The remainder of the proof is precisely the same as for theorem 3.

Corollary. When the biorthogonal system (u_i, v_i) is complete as to u, v, respectively, a limited matrix $(a)_u$, $(a)_v$, respectively, implies a limited matrix $(b)_v$, $(b)_u$, respectively, and conversely.

Theorem 5. *If* $\{f_i\}$ *is any sequence of linearly independent functions of* $\mathfrak{F}$, *the necessary and sufficient condition, that for every sequence of constants* $\{c_i\}$ *of finite norm there may exist a function* f *of* $\mathfrak{F}$ *such that*

$$c_i = \int f f_i \qquad (i),$$

is that the matrix $(a)_f$ *be limited.*

Multiply the first set of equations (1) by an arbitrary function f of $\mathfrak{F}$ and integrate:

$$\int f\phi_i = \sum_{k=1}^{i} a_{ik} \int f_k f \qquad (i). \tag{8}$$

Suppose first that the matrix $(a)_f$ is limited; then the sequence of constants

$$\left\{ \sum_{k=1}^{i} a_{ik} c_k \right\}$$

is of finite norm and there exists [lemma 3] a function f of $\mathfrak{F}$ such that

$$\int f \phi_i = \sum_{k=1}^{i} a_{ik} c_k \qquad (i).$$

Taking into consideration the properties (2), § 1, it is clear from the equations (8) that

$$c_i = \int f_i f \qquad (i).$$

We now make the assumption that to every sequence of constants $\{c_i\}$ of finite norm there corresponds a function f of $\mathfrak{F}$ such that

$$c_i = \int f_i f \qquad (i).$$

By means of the equations (8) and lemma 1 we find that

$$\sum_{k=1}^{i} a_{ik} c_k$$

is of finite norm for every sequence $\{c_i\}$ of finite norm, and thus by definition the matrix $(a)_f$ is limited.

If the system $\{f_i\}$ is complete, then

$$\int f^2 \leqq M \sum_i c_i^2.$$

Corollary. If $\{u_i\}$ and $\{v_i\}$ form a biorthogonal system, the necessary and sufficient condition, that for every sequence of constants $\{c_i\}$ of finite norm there exists a function f of $\mathfrak{F}$ such that for every i, $c_i = \int u_i f$, $c_i = \int v_i f_i$, respectively, is that the matrix $(a)_u$, $(a)_v$, respectively, be limited.

Theorem 6. *If* $\{u_i\}$ *and* $\{v_i\}$ *form a biorthogonal system of functions, a limited matrix* $(b)_v$, $(b)_u$, *respectively, implies a limited matrix* $(a)_u$, $(a)_v$, *respectively.*

We give the proof for a limited matrix $(b)_v$. The matrix $(b)_v$ is formed from the coefficients in the equations

$$v_i = \sum_{j=1}^{i} b_{ij}\psi_j \qquad (i).$$

Multiply these equations by φ_k and integrate:

$$a_{ki} = \sum_{j=1}^{i} b_{ij} \int \psi_j \varphi_k \qquad (i, k), \tag{9}$$

where by equations (4) a_{ki} are the elements of the matrix $(a_{ki})_u$. The matrix $(a_{ki})_u$ is the product of the two matrices $(b_{ij})_v$ and $\left(\int \psi_j \varphi_k\right)$ which are both limited (lemma 7), and is therefore (lemma 6) itself limited.

For a biorthogonal system (u_i, v_i), the functions u_i can be multiplied by constants $\lambda_i \neq 0$ so that the resulting matrix $(a)_{\lambda u}$ is limited.

Corollary 1. If $\{u_i\}$ and $\{v_i\}$ form a biorthogonal system such that the sequence $\left\{\int f v_i\right\}$ is of finite norm for every f of $\mathfrak{F}$, then for every sequence of constants $\{c_i\}$ of finite norm there exists a function g of $\mathfrak{F}$ such that

$$c_i = \int g u_i.$$

Corollary 2. When the biorthogonal system (u_i, v_i) is complete as to u, the elements of the matrices $(a)_u$ and $(b)_v$ are connected by the relations

$$\sum_k a_{ki}^2 = \sum_k b_{ik}^2 \qquad (i). \tag{10}$$

A similar relation exists between the elements of the matrices $(a)_v$ and $(b)_u$ when the biorthogonal system is complete as to v.

We give below four examples to illustrate the different cases which may arise.

Example 1. Let $\{\phi_i\}$ be a system of normalized orthogonal functions for the interval I, and $\{a_i\}$ a system of constants such that $|a_i| \leqq M$ where M is some finite positive quantity. Then $\{u_i\}$ and $\{v_i\}$, defined by

$$u_{2j-1} = \phi_{2j-1}, \qquad u_{2j} = \phi_{2j} + a_j \phi_{2j-1}, \qquad v_{2j-1} = \phi_{2j-1} - a_j \phi_{2i}, \qquad v_{2j} = \phi_{2j},$$

form a biorthogonal system (u_i, v_i) whose matrices $(b)_u$, $(b)_v$ and therefore $(a)_v$, $(a)_u$ are limited.

Example 2. The same biorthogonal system as in example 1, except that the constants $\{a_i\}$ are such that $\{1/a_i\}$ is of finite norm. In this case neither of the matrices $(b)_u$ and $(b)_v$ is limited.

Example 3. Let $\{\phi_i\}$ be a system of normalized orthogonal functions for the interval I: $\{a_i\}$ a sequence of constants such that $\{1/a_i\}$ is of finite norm. For the biorthogonal system (u_i, v_i) defined by the equations

$$u_i = a_i \phi_i \qquad (i),$$

$$v_i = \frac{\phi_i}{a_i} \qquad (i),$$

the matrix $(b)_v$ is limited, but the matrix $(b)_u$ is not.

Example 4. The systems $\{\phi_i\}$ and $\{a_i\}$ are conditioned as in example (3). For the biorthogonal system (u_i, v_i) defined by the equations

$$u_i = \phi_{2i-1} \qquad (i),$$

$$v_i = \phi_{2i-1} + a_i \phi_{2i} \qquad (i),$$

the matrix $(a)_u$ is limited but the matrix $(b)_v$ is not. In this example the matrices $(a)_v$ and $(b)_u$ are also limited but the system $\{u_i\}$ is not complete: compare with theorem 4.

§ 5. *Convergence and evaluation of*

$$\sum_{i=1}^{\infty} \int f_1 u_i \int v_i f_2 .$$

Definition. For the biorthogonal system (u_i, v_i), $\mathfrak{G}$ is the class of all functions g for which there exists a system of constants $\{\mu_i \neq 0\}$ such that $\left\{\mu_i \int g u_i\right\}$ is of finite norm, and $\left\{\int f v_i/\mu_i\right\}$ is of finite norm for *every function* f of $\mathfrak{F}$.

Example. For the biorthogonal system given in example 2, § 4, the class $\mathfrak{G}$ is the class of all functions g such that $\left\{c_i \int g u_i\right\}$ is of finite norm, where $c_{2i-1} = a_i$, $c_{2i} = 1$. It is evident that the matrix $(b)_{v/\mu}$ is limited. Any other system of constants $\{\mu_i\}$ which would make $(b)_{v/\mu}$ limited is such that the ratio c_i/μ_i is less than a fixed constant for every i, and therefore for any function g such that $\left\{\mu_i \int g u_i\right\}$ is of finite norm, the sequence $\left\{c_j \int g u_i\right\}$ is also of finite norm.

When $(b)_v$ is limited, $\mathfrak{G}$ contains the class of all functions g of $\mathfrak{F}$ such that $\left\{\int g u_i\right\}$ is of finite norm.

Definition. For the biorthogonal system (u_i, v_i), $\mathfrak{H}$ is the class of all functions h for which there exists a system of constants $\{\nu_i \neq 0\}$ such that $\left\{\nu_i \int h v_i\right\}$ is of finite norm, and $\left\{\int f u_i/\nu_i\right\}$ is of finite norm for *every function* f of $\mathfrak{F}$.

When $(b)_u$ is limited, $\mathfrak{H}$ contains the class of all functions h of $\mathfrak{F}$ such that $\left\{\int h v_i\right\}$ is of finite norm.

In the definition the function g is assumed to belong to the class $\mathfrak{F}$, but any function g satisfying those conditions must belong to $\mathfrak{F}$ since the matrix $(b)_{v/\mu}$ and therefore the matrix $(a)_{\mu\nu}$ corresponding to g is limited, and by the equations (5), $\left\{\int g\varphi_i\right\}$ is of finite norm and therefore g belongs to $\mathfrak{F}$. Similarly the functions h must necessarily belong to $\mathfrak{F}$.

Theorem 7. *If (u_i, v_i) is a biorthogonal system, then for every function g of $\mathfrak{G}$ and for every function f of $\mathfrak{F}$ the expression*

$$\sum_{i=1}^{\infty} \int g u_i \int v_i f$$

is convergent; and for every function h of $\mathfrak{H}$ and every function f of $\mathfrak{F}$ the expression

$$\sum_{i=1}^{\infty} \int f u_i \int v_i h$$

is convergent.

The convergence follows directly from the two inequalities:

$$\left(\sum_{i=1}^{\infty} \int g u_i \int v_i f\right)^2 \leqq \sum_{i=1}^{\infty} \left(\mu_i \int g u_i\right)^2 \sum_{i=1}^{\infty} \left(\frac{\int f v_i}{\mu_i}\right)^2 \qquad (f, g),$$

$$\left(\sum_{i=1}^{\infty} \int f u_i \int v_i h\right)^2 \leqq \sum_{i=1}^{\infty} \left(\frac{\int f u_i}{\nu_i}\right)^2 \sum_{i=1}^{\infty} \left(\nu_i \int h v_i\right)^2 \qquad (f, h).$$

Theorem 8. *When the biorthogonal system (u_i, v_i) is complete as to u, or when the functions v_i and g are orthogonal to the functions which together with the u_i form a complete system, the integral $\int g f$ may be represented as follows:*

$$\int g f = \sum_{i=1}^{\infty} \int g u_i \int v_i f \qquad (f, g); \tag{11}$$

when it is complete as to v, or when the functions u_i and h are orthogonal to the functions which together with the v_i form a complete system, the integral $\int h f$ may similarly be represented:

$$\int h f = \sum_{i=1}^{\infty} \int f u_i \int v_i h \qquad (f, h). \tag{12}$$

We give the proof for the first part of the theorem.

From the definition of $\mathfrak{G}$ the matrix $(b)_{\nu|\mu}$ is limited, and therefore by theorem 6 the matrix $(a)_{\mu u}$ is limited.

The elements of the limited matrix $(a)_{\mu u}$ are a_{ik}/μ_k. Consider the bilinear form

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{ik}}{\mu_k}x_k y_i.$$

Make the substitution

$$x_k = \mu_k \int g u_k \qquad (k),$$

$$y_i = \int \varphi_i f \qquad (i).$$

Since

$$\frac{\int f v_k}{\mu_k} = \sum_{k=1}^{\infty}\frac{a_{ik}}{\mu_k}\int \varphi_i f \qquad (k)$$

we obtain from summation by rows

$$\sum_{k=1}^{\infty}\int g u_k \int v_k f.$$

Since

$$\int f\varphi_i = \sum_{k=1}^{i}\frac{a_{ik}}{\mu_k}\mu_k \int u_k f$$

we obtain from summation by columns

$$\sum_{i=1}^{\infty}\int g\varphi_i \int \varphi_i f.$$

By lemma 7 these two sums are equal, and applying lemma 2 we have the desired result:

$$\int gf = \sum_{i=1}^{\infty}\int g u_i \int v_i f \qquad (f, g).$$

Corollary 1. When the biorthogonal system (u_i, v_i) is complete as to u, then

$$\int g^2 = \sum_{i-1}^{\infty}\int g u_i \int v_i g \qquad (g) \tag{13}$$

when it is complete as to v, then

$$\int h^2 = \sum_{i=1}^{\infty}\int h u_i \int v_i h \qquad (h). \tag{14}$$

These relations do not necessarily hold for every function f of $\mathfrak{F}$ unless $\mathfrak{G} = \mathfrak{F}$ and $\mathfrak{H} = \mathfrak{F}$, respectively. This is illustrated by the example at the end of this section.

Corollary 2. If the function p is orthogonal to all the functions v_i, u_i respectively, then it is orthogonal to all the functions of the classes $\mathfrak{G}$, $\mathfrak{H}$ respectively.

Corollary 3. If the biorthogonal system (v_i, u_i) is complete as to u, v, respectively, and the function p is orthogonal to all the functions v_i, u_i, respectively, then the matrices $(b)_u$ and $(b)_v$ cannot both be limited.

Theorem 9. *When the biorthogonal system (u_i, v_i) is complete as to u, v respectively there exists no function in $\mathfrak{G}$, $\mathfrak{H}$, respectively, essentially different from zero which is orthogonal to the system $\{v_i\}$, $\{u_i\}$, respectively.*

This is an immediate consequence of the equations (13) and (14).

Corollary. When $\mathfrak{G} = \mathfrak{F}$, $\mathfrak{H} = \mathfrak{F}$, respectively, a complete system $\{u_i\}$, $\{v_i\}$ respectively, implies a complete system $\{v_i\}$, $\{u_i\}$ respectively.

Theorem 10. *If the biorthogonal system (u_i, v_i) is complete as to u, v, respectively and the functions u_i, v_i, respectively, belong to the class $\mathfrak{G}$, $\mathfrak{H}$, respectively, then the system $\{v_i\}$, $\{u_i\}$ respectively is also complete.*

A function f of $\mathfrak{F}$, orthogonal to the functions v_i, would be orthogonal to the functions $\{u_i\}$ also, as is seen from the equations (11).

The following example is that of a biorthogonal system (u_i, v_i) complete as to u but not complete as to v.

Example. Let $\{\phi_i\}$ form a complete system of normalized orthogonal functions, and let $p \neq 0$ be a function belonging to $\mathfrak{F}$ but linearly dependent on no finite number of functions ϕ_i and therefore

$$\int p^2 - \sum_{k=i}^{n} \left(\int p\phi_k \right)^2 > 0 \qquad (n).$$

Construct the functions

(15)
$$u_i = \frac{\left[\int p^2 - \sum_{k=1}^{i-1} \left(\int p\phi_k \right)^2 \right] \phi_i + \int p\phi_i \sum_{k=1}^{i-1} \phi_k \int p\phi_k}{\sqrt{\int p^2 - \sum_{k=1}^{i} \left(\int p\phi_k \right)^2} \sqrt{\int p^2 - \sum_{k=1}^{i-1} \left(\int p\phi_k \right)^2}} \qquad (i)$$

$$v_i = u_i - \frac{p \int p u_i}{\int p^2} \qquad (i).$$

The functions u_i and v_i form a biorthogonal system (u_i, v_i) complete as to u but the function p is orthogonal to all the functions v_i. The functions v_i form a system of orthogonal and normalized functions and therefore the matrix $(b)_v$ is limited. The matrix $(b)_u$ is not limited; otherwise $\int p^2$ could be expressed by

$$\int p^2 = \sum_{i=1}^{\infty} \int p u_i \int v_i p,$$

which is impossible.

This construction could be generalized, so that the system $\{u_i\}$ would be complete and all the functions v_i orthogonal to a finite number of functions $p_1, p_2, \cdots, p_n$.

§ 6. *Associated functions.*

Definition. The function f_1 is a *vu-associated* function of the function f with respect to the biorthogonal system (u_i, v_i) if it satisfies the following relation:

$$\int f v_i = \int u_i f_1 \qquad (i). \tag{16}$$

Remark 1. If the system $\{u_i\}$ is complete, then f_1 is the only *vu*-associated function of f, and for a function p orthogonal to all the functions v_i the *vu*-associated function is zero, and the function $f + \mu p$, where μ is an arbitrary constant, has the same *vu*-associated function as f.

Remark 2. The functions v_i are *vu*-associated functions of u_i. For an orthogonal system a *vu*-associated function is the function itself.

Remark 3. The sum of two functions, which are *vu*-associated functions of two given functions, is a *vu*-associated function of the sum of the two given functions.

Definition. The function f_1 is a *uv-associated* function of the function f with respect to the biorthogonal system (u_i, v_i) if it satisfies the following relation:

$$\int f_1 v_i = \int u_i f \qquad (i). \tag{17}$$

Remark 4. A statement of the immediate consequences of this definition is obtained by interchanging u and v in the remarks (1), (2), and (3).

Theorem 11. *If the biorthogonal system (u_i, v_i) has a limited matrix $(b)_v$, $(b)_u$, respectively, then for every function f of $\mathfrak{F}$ there exists a vu-associated g which belongs to $\mathfrak{G}$, a uv-associated function h which belongs to $\mathfrak{H}$, respectively.*

This theorem is a consequence of the theorems 3, 5, and 6. Take any function f of $\mathfrak{F}$ and let $c_i = \int f v_i$; by theorem 3, the sequence $\{c_i\}$ is of finite norm, and by theorems 6 and 5 there exists a function g of $\mathfrak{G}$ such that

$$c_i = \int f v_i = \int u_i g \qquad (i).$$

Corollary. If the biorthogonal system (u_i, v_i) has a limited matrix $(b)_v$, then for every function f of $\mathfrak{F}$ there exists a *uv*-associated function when and only when the matrix $(b)_u$ is also limited.

Theorem 12. *If for a biorthogonal system, complete as to u, the vu-associated function g exists for every function f of $\mathfrak{F}$, then the matrix $(b)_v$ is limited.*

This theorem is a consequence of lemma 3 and the equations (5) and (7). Multiply the equations (5) by g and integrate and then substitute from the equations (7) for $\int gu_i$. By lemma 3 the matrix $(\sum u_{ik}u_{jk})$ is limited and by reason of the properties (2), § 2, it follows that $(a)_u$ is also limited. Corollary to theorem 4 shows that $(b)_v$ is limited.

Theorem 13. *If the biorthogonal system (u_i, v_i) is complete as to u and has a limited matrix $(b)_v$, and if g is the vu-associated function of f, then the integral $\int fg$ can be expressed as the sum of squares:*

$$\int fg = \sum_{i=1}^{\infty} \left(\int fv_i \right)^2 \qquad (f). \tag{18}$$

This follows from a direct application of the theorem 8 and equations (11).

Remark. If the biorthogonal system (u_i, v_i) is complete as to u and has a limited matrix $(b)_v$ and if g is the vu-associated function of some function f, and the function p is orthogonal to all the functions v_i, then p is also orthogonal to g. This is a special case of the corollary to theorem 8.

Theorem 14. *If the biorthogonal system (u_i, v_i) is complete as to u and has a limited matrix $(b)_v$, and if g_1 is the vu-associated function of f_1, and g_2 that of f_2, then*

$$\int f_1 g_2 = \int f_2 g_1 \qquad (f_1, f_2). \tag{19}$$

This is a consequence of theorem 8.

If the biorthogonal system (u_i, v_i) has a limited matrix $(b)_v$, then by theorem 11 the vu-associated functions define a functional transformation T,

$$g = T(f) \qquad (f), \tag{20}$$

which is applicable to the class $\mathfrak{F}$, and is single-valued if the system $\{u_i\}$ is complete.

A particular case of equation (20) is

$$v_i = T(u_i) \qquad (i). \tag{21}$$

When the system $\{u_i\}$ is complete, T is linear:

$$T(a_1 f_1 + a_2 f_2) = a_1 T(f_1) + a_2 T(f_2) \qquad (f_1, f_2). \tag{22}$$

In terms of T the theorems 14 and 13 may be expressed as follows:

$$\int [f_1 T(f_2) - f_2 T(f_1)] = 0 \qquad (f_1, f_2) \tag{23}$$

and

$$\int f T(f) \geqq 0 \qquad (f), \tag{24}$$

the equality sign in (24) holding only for $f = 0$ or $T(f) = 0$.

We give below some examples of this functional transformation T.

Example 1. For a complete orthogonal system $T(f)=f$.

Example 2. For the complete biorthogonal system

$$v_i = \alpha u_i \qquad (i),$$

where α is a positive function of $\mathfrak{F}$, the transformation T is

$$T(f) = \alpha f.$$

Example 3. Let (u_i, v_i) be the biorthogonal system defined in the example in § 5. The transformation T is given by

$$T(f) = f - \frac{p\int pf}{\int p^2}.$$

Example 4. Let $K(s, t)$ be a kernel which is continuous, symmetric, has positive characteristic numbers $\{\mu_i^2\}$, and for which there exists no function f in $\mathfrak{F}$ such that $\int K(s, t)f(t)dt = 0$. Let the characteristic functions be $\{\phi_i\}$:

$$\phi_i(s) = \mu_i^2 \int K(s, t)\phi_i(t)dt \qquad (i).$$

The biorthogonal system

$$u_i = \mu_i \phi_i, \qquad v_i = \frac{\phi_i}{\mu_i} \qquad (i)$$

is complete as to u, and has a limited matrix $(b)_v$ because the sequence of numbers $\{1/\mu_i\}$ is of finite norm. The functional transformation T is

$$g(s) = \int K(s, t)f(t)dt.$$

§ 7. *Equivalent biorthogonal systems.*

Definition. The biorthogonal systems (u_i, v_i) and $(\bar{u}_i, \bar{v}_i)$ complete as to u and $\bar{u}$, respectively, and satisfying the relation

$$\int u_i \bar{v}_k = \int v_i \bar{u}_k \qquad (i, k),$$

are called equivalent.

As a trivial case, all orthogonal systems are equivalent. The definition may be applied to some biorthogonal systems which are not complete.

Theorem 15. *If the biorthogonal system (u_i, v_i) is complete as to u and has a limited matrix $(b)_v$, then two biorthogonal systems $(\bar{u}_i, \bar{v}_i)$ and $(\overline{u}_i, \overline{v}_i)$ each equivalent to (u_i, v_i) are equivalent to each other.*

This follows directly from the equations (19).

Theorem 16. *If the biorthogonal systems* (u_i, v_i) *and* $(\bar{u}_i, \bar{v}_i)$ *are complete as to* u *and* $\bar{u}$ *respectively, and if the matrices* $(b)_v$ *and* $(b)_{\bar{v}}$ *are limited, then the necessary and sufficient condition that the matrix* $\int \bar{u}_i v_k$ *be orthogonal, is that the two given systems be equivalent.*

If the two systems are equivalent, the conditions for the orthogonality (§ 2) of the matrix $\int \bar{u}_i v_k$ are satisfied; for

$$\sum_{k=1}^{\infty} \int \bar{u}_i v_k \int v_k \bar{u}_j = \sum_{k=1}^{\infty} \int \bar{u}_i v_k \int u_k \bar{v}_j = \int \bar{u}_i \bar{v}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (i, j),$$

$$\sum_{k=1}^{\infty} \int v_i \bar{u}_k \int \bar{u}_k v_j = \sum_{k=1}^{\infty} \int v_i \bar{u}_k \int \bar{v}_k u_j = \int u_i v_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (i, j).$$

To prove the converse, let $\bar{\bar{v}}_i$ be the vu-associated function of $\bar{u}_i$. Then

$$\int \bar{u}_i \bar{\bar{v}}_j = \sum_{k=1}^{\infty} \int \bar{u}_i v_k \int u_k \bar{\bar{v}}_j = \sum_{k=1}^{\infty} \int \bar{u}_i v_k \int v_k \bar{u}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and therefore

$$\int \bar{u}_i \bar{\bar{v}}_j = \int \bar{v}_j \bar{u}_i \qquad (i, j).$$

Since the system $\{\bar{u}_i\}$ is complete we obtain

$$\bar{\bar{v}}_i = \bar{v}_i \qquad (i)$$

and the two systems are equivalent.

Theorem 17. *If the biorthogonal system* (u_i, v_i) *is complete as to* u *and the matrix* $(b)_v$ *is limited, then for any biorthogonal system* $(\bar{u}_i, \bar{v}_i)$ *complete as to* u *and equivalent to the system* (u_i, v_i) *the matrix* $(b)_{\bar{v}}$ *is limited.*

Let f be any function of $\mathfrak{F}$ and write the equations

$$\int f \bar{v}_i = \sum_{k=1}^{\infty} \int f v_k \int u_k \bar{v}_i \qquad (i).$$

From this system of equations we see that $\left\{\int f \bar{v}_i\right\}$ is of finite norm for every f and therefore the matrix $(b)_{\bar{v}}$ is limited.

Theorem 18. *If the equivalent biorthogonal systems* (u_i, v_i) *and* $(\bar{u}_i, \bar{v}_i)$ *are complete as to* u *and* $\bar{u}$ *respectively, and the matrices* $(b)_v$ *and* $(b)_{\bar{v}}$ *are limited, then the* vu*-associated function* g *of any given function* f *is equal to the* $\bar{v}\bar{u}$*-associated function* $\bar{g}$ *of* f.

We have

$$\int \bar{g} \bar{u}_i = \int f \bar{v}_i = \sum_{k=1}^{\infty} \int f v_k \int u_k \bar{v}_i = \sum_{k=1}^{\infty} \int g u_k \int v_k \bar{u}_i = \int g \bar{u}_i.$$

Since the system $\bar{u}_i$ is complete, $\bar{g} = g$.

Corollary. The functional transformations T and $\bar{T}$, corresponding to two biorthogonal systems (u_i, v_i) and $(\bar{u}_i, \bar{v}_i)$ which satisfy the conditions of theorem 17, are equal.

Remark. If two biorthogonal systems have the same functional transformation T, they are equivalent.

§ 8. *Functional transformations T.*

In this section we consider functional transformations T, which transform every function of $\mathfrak{F}$ into a function of $\mathfrak{F}$ and which are single-valued and possess the following three properties: *

$$T(a_1 f_1 + a_2 f_2) = a_1 T(f_1) + a_2 T(f_2) \qquad (a_1, a_2, f_1, f_2), \tag{22}$$

$$\int [f_1 T(f_2) - f_2 T(f_1)] = 0 \qquad (f_1, f_2), \tag{23}$$

$$\int f T(f) \geqq 0 \qquad (f), \tag{24}$$

the equality sign in (24) holding only for $f = 0$ or $T(f) = 0$.

Theorem 19. *Any biorthogonal system* (u_i, v_i) *such that*

$$v_i = T(u_i) \qquad (i),$$

where T has the three properties (22), (23), (24), *has a limited matrix* $(b)_v$.

Let f be any function of $\mathfrak{F}$; then by the property (24)

$$\int \left(f - \sum_{k=1}^{i} u_k \int u_k T(f) \right) \left(T(f) - \sum_{k=1}^{i} v_k \int v_k f \right) \geqq 0 \qquad (i, f).$$

Transforming this inequality, we obtain

$$\sum_{k=1}^{i} \left(\int v_k f \right)^2 \leqq \int f T(f) \qquad (i, f), \tag{25}$$

and therefore the matrix $(b)_v$ is limited.

Corollary. For any two functions f_1 and f_2 of $\mathfrak{F}$,

$$\left(\int f_1 T(f_2) \right)^2 \leqq \int f_1 T(f_1) \int f_2 T(f_2) \qquad (f_1, f_2). \tag{26}$$

In the inequality (25) make the substitution $i = 1$, $f = f_2$, and

$$v_1 = \frac{T(f_1)}{\int f_1 T(f_1)},$$

where f_1 and f_2 are any two functions of $\mathfrak{F}$, such that $T(f_1) \neq 0$ and $T(f_2) \neq 0$.

* Since T is single-valued, the linearity (22) is a consequence of (23).

Theorem 20. *For any functional transformation T which has the three properties* (22), (23), (24), *and which does not transform every function into zero, there exist biorthogonal systems* (u_i, v_i) *for which*

$$v_i = T(u_i) \qquad (i).$$

Let $\{f_i\}$ be any sequence of linearly independent functions of $\mathfrak{F}$, for which $T(f_i) \neq 0$ and also are linearly independent. Construct the following system:

$$u_i = \frac{f_i - \sum_{k=1}^{i-1} u_k \int u_k T(f_i)}{\sqrt{\int \left(f_i - \sum_{k=1}^{i-1} u_k \int u_k T(f_i)\right)\left(T(f_i) - \sum_{k=1}^{i-1} v_k \int v_k f_i\right)}} \qquad (i),$$

(27)

$$v_i = \frac{T(f_i) - \sum_{k=1}^{i-1} v_k \int v_k f_i}{\sqrt{\int \left(f_i - \sum_{k=1}^{i-1} u_k \int u_k T(f_i)\right)\left(T(f_i) - \sum_{k=1}^{i-1} v_k \int v_k f_i\right)}} \qquad (i).$$

For $i = 1$ it is clear that the denominators do not vanish and that $v_i = T(u_i)$. Assume that the statement is true for $i - 1$; then

$$T(f_i) - \sum_{k=1}^{i-1} v_k \int v_k f_i = T\left(f_i - \sum_{k=1}^{i-1} u_k \int u_k T(f_i)\right),$$

and therefore by (24) the denominators of u and v cannot vanish unless

$$f_i - \sum_{k=1}^{i-1} u_k \int u_k T(f_i) = 0$$

or

$$T(f_i) - \sum_{k=1}^{i-1} v_k \int v_k f_i = 0.$$

Either one of these equations is contrary to the hypothesis on $\{f_i\}$. Therefore

$$v_i = T(u_i) \qquad (i).$$

It can now be argued that the systems $\{u_i\}$ and $\{v_i\}$ form a biorthogonal system. It is easily seen that the relations

$$\int u_i v_j = \begin{cases} 0 & (i \neq j), \\ 1 & (i = j), \end{cases}$$

hold for $i, j = 1, 2$. Assume that they hold for $i, j = 1, 2, \cdots, n-1$ and prove that they then hold for $i, j = 1, 2, \cdots, n$.

Since in a biorthogonal system (u_i, v_i) the functions u_i and also v_i are necessarily linearly independent, the vanishing of the denominator of u_i and v_i gives a sufficient condition that $\{f_i\}$ and $\{T(f_i)\}$ be linearly dependent.

The biorthogonal system (u_i, v_i) defined by the equations (27) has by theorem 18 a limited matrix $(b)_v$; a vu-associated function g of any function f exists and

$$g = T(f).$$

Biorthogonal systems constructed from different sequences $\{f_i\}$ are equivalent or belong to the same type.

There exist functional transformations T with the properties (22), (23), (24) for which there does not exist a complete system of linearly independent functions $\{f_i\}$ such that the functions $T(f_i)$ are linearly independent. For example, the functional transformation

$$T(f) = \int K(s, t) f(t)\, dt,$$

where

$$K(s, t) = \sum_{i=1}^{n} \frac{\phi_i(s)\,\phi_i(t)}{\mu_i^2}$$

transforms every function f into a linear combination of the n functions ϕ_i $(i = 1, \cdots, n)$. If, however, there exists a complete system of linearly independent functions $\{f_i\}$ such that the functions $T(f_i)$ are linearly independent, then a biorthogonal system can be constructed by (27) which is complete as to u. In all cases however,

$$\int gf = \sum_{i=1}^{\infty} \int g u_i \int v_i f.$$

Theorem 21. *A single-valued functional transformation T which has the three properties* (22), (23), *and* (24), *and for which there exists a complete system of linearly independent functions $\{f_i\}$ such that $T(f_i)$ are also linearly independent, is equivalent to a type of biorthogonal systems (u_i, v_i) complete as to u and with limited matrices $(b)_v$.*

The formulæ for u_i and v_i may be expressed in the following form:

$$u_i = \frac{\begin{vmatrix} \int f_1 T(f_1) & \int f_1 T(f_2) & \cdots & \int f_1 T(f_{i-1}) & f_1 \\ \int f_2 T(f_1) & \int f_2 T(f_2) & \cdots & \int f_2 T(f_{i-1}) & f_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \int f_i T(f_1) & \int f_i T(f_2) & \cdots & \int f_i T(f_{i-1}) & f_i \end{vmatrix}}{\sqrt{\begin{vmatrix} \int f_1 T(f_1) & \int f_1 T(f_2) & \cdots & \int f_1 T(f_{i-1}) \\ \int f_2 T(f_1) & \int f_2 T(f_2) & \cdots & \int f_2 T(f_{i-1}) \\ \cdots & \cdots & \cdots & \cdots \\ \int f_{i-1} T(f_1) & \int f_{i-1} T(f_2) & \cdots & \int f_{i-1} T(f_{i-1}) \end{vmatrix} \cdot \begin{vmatrix} \int f_1 T(f_1) & \int f_1 T(f_2) & \cdots & \int f_1 T(f_i) \\ \int f_2 T(f_1) & \int f_2 T(f_2) & \cdots & \int f_2 T(f_i) \\ \cdots & \cdots & \cdots & \cdots \\ \int f_i T(f_1) & \int f_i T(f_2) & \cdots & \int f_i T(f_i) \end{vmatrix}}} \quad (i),$$

$$v_i = \frac{\begin{vmatrix} \int f_1 T(f_1) & \int f_2 T(f_1) & \cdots & \int f_{i-1} T(f_1) & T(f_i) \\ \int f_1 T(f_2) & \int f_2 T(f_2) & \cdots & \int f_{i-1} T(f_2) & T(f_i) \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \int f_1 T(f_i) & \int f_2 T(f_i) & \cdots & \int f_{i-1} T(f_i) & T(f_i) \end{vmatrix}}{\sqrt{\begin{vmatrix} \int f_1 T(f_1) & \int f_2 T(f_1) & \cdots & \int f_{i-1} T(f_1) \\ \int f_1 T(f_2) & \int f_2 T(f_2) & \cdots & \int f_{i-1} T(f_2) \\ \cdot & \cdot & \cdots & \cdot \\ \int f_1 T(f_{i-1}) & \int f_2 T(f_{i-1}) & \cdots & \int f_{i-1} T(f_{i-1}) \end{vmatrix} \cdot \begin{vmatrix} \int f_1 T(f_1) & \int f_2 T(f_1) & \cdots & \int f_i T(f_1) \\ \int f_1 T(f_2) & \int f_2 T(f_2) & \cdots & \int f_i T(f_2) \\ \cdot & \cdot & \cdots & \cdot \\ \int f_1 T(f_i) & \int f_2 T(f_i) & \cdots & \int f_i T(f_i) \end{vmatrix}}} \quad (i).$$

The necessary and sufficient condition that both the systems $\{f_i\}$ and $\{T(f_i)\}$ be linearly dependent is the vanishing of the determinant

$$\begin{vmatrix} \int f_1 T(f_1) & \int f_1 T(f_2) & \cdots & \int f_1 T(f_i) \\ \int f_2 T(f_1) & \int f_2 T(f_2) & \cdots & \int f_2 T(f_i) \\ \cdot & \cdot & \cdots & \cdot \\ \int f_i T(f_1) & \int f_i T(f_2) & \cdots & \int f_i T(f_i) \end{vmatrix} \tag{28}$$

Theorem 22. *If, for a functional transformation T with the properties* (22), (23), *and* (24), *there exists a function p such that $p = \lambda T(p)$, then there exists a biorthogonal system $(\bar{u}_i, \bar{v}_i)$ for which the $\bar{v}\bar{u}$-associated function g of a function f orthogonal to p is given by*

$$g = T(f).$$

Taking into consideration the inequality (26) and the condition imposed on p, we see that the functional transformation

$$\bar{T}(f) = T(f) - \frac{p \int p T(f)}{\int p^2}$$

satisfies the three conditions (22), (23), and (24). By theorem 20 we can construct a biorthogonal system $(\bar{u}_i, \bar{v}_i)$ such that

$$\bar{v}_i = \bar{T}(\bar{u}_i).$$

The $\bar{v}\bar{u}$-associated function g of f is given by

$$g = T(f) - \frac{T(p) \int pf}{\int p^2} = T(f) - \frac{p \int p T(f)}{\int p^2}$$

and therefore if f_1 is orthogonal to p,

$$g_1 = T(f_1).$$

§ 9. *Solutions of* $f = \lambda T(f)$.

Let T be a single-valued functional transformation which transforms every function of $\mathfrak{F}$ into a function of $\mathfrak{F}$ and which has the properties (22), (23), and (24); and let (u_i, v_i) be a biorthogonal system (theorem 20) for which

$$v_i = T(u_i) \qquad (i).$$

Then the matrix $(b)_v$ is limited (theorem 19) and therefore also the matrix $(a)_u$.

The matrix (α_{ik}), where

$$\alpha_{ik} = \sum_{j=1}^{\infty} a_{ij} a_{kj},$$

is limited, by lemma 6, and is symmetric.

In this section we consider the functional equation

$$f = \lambda T(f), \tag{31}$$

where λ is a parameter, and f a function to be determined.

Theorem 23. *If there exists a solution* f *of the equation*

$$f = \lambda T(f),$$

then

$$z_i = \int f\varphi_i \qquad (i)$$

is a solution of finite norm of the infinite system of homogeneous linear equations

$$z_i - \lambda \sum_{k=1}^{\infty} \alpha_{ik} z_k = 0 \qquad (i), \tag{32}$$

and conversely a solution of finite norm of the equations (32) *leads to a solution of the equation* (31).

The functional equation (31) is equivalent to the set of equations

$$\int f u_i = \lambda \int v_i f \qquad (i). \tag{33}$$

By means of the equations (5) and (7) we obtain the result that the equations (33) are equivalent to (32).

Corollary. The functional equation (31) has a solution f when and only when the quadratic form $\sum \alpha_{ik} x_i x_k$ has a characteristic number; the number of the solutions f depends on the multiplicity of the characteristic number.

This corollary follows from lemma 8.

Theorem 24. *Two solutions of*

$$f = \lambda T(f),$$

corresponding to two different values of λ, *are orthogonal.*

Let the two values of λ be λ_1 and λ_2, and the corresponding solutions f_1 and f_2:

$$f_1 = \lambda_1 T(f_1), \qquad f_2 = \lambda_2 T(f_2).$$

Multiply the first equation by f_2 and integrate. By means of the second equation and (23) we obtain

$$\int f_1 f_2 = \frac{\lambda_1}{\lambda_2} \int f_1 f_2,$$

and therefore

$$\int f_1 f_2 = 0,$$

when λ_1 and λ_2 are distinct.

If the kernel $K(s, t)$ is symmetric, positive semi-definite, and satisfies certain continuity conditions, the integral equation

$$f(s) = \lambda \int K(s, t) f(t)\, dt$$

is a special case of the functional equation (31). This integral equation has at least one solution, the multiplicity of a characteristic number is always finite, and if there are an infinite number of characteristic numbers the limit point is infinite.

We give examples in which the spectrum * of the quadratic form $\sum \alpha_{ik} x_i x_k$ consists of a point spectrum with finite or infinite multiplicity, or with a finite or infinite limit point, or of a continuous spectrum.

Example 1. Same as the example in § 5.

The spectrum consists of the one point unity with an infinite multiplicity. Any function orthogonal to p is a solution of the equation (31).

Example 2. Let $\{\phi_i\}$ be a complete system of normalized and orthogonal functions; and let (u_i, v_i) be the biorthogonal system defined by

$$u_i = \sum_{k=1}^{i} \phi_k \qquad (i),$$

$$v_i = \phi_i - \phi_{i+1} \qquad (i).$$

The quadratic form $\sum \alpha_{ik} x_i x_k$ becomes

$$x_1^2 + 2 \sum_{i=2}^{\infty} x_i^2 - 2 \sum_{i=1}^{\infty} x_i x_{i+1}$$

* *H.* IV, p. 172.

and its spectrum* is a continuous spectrum consisting of the values λ between $+1$ and $+\infty$. The equation (31) has no solution f belonging to $\mathfrak{F}$.

Example 3. Same as example 1, § 4, in which the constants a_i approach zero as a limit, and the functions ϕ_i are complete and continuous.

The spectrum consists of a point spectrum with limit point at 1:

$$\lambda_{2j-1} = \frac{2 + a_j^2 + a_j\sqrt{a_j^2+4}}{2}, \qquad \lambda_{2j} = \frac{2 + a_j^2 - a_j\sqrt{a_j^2+4}}{2} \qquad (j).$$

The corresponding solutions of (31) are

$$\alpha_{2j-1} = \frac{\phi_{2j-1}(1-\lambda_{2j-1}) - a_j\phi_{2j}}{(1-\lambda_{2j-1})^2 + a_j^2}, \qquad \alpha_{2j} = \frac{\phi_{2j-1}(1-\lambda_{2j}) - a_j\phi_{2j}}{(1-\lambda_{2j})^2 + a_j^2} \qquad (j).$$

The functions $\{\alpha_i\}$ form a complete system of normalized orthogonal functions.

It is possible to take a_i so small that

$$K(s,t) = \sum_{i=1}^{\infty} \frac{\alpha_i(s)\alpha_i(t)}{1-\lambda_i}$$

converges uniformly; $1/1-\lambda_i$ are then the characteristic numbers of $K(s,t)$. With this restriction imposed on a_i, we obtain for any function f and its vu-associated function g

$$f(s) = g(s) - \int K(s,t)g(t)dt.$$

Since 1 is not a characteristic number of $K(s,t)$ we can solve this equation for g and obtain for T

$$g(s) = f(s) - \int k(s,t)f(t)dt,$$

where $k(s,t)$ is the resolvent of $K(s,t)$. The first equation gives the relation between any function g and its uv-associated function f.

Example 4. Same as example 1, § 4, in which however the constants a_i are all equal to the constant a, the system $\{\phi_i\}$ is complete, and the functions ϕ_i are continuous.

In this case the spectrum consists of the two points:

$$l = \frac{2 + a^2 + a\sqrt{a^2+4}}{2}, \qquad \frac{1}{l} = \frac{2 + a^2 - a\sqrt{a^2+4}}{2},$$

* H. WEYL, *Singuläre Integralgleichungen*, Dissertation, Göttingen, 1908, p. 69.

each of infinite multiplicity. Let the solutions of (31), corresponding to l and $1/l$, be $\{\alpha_i\}$ and $\{\alpha_i\}$ respectively. Each of these systems we may consider as orthogonal and normalized; and each system is orthogonal to the other. Further, the two systems together form a complete orthogonal system of continuous functions; and it is possible to choose systems of constants $\{l_i\}$ and $\{l_i^*\}$ such that they are the characteristic numbers of two symmetric kernels $K(s, t)$ and $K^*(s, t)$, whose corresponding characteristic functions are the $\{\alpha_i\}$ and $\{\alpha_i^*\}$ respectively. Then the functions u_i and v_i satisfy the equation

$$\int [K(s, t) + lK^*(s, t)]\, u_i(t)\, dt = \int [lK(s, t) + K^*(s, t)]\, v_i(t)\, dt \quad (i).$$

This is an integral equation of the first kind in both u_i and v_i. The functional transformation is given by *

$$v_i(s) = 2\int_0^\infty \left(\sum_k (-1)^k \frac{t^{2k+1}}{k!}\, \beta_{2k+1}^{(i)}(s) \right) dt \qquad (i),$$

where

$$\beta_k^{(i)}(s) = l\int [l^{2k} K^{(2k+2)}(s, t) + K^{*(2k+2)}(s, t)]\, u_i(t)\, dt$$

and $K^{(2k+2)}(s, t)$ and $K^{*(2k+2)}(s, t)$ are iterated kernels of K and K^* respectively.

§ 10. *Complete† elementary theory of six properties.*

A biorthogonal system (u_i, v_i) may have one or more of the following six properties: 1) $\{u_i\}$ complete; 2) $\{v_i\}$ complete; 3) $(a)_u$ limited; 4) $(b)_u$ limited; 5) $(a)_v$ limited; 6) $(b)_v$ limited. If we consider all the possibilities of a biorthogonal system possessing or not possessing each one of these six properties, there are $2^6 = 64$ cases to be considered. To establish a complete elementary theory we must, for each of the 64 cases, either show by theorems the impossibility of the case or exhibit an example. In this way we show that there are no general theorems relating to these six properties other than those already obtained.

Theorem 25. *There are no other theorems which are analogous to the corollary to theorem 4, theorem 6, and corollary 3 to theorem 8, and which express interdependence among the six properties mentioned above.*

To denote that a biorthogonal system possesses or does not possess the ith property $(i = 1, 2, \cdots, 6)$ we write $+$ or $-$ in the ith place.

Since by theorem 6 a limited matrix $(b)_v$, $(b)_u$ respectively implies a limited matrix $(a)_u$, $(a)_v$ respectively, the following cases are excluded

$$(\pm, \pm, -, \pm, \pm, +),$$
$$(\pm, \pm, \pm, +, -, \pm),$$

of which 28 are distinct.

* H. Bateman, *Inversion of a definite integral*, Mathematische Annalen, vol. 63 (1907).

† E. H. Moore, *Introduction to a form of General Analysis*, The New Haven Mathematical Colloquium, p. 82.

The corollary to theorem 4 excludes the cases

$$(\pm, \pm, +, \pm, \pm, -),$$

$$(\pm, \pm, \pm, -, +, \pm).$$

Of these there are 11 cases which are new and distinct.

Corollary 3 to theorem 8 excludes the two cases

$$(+, -, +, +, +, +),$$

$$(-, +, +, +, +, +).$$

There remain 23 cases, for which we exhibit examples. Two cases which differ only by an interchange of u and v may be considered as one case.

1) $(+, +, +, +, +, +)$.

Example 1 in § 4, in which the system $\{\phi_i\}$ is complete.

This example shows the impossibility of a theorem to the effect that a biorthogonal system complete both as to u and v cannot have all the matrices limited.

2) $(+, +, +, -, -, +)$,

3) $(+, +, -, +, +, -)$.

Example 3 in § 4, in which the system $\{\phi_i\}$ is complete.

4) $(+, +, -, -, -, -)$.

Example 3 in § 4, in which the system $\{\phi_i\}$ is complete.

5) $(-, -, +, +, +, +)$.

Example 1 in § 4, in which the system $\{\phi_i\}$ is incomplete.

6) $(-, -, -, +, +, -)$,

7) $(-, -, +, -, -, +)$.

Example 3 in § 4, in which the system $\{\phi_i\}$ is incomplete.

8) $(-, -, -, -, -, -)$.

Example 2 in § 4, in which the system $\{\phi_i\}$ is incomplete.

9) $(-, -, +, +, +, -)$,

10) $(-, -, +, -, +, +)$.

Example 4 in § 4, in which the system $\{\phi_i\}$ is incomplete.

11) $(-, -, +, -, -, -)$,

12) $(-, -, -, -, +, -)$.

Examples for the cases 20) and 21) below, in which the functions u_1 and v_1 are omitted.

13) $(-, -, +, -, +, -)$.

$\{\phi_i\}$ is an incomplete system of normalized orthogonal functions. $\{a_i\}$ is a sequence of constants such that $\{1/a_i\}$ is of finite norm. $\{\lambda_i\}$ is a sequence of constants such that $\{1/\lambda_i\}$ is of finite norm and also $\{\lambda_i/a_i\}$ is of finite norm. The biorthogonal system

$$u_i = \lambda_i \phi_{2i-1}, \qquad v_i = (\phi_{2i-1} + a_i \phi_{2i}) \frac{1}{\lambda_i} \tag{i}$$

clearly has a limited matrix $(a)_u$ and unlimited matrices $(b)_u$ and $(b)_v$. That the matrix $(a)_v$ is limited follows from the fact that, for the biorthogonal system $(\bar{u}_i, \bar{v}_i)$:

$$\bar{u}_i = \frac{\lambda_i \phi_{2i}}{a_i}, \qquad \bar{v}_i = v_i \tag{i},$$

the matrix $(b)_{\bar{u}}$ is limited.

14) $(+, -, +, -, +, +)$,

15) $(-, +, +, +, +, -)$.

Example in § 5,

16) $(+, -, +, -, -, +)$,

17) $(-, +, -, +, +, -)$.

The biorthogonal system is given by

$$u_i = \lambda_i \bar{u}_i, \qquad v_i = \frac{\bar{v}_i}{\lambda_i} \tag{i},$$

where $(\bar{u}_i, v_i)$ is the biorthogonal system in the example in § 5, and the sequence $\{1/\lambda_i\}$ is of finite norm.

18) $(+, -, -, +, +, -)$,

19) $(-, +, +, -, -, +)$.

The biorthogonal system is given by

$$u_i = \frac{\bar{u}_i}{\lambda_i}, \qquad v_i = \lambda_i \bar{v}_i \tag{i},$$

where $(\bar{u}_i, \bar{v}_i)$ is the biorthogonal system in the example in § 5, and λ_i are chosen so that $(b)_v$ is not limited and $(b)_u$ is limited.

20) $(+, -, -, -, +, -)$,

21) $(-, +, +, -, -, -)$.

The biorthogonal system is given by

$$u_i = \frac{\bar{u}_i - p}{\lambda_i}, \qquad v_i = \lambda_i \bar{v}_i \tag{i},$$

where $(\bar{u}_i, \bar{v}_i)$ is the biorthogonal system in the example in § 5 and the λ_i are constants such that $\{1/\lambda_i\}$ is not, but $\{1/\lambda_i^2\}$ is, of finite norm.

22) $\qquad (+, -, -, -, -, -),$

23) $\qquad (-, +, -, -, -, -).$

The biorthogonal system is given by

$$u_i = \frac{\bar{u}_i\left(\int p^2 - \sum_{k=1}^{i-1}\int p\bar{u}_k\int \bar{v}_k p\right) + \int p\bar{u}_i \sum_{k=1}^{i-1}\bar{u}_k\int p\bar{v}_k}{\sqrt{\left(\int p^2 - \sum_{k=1}^{i}\int p\bar{u}_k\int \bar{v}_k p\right)\left(\int p^2 - \sum_{k=1}^{i-1}\int p\bar{u}_k\int \bar{v}_k p\right)}} \qquad (i),$$

$$v_i = \frac{\left(\bar{v}_i - \frac{p\int p\bar{v}_i}{\int p^2}\right)\left(\int p^2 - \sum_{k=1}^{i-1}\int p\bar{u}_k\int \bar{v}_k p\right) + \int p\bar{v}_i \sum_{k=1}^{i-1}\left(\bar{v}_k - \frac{p\int p\bar{v}_k}{\int p^2}\right)\int p\bar{u}_k}{\sqrt{\left(\int p^2 - \sum_{k=1}^{i}\int p\bar{u}_k\int \bar{v}_k p\right)\left(\int p^2 - \sum_{k=1}^{i-1}\int p\bar{u}_k\int \bar{v}_k p\right)}} \qquad (i),$$

where

$$\bar{u}_{2j} = a_j\phi_{2j}, \qquad \bar{u}_{2j+1} = \phi_{2j+1} + a_j^2\phi_{2j},$$
$$\bar{v}_{2j} = \frac{\phi_{2j} - a_j^2\phi_{2j+1}}{a_j}, \qquad \bar{v}_{2j+1} = \phi_{2j+1} \qquad (j),$$

and $\{\phi_i\}$ is a complete system of normalized orthogonal functions, $\{a_i\}$ a sequence of constants such that $\{1/a_i\}$ is of finite norm, and p a function such that

$$\int p\phi_{2j} = \frac{1}{a_j^3}, \qquad \int p\phi_{2j+1} = \frac{1}{a_j^5}.$$

The two matrices $(b)_{\bar{u}}$ and $(b)_{\bar{v}}$ are unlimited. The function p is orthogonal to all the functions v_i; the system $\{u_i\}$ is complete. The matrices $(b)_v$ and $(a)_u$ are unlimited for $\int p\bar{v}_{2j} = 0$ and therefore $v_{2j} = \bar{v}_{2j}$. Functions f orthogonal to p can be found for which $\left\{\int fu_i\right\}$ is not of finite norm and therefore $(b)_u$ is unlimited. By considering the biorthogonal system $(\bar{\bar{u}}_i, \bar{\bar{v}}_i)$:

$$\bar{\bar{u}}_1 = \frac{p}{\sqrt{\int p^2}}, \qquad \bar{\bar{u}}_{i+1} = u_i - \frac{p\int pu_i}{\int p^2},$$
$$\bar{\bar{v}}_1 = \frac{p}{\sqrt{\int p^2}}, \qquad \bar{\bar{v}}_{i+1} = v_i \qquad (i),$$

which is complete both as to $\bar{\bar{u}}$ and as to $\bar{\bar{v}}$, we find that $(a)_v$ is unlimited.

March, 1910.

APPLICATIONS OF BIORTHOGONAL SYSTEMS OF FUNCTIONS TO THE THEORY OF INTEGRAL EQUATIONS*

BY

ANNA JOHNSON PELL

Introduction.

In this paper we give a sufficient condition that the characteristic numbers of an unsymmetric kernel exist and be real, and prove the expansibility of arbitrary functions in terms of the corresponding characteristic functions. This sufficient condition is stated in terms of a functional transformation† $T(f)$ defined by certain general properties (§ 1), and for the special case $T(f) = f$ we obtain the known theory of the orthogonal integral equations. The method employed is that of infinitely many variables and is based to some extent on an earlier paper.‡ However, the present paper, with the exception of a few footnotes, can be read independently of (I) if Theorems 20 and 8 which are here stated in § 1 are accepted; and if the suggestion in Remark 1 of § 1 is carried out, the only reference to (I) that is necessary is Theorem 20.

§ 1. *Preliminary notation and theorems.*

We denote by I the interval $a \leqq s \leqq b$ of the real variable s, and by S the square

$$a \leqq s \leqq b, \qquad a \leqq t \leqq b,$$

* Presented to the Society, September, 1909.

† Since the original manuscript (the present paper is essentially unchanged in content and method, but somewhat revised in form) was sent to the editors in March, 1910, J. MARTY has announced some results for unsymmetric kernels in the Comptes Rendus. His sufficient condition (April, 1910), which is also necessary (June, 1910) is stated in terms of a special $T(f)$, namely $T(f) = \int K(s, t) f(t) dt$, which satisfies the conditions of § 1 imposed on the $T(f)$ of this paper. Special examples show that the condition (c) of § 1 is more or less essential, see also MARTY's correction (ibid., June, 1910). The method indicated by him is a generalization of that used by E. SCHMIDT to prove the existence and is not essentially different from the method referred to by the author in the footnote at the beginning of § 3. MARTY states no expansion theorems. In May, 1910, the author sent a note to the Bulletin of the American Mathematical Society which appeared in July, 1910, and which gives necessary and sufficient conditions in terms of the general $T(f)$.

‡ A. J. PELL, these Transactions, this number. It will be referred to as (I).

of the real variables s and t. We denote by $\mathfrak{F}$ the class of all real functions of s which are integrable and whose squares are integrable on the interval I in the sense of LEBESGUE, and regard two functions of $\mathfrak{F}$ as equal if they differ only on a set of points of content zero. The argument of the function is omitted whenever possible. The integral

$$\int_a^b \varphi(s)\,ds$$

is denoted by $\int \varphi$. A sequence of real numbers is of finite norm if the sum of the squares is convergent.

Let $T(f)$ be a single valued functional transformation with the following properties:

(*a*) $T(f)$ transforms every function of $\mathfrak{F}$ into a function of $\mathfrak{F}$.

(*b*) For any two functions f_1 and f_2 of $\mathfrak{F}$,

$$\int f_1\,T(f_2) = \int f_2\,T(f_1).$$

From this property follows the linearity of $T(f)$:

$$T(a_1 f_1 + a_2 f_2) = a_1\,T(f_1) + a_2\,T(f_2).$$

(*c*) For every function f of $\mathfrak{F}$,

$$\int f T(f) \geqq 0,$$

the equality sign holding only for functions f such that $T(f) \equiv 0$.

Theorems 20 and 8 of (I) give the existence* of a biorthogonal system (U_i, V_i) such that

(*d*) $$V_i = T(U_i).$$

(*e*) For every function f of $\mathfrak{F}$,

$$\sum_i \left(\int f V_i\right)^2 \leqq \int f T(f).$$

(*f*) For any functions† f_1 and $g_2 = T(f_2)$,

$$\int f_1 g_2 = \sum_i \int f_1 V_i \int U_i g_2.$$

For the purposes of this paper we suppose further that $T(f)$ has the following properties:

* The theory developed in (I) shows that every biorthogonal system (u_i, v_i) such that $\sum_i \left(\int f v_i\right)^2$ converges for every f of $\mathfrak{F}$ gives rise to a functional transformation $T(f)$ with the properties (*a*), (*b*), (*c*) and such that $v_i = T(u_i)$.

† Theorem 8 of (I) is more general than this in some cases.

(a_1) $T(f)$ transforms every continuous function into a continuous function; and for every continuous function f

$$\int |T(f)| \leqq k \int |f|,$$

where k is a constant independent of f.

(c_1) There is at most* one function p, not essentially zero, such that $T(p) \equiv 0$ and this function p is continuous.

In virtue of (a_1) and (c_1) we can assume that the biorthogonal system (U_i, V_i) is such that

(h) The functions U_i and V_i are continuous and the system U_i is complete.

Remark 1. If the conditions on $T(f)$ were stated for continuous functions only in the form of (a_1), (b) and (c), we could exhibit a biorthogonal system (u_i, v_i) such that the properties (d), (e) and (h) hold for continuous functions by referring to Theorem 20 of (I), and the property (f) could be shown to hold for all continuous functions f_1 and f_2 by a simple generalization of HILBERT's proof for the analogous theorem for orthogonal functions (H, V, pp. 443–445).† The generalization consists of the systematic substitution of $\int fT(f)$ for $\int f^2$.

Remark 2. If $K(s, t)$ is continuous on S, then the transformed function with respect to either variable is also continuous on S. This follows from the conditions ‡ (a_1) and (b) and a theorem § by Professor E. H. MOORE, namely; the necessary and sufficient condition that the sequence of continuous functions $f_n(s)$ converge uniformly is, that for every e there exist an n_e depending on e such that for $n_1 \geqq n_e$ and $n_2 \geqq n_e$

$$\left| \int (f_{n_1} - f_{n_2}) f \right| \leqq k_1 \int |f|$$

for every continuous function f.

Remark 3. For every function $K(s, t)$ continuous on S, we have

$$T_s \int K(s, t)\, dt - \int T_s K(s, t)\, dt = 0,$$

* Throughout the paper p denotes a function such that $T(p) \equiv 0$. The condition (c_1) is added for simplicity; the results obtained would hold with slight modifications if there were more than one such function p. See Remark (7), § 3.

† HILBERT's fourth and fifth memoirs in Göttinger Nachrichten are referred to as H, IV, and H, V.

‡ These conditions are used as follows:

$$\left| \int f(t)[T_t K(s', t) - T_t K(s, t)]\, dt \right| = \left| \int T[f(t)][K(s', t) - K(s, t)]\, dt \right|$$
$$\leqq k_1 \int |T(f)| \leqq k k_1 \int |f|.$$

§ *Introduction to a form of General Analysis*, in The New Haven Mathematical Colloquium (1910), p. 5; Atti del IV congresso internazionale dei matematici, vol. 2 (1909), p. 103.

where T_s denotes that T operates on the argument s. To prove this, multiply the left-hand side of the equation by an arbitrary continuous function, integrate and apply property (b).

In a similar manner it can be shown that

$$T_s T_t K(s, t) = T_t T_s K(s, t).$$

A particular case of this is that $T_s T_t K(s, t)$ is symmetric if $K(s, t)$ is symmetric.

§ 2. *Integral equation with general kernel.*

Let $K(s, t)$ be a function continuous in S, f a function continuous on I, $T(f)$ the functional transformation defined in § 1and u an unknown function. Consider the integral equations

$$u(s) + \int K(s, t)\, T[u(t)]\, dt = f(s), \tag{1}$$

$$u(s) + \int K(s, t)\, T[u(t)]\, dt = 0, \tag{2}$$

and their adjoints

$$u(s) + \int K(t, s)\, T[u(t)]\, dt = f(s), \tag{3}$$

$$u(s) + \int K(t, s)\, T[u(t)]\, dt = 0. \tag{4}$$

By means of the biorthogonal system (U_i, V_i) defined in § 1 we establish a one-to-one correspondence between the equations (1), (2), (3), (4) and a system of linear equations in infinitely many variables α_i,

$$\alpha_i + \sum_{j=1}^{\infty} k_{ji}\alpha_j = a_i, \tag{5}$$

$$\alpha_i + \sum_{j=1}^{\infty} k_{ji}\alpha_j = 0, \tag{6}$$

and their adjoints. In these equations the sequence $\{a_i\}$ is of finite norm and

$$k_{ij} = \int\int K(s, t) V_i(t) V_j(s)\, ds\, dt.$$

We proceed by forming the following functions and constants

$$k_i(s) = \int K(s, t) V_i(t)\, dt, \qquad k_{ij} = \int k_i V_j, \qquad a_i = \int f V_i,$$

$$L(s, t) = T_t K(s, t), \qquad L_1(s, t) = T_s K(s, t),$$

$$M(s, t) = T_s L(s, t) = T_t L_1(s, t),$$

$$l_i(s) = T[k_i(s)] = \int L(s, t) V_i(t)\, dt = \int M(s, t) U_i(t)\, dt.$$

The functions k_i and l_i are continuous on I and $L(s,t)$, $L_1(s,t)$ and $M(s,t)$ are continuous on S. From the property (f) of § 1 we obtain

$$\sum_{j=1}^{\infty}[k_j(s)]^2=\int K(s,t)L(s,t)dt,$$

$$\sum_{j=1}^{\infty}[l_j(s)]^2=\int M(s,t)L_1(s,t)dt,$$

$$\sum_{j=1}^{\infty}k_{ij}^2=\int k_i l_i,$$

and since

$$\sum_{i=1}^{n}\sum_{j=1}^{\infty}k_{ij}^2=\sum_{i=1}^{n}\int k_i l_i\leqq\sum_{i=1}^{n}\int(k_i^2+l_i^2)$$

we obtain finally

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}k_{ij}^2\leqq\int\int K(s,t)L(s,t)ds\,dt+\int\int M(s,t)L_1(s,t)ds\,dt.$$

This last inequality shows that the bilinear form

$$\sum k_{ij}x_i y_j$$

is continuous (*H*, IV, p. 203). Therefore Hilbert's theorems (*H*, IV, p. 219, and *H*, V, p. 449) can be applied to the equations (5) and (6).

Theorem 1. *The non-homogeneous equation* (1) *has a unique continuous solution if the homogeneous equation* (2) *has no solution. If the homogeneous equation* (2) *has a continuous solution* u, *it is of finite multiplicity* n, *and the adjoint equation* (4) *has the same number* n *of linearly independent solutions* $\bar{u}$; *and the necessary and sufficient conditions that the non-homogeneous equations* (1) *and* (3) *have solutions are respectively*

$$\int fT\bar{u}_k=0 \qquad (k=1,2,\cdots,n),$$

and

$$\int fT(u_k)=0 \qquad (k=1,2,\cdots,n).$$

After the correspondence has been established the proof follows from an application of Hilbert's theorems mentioned above. If the equation (5) has a solution $\{\alpha_i\}$ of finite norm the function

$$\alpha=\sum_{j=1}^{\infty}\alpha_j k_j \tag{7}$$

is continuous, since from property (e) the series converges uniformly (*H*, V, p. 442). Multiply (7) by V_i and integrate,

$$\int\alpha V_i=\sum_{i=1}^{\infty}\alpha_j k_{ji}=a_i-\alpha_i.$$

Let

$$u = f - \alpha,$$

then

$$\int u V_i = \alpha_i.$$

The function u is continuous and is a solution of the equation (1), for making the substitution

$$f_1(t) = K(s, t), \qquad g_2 = T(u)$$

in the equation (f), § 1, we obtain

$$\int K(s, t)\, T[u(t)]\, dt = \sum_{j=1}^{\infty} \alpha_j k_j(s) = f(s) - u(s).$$

Conversely if u is a continuous solution of (1), the sequence of constants

$$\alpha_i = \int u V_i$$

is of finite norm and satisfies the equation (5), for the series

$$\sum_{i=1}^{\infty} \alpha_i k_i(s) = \int K(s, t) T[u(t)]\, dt$$

is uniformly convergent, and after multiplication by V_i and integration, we obtain the equation (5).

In exactly the same manner the equivalence * of the other sets of equations can be shown.

Applying property (b) of $T(f)$ on the equations (1) and (2), and operating with T on the equations (3) and (4), theorem 1 can be argued from the Fredholm theory.

§ 3. *Integral equation with symmetric kernel.*†

From the hypothesis

$$K(s, t) = K(t, s)$$

follows immediately

$$k_{ij} = k_{ji},$$

* In virtue of theorem 16, (I), any other biorthogonal system of functions satisfying (d), (e) and (f) leads to the same solutions of the integral equations as that obtained by using the system (U_i, V_i) (H, V, p. 451).

† The results obtained in §§ 3, 4 could be obtained by a proper generalization of the method developed by E. SCHMIDT in his dissertation and in Mathematische Annalen, vol. 63. In place of SCHWARZ's inequality we have

$$\left(\int f_1 T(f_2)\right)^2 \leqq \int f_1 T(f_1) \int f_2 T(f_2)$$

(see § 8, I), and for the iterated kernels

$$K^{(n)}(s, t) = \int K(s, r)\, T_r K^{n-1}(r, t)\, dr.$$

where the constants k_{ij} are defined in § 2. By means of an orthogonal transformation of the variables* $\{x_i\}$ the continuous (§ 2) quadratic form,

$$K(x) = \sum k_{ij} x_i x_j,$$

if one assumes that not all of the coefficients k_{ij} are zero, is transformed into the normal form (H, IV, p. 201):

$$K(x) = \sum_{i=1}^{\infty} c_i x_i'^2.$$

All of the coefficients c_i are real. But some may have the value zero, let the corresponding linear forms be

$$x'_{h_i} = \sum_{k=1}^{\infty} m_{ik} x_k = M_i(x).$$

Denote the other values of c_i by $1/\lambda_i$, and the corresponding linear transformations by

$$x'_{g_i} = \sum_{k=1}^{\infty} l_{ik} x_k = L_i(x).$$

The linear forms $L_i(x)$ and $M_i(x)$ are connected by the following relations (H, IV, p. 202):

$$\sum_{j=1}^{\infty} l_{ij} l_{kj} = \begin{cases} 1 & (i = k), \\ 0 & (i \neq k), \end{cases} \tag{8}$$

$$\sum_{j=1}^{\infty} l_{ij} m_{kj} = 0, \tag{9}$$

$$\sum_{i=1}^{\infty} x_i y_i = \sum_{i=1}^{\infty} L_i(x) L_i(y) + \sum_{i=1}^{\infty} M_i(x) M_i(y). \tag{10}$$

The relations connecting the linear forms $L_i(x)$ and $M_i(x)$ and the quadratic form $K(x)$ are as follows:

$$\sum_{j=1}^{\infty} l_{jk} k_{ji} = \frac{l_{ki}}{\lambda_k}, \tag{11}$$

$$\sum_{j=1}^{\infty} m_{jk} k_{ji} = 0. \tag{12}$$

Remark 4. We need to consider the exceptional cases $k_{ij} = 0$ for all i and j. Since

$$k_{ij} = \int\int K(s,t) V_i(s) V_i(t)\, ds\, dt = \int\int T_s T_t K(s,t) U_i(s) U_i(t)\, ds\, dt,$$

it follows from (h) that

$$T_s T_t K(s,t) \equiv 0,$$

* By $\{x_i\}$ and $\{y_i\}$ we denote sequences of finite norm.

and therefore, if there exists a p, $K(s, t)$ has the form

$$K(s, t) = \alpha(s)p(t) + \alpha(t)p(s) + kp(s)p(t),$$

where α is any continuous function and k is any constant.

Theorem 2. *For a symmetric continuous kernel* $K(s, t)$, *which is not identically zero and which is not of the form* $\alpha(s)p(t)+\alpha(t)p(s)+kp(s)p(t)$ *if there is a function* p, *there exists at least one value of* λ *which is necessarily real and for which the integral equation*

$$u(s) = \lambda \int K(s, t)\, T[u(t)]\, dt \tag{13}$$

has a continuous solution $u(s)$ *which is not identically zero. If there is a* p *and if* $K(s, t)$ *has the form*

$$\alpha(s)p(t) + \alpha(t)p(s) + kp(s)p(t),$$

there are no solutions u.

The theorem follows directly from the existence of the constants λ_i and the solutions l_{ik} of (11), and an application of the process employed in § 2. We put

$$u_i(s) = \lambda_i \sum_{j=1}^{\infty} l_{ij} k_j(s) \tag{14}$$

and show that

$$u_i(s) = \lambda_i \int K(s, t) T[u_i(t)]\, dt.$$

Remark 5. If there is a function p, then the existence theorem for the equation (13) is equivalent to an existence theorem for the equation

$$w(s) + \mu p(s) = \lambda \int K(s, t) T[w(t)]\, dt, \qquad \int wp = c, \tag{15}$$

where μ is a parameter and c is any given constant.

Corollary 1. There exists at least one characteristic number λ, which is necessarily real, for an unsymmetric kernel $L(s, t)$ which is of the form $T_t K(s, t)$, where $K(s, t)$ is symmetric and is not of the form

$$\alpha(t)p(s) + \alpha(s)p(t) + kp(s)p(t)$$

if a function p exists. Corresponding to the characteristic numbers λ_i the kernel $L(s, t)$, and the transposed kernel $L(t, s)$, have continuous characteristic functions u_i, and v_i, respectively, which form a biorthogonal system (u_i, v_i) of the type* $T(f)$.

Applying property (b) to the right-hand side of the equation (13) we obtain

$$u(s) = \lambda \int L(s, t) u(t)\, dt. \tag{16}$$

* That is, $v_i = T(u_i)$.

Operating with $T(f)$ on the equation (15) we obtain after substituting $v = T(u) = T(w + \mu p)$

$$v(s) = \lambda \int L(t, s) v(t)\, dt. \tag{17}$$

In both cases the processes are reversible. That the functions u_i and v_i form a biorthogonal system follows from the equations (8).

Remark 6. The existence theorem for the equation (15) is equivalent to an existence theorem for the equations

$$T(w) = \lambda \int M(s, t) w(t)\, dt, \qquad \int wp = c, \tag{18}$$

where c is any given constant, and $M(s, t) = T_s T_t K(s, t)$ and is therefore symmetric (Remark 3).

Remark 7. If there were more than one function p, the exceptional cases in theorem 2 would be given * by $T_s T_t K(s, t) \equiv 0$.

If the functions $f(s)$ and $K(s, t)$ were integrable and with integrable squares, the theorems 1 and 2 would hold for any functional transformation $T(f)$ with the properties (a), (b) and (c); the solutions being integrable with integrable squares.

§ 4. *Development of arbitrary functions.*

Theorem 3. *Any function f expressible in the form* †

$$f(s) = \int K(s, t) g_1(t)\, dt = \int L(s, t) f_1(t)\, dt,$$

where f_1 is any continuous function and $g_1 = T(f_1)$, can be developed into the uniformly convergent series

$$f(s) = \sum_i u_i(s) \int f v_i + \nu_f p(s). \tag{19}$$

In the linear form $M(x)$ make the substitution

$$x_i = k_i(s);$$

the resulting series is uniformly convergent. Multiply by V_j and integrate; from the relation (9) we obtain

$$\int M_i[k(s)]\, V_j(s)\, ds = 0$$

* The functions U_i and p_i would form a complete system.

† The theorem could be stated for any function g_1 such that $\Sigma\left(\int g_1 u_i\right)^2$ converges, or even more broadly in some cases (Theorem 8, (I)).

and therefore, if there is a p,

$$M_i[k(s)] = \sum_{j=1}^{\infty} m_{ij} k_j(s) = \nu_i p(s),$$

otherwise, $M_i[k(s)] = 0$.

In the equation (10) make the substitution

$$x_i = \int g_1 U_i, \qquad y_i = \int K(s, t) V_i(t)\, dt.$$

The result is

$$f(s) = \int K(s, t) g_1(t)\, dt = \int L(s, t) f_1(t)\, dt$$

$$= \sum_i \left(\sum_{j=1}^{\infty} l_{ij} \int g_1 u_j \right) \left(\sum_{j=1}^{\infty} l_{ij} k_j(s) \right) + \nu_f p(s)$$

$$= \sum_i \frac{u_i(s) \int g_1 u_i}{\lambda_i} + \nu_f p(s)$$

$$= \sum_i u_i(s) \int v_i f + \nu_f p(s).$$

Theorem 4. *Any function f expressible in the form*

$$f(s) = \int M(s, t) f_1(t)\, dt = \int L(t, s) g_1(t)\, dt,$$

where f_1 is any continuous function and $g_1 \equiv T(f_1)$, can be developed into the uniformly convergent series*

$$f(s) = \sum_i v_i(s) \int u_i f. \tag{20}$$

This development may be obtained either by substituting in the equation (10)

$$x_i = \int f_1 V_i, \qquad y_i = \int M(s, t) U_i(t)\, dt,$$

or by operating on the development in theorem 3 with the transformation $T(f)$.

Remark 8. The kernel $L(s, t)$ has no characteristic number λ not equal to λ_i; suppose it had, and let u be the corresponding characteristic function, then u could be developed into a uniformly convergent series, but the coefficients $\int u v_i$ would be zero, and therefore $u = 0$ or $u = p$.

* This restriction on the function g_1 is not necessary. See footnote to theorem 3.

Similarly we can show that there are no solutions corresponding to a characteristic number λ_i, which are not linear in the characteristic functions belonging to λ_i.

Remark 9. If there are only a finite number n of characteristic numbers λ_i it follows from the developments (19) and (20) that the kernels have the form

$$K(s,t) = \sum_{i=1}^{n} \frac{u_i(s)u_i(t)}{\lambda_i} + \alpha(s)p(t) + \alpha(t)p(s) + kp(s)p(t),$$

$$L(s,t) = \sum_{i=1}^{n} \frac{u_i(s)v_i(t)}{\lambda_i} + p(s)\beta(t),$$

$$M(s,t) = \sum_{i=1}^{n} \frac{v_i(s)v_i(t)}{\lambda_i}.$$

Theorem 5. *The maximum value which the expression*

$$\left| \int\int K(s,t)\, T[f(s)]\, T[f(t)]\, dsdt \right|$$

can assume for all continuous functions f such that

$$\int f T(f) = 1$$

is equal to the reciprocal of that λ_i which is smallest in absolute value, and this maximum is attained for the corresponding characteristic function.

§ 5. *Examples.*

As examples of the different forms of integral equations for which § 3 gives existence theorems we give the following:

Example 1.

$$u(s) = \lambda \int K(s,t)\alpha(t)u(t)dt,$$

where α is a continuous positive function.

Example 2.

$$u(s) = \lambda \int\int K(s,t)K^*(s,t)u(t_1)\,dt\,dt_1,$$

where $K^*(s,t)$ is a positive-definite continuous kernel.

Example 3.

$$u(s) + \mu q(s) = \lambda \int K(s,t)T[u(t)]dt, \qquad \int uq = 0,$$

where q is an an invariant function of the functional transformation $T(f)$, i. e., $q = kT(q)$. The equations are derived from the equations (15) where

$c = 0$ and the functional transformation * is

$$\bar{T}(f) = T(f) - \frac{q \int q T(f)}{\int q^2},$$

which satisfies all the conditions of § 1.

As a special case of this, for $T(f) = f$, we have the integral equation treated by CAIRNS in his dissertation. †

Example 4. Problems in the calculus of variations in which the condition

$$\int f T(f) = 1$$

replaces the usual condition

$$\int f^2 = 1,$$

give rise to differential-integral equations of the form

$$L(u) + \lambda T(u) = 0,$$

where

$$L(u) = \frac{d\left(p_1 \frac{du}{ds}\right)}{ds} + p_2 u(s),$$

and p_1 is a positive function which together with its first derivative is continuous, and p_2 is a continuous function. The equation above reduces to the integral equation

$$u(s) = \lambda \int G(s, t)\, T[u(t)]\, dt,$$

where $G(s, t)$ is a Green's function for $L(u)$ satisfying certain boundary conditions. This equation has infinitely many solutions, satisfying the same boundary conditions as $G(s, t)$, corresponding to real values of λ, if there are only a finite number of functions p_i such that $T(p_i) \equiv 0$.

A special case is the non-self-adjoint differential equation

$$L[\bar{L}(v)] - \lambda v = 0,$$

where $\bar{L}(v)$ is a self-adjoint differential expression. The adjoint differential equation is

$$\bar{L}[L(u)] + \lambda u = 0.$$

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*See Theorem 16, (I).

† W. D. CAIRNS, *Die Anwendung der Integralgleichungen auf die zweite Variation bei isoperimetrischen Problemen*, Göttingen, 1907.

Example 5. In example (2) let the function $K^*(s, t)$ be equal to the Green's function $G(s, t)$ of example (4). Then there exists at least one value of λ for which the equation

$$L(v) + \lambda \int K(s, t)v(t)dt = 0$$

has a solution v which is continuous together with its first and second derivatives and satisfies the boundary conditions of $G(s, t)$.

The two systems ψ_i and $-\psi_i''$, considered by E. SCHMIDT,* form a biorthogonal system of the type

$$T(f) = \int G(s, t)f(t)dt,$$

where $G(s, t)$ is the Green's function, for $L(v) = -v''$, which vanishes at the ends of the interval.

Example 6. Let $\bar{T}(f)$ be a functional transformation which satisfies the conditions (a), (b), and (a_1) of § 1, but not necessarily (c) and (c_1). Let $K(s, t)$ be a symmetric, continuous kernel which is positive-semi-definite:

$$\int\int_t K(s, t)f(s)f(t)ds\,dt \geqq 0,$$

the equality sign holding at most for one function p. Then the functional transformation

$$T(f) = \int K(s, t)f(t)dt$$

satisfies all the conditions of § 1.

If the integral equation †

$$(A) \qquad u(s) = \lambda \int \bar{T}_t K(s, t)u(t)dt$$

has a solution u for some value of λ, then the adjoint integral equation

$$v(s) = \lambda \int [\bar{T}_t K(s, t)]\, v(t)dt$$

has a solution v such that

$$v = T(u).$$

Two sets of solutions (u_i, v_i) and (u_j, v_j) corresponding to two different values of λ satisfy the condition

$$\int u_i v_j = 0;$$

* Mathematische Annalen, vol. 63 (1907), p. 473.

† A special case, namely $\bar{T}(f) = a \cdot f$, has been considered by MARTY, Comptes Rendus, February 28, 1910, p. 515.

that is, the solutions form a biorthogonal system of the type T, and therefore the values of λ, for which there exist solutions, are real. The existence of real characteristic numbers λ is shown by means of the integral equation

$$(B) \qquad u(s) = \nu \int \bar{T}_s \bar{T}_t K(s, t) T[u(t)]\, dt,$$

which is of the type of the equation (14), for the kernel $\bar{T}_s \bar{T}_t K(s, t)$ is symmetric (Remark 3). It is evident that any solution of (A) for the value λ is a solution of (B) for $\nu = \lambda^2$. And unless $\bar{T}_s \bar{T}_t K(s, t)$ is of the form

$$\alpha(s) p(t) + \alpha(t) p(s) + k p(s) p(t),$$

there exists at least one value of ν which is real and positive and for which the equation (B) has a continuous solution u.

Construct the two functions

$$2u_1(s) = u(s) + \sqrt{\nu} \int T_s K(s, t) u(t)\, dt,$$

$$2u_2(s) = u(s) - \sqrt{\nu} \int T_s K(s, t) u(t)\, dt.$$

The continuous functions u_1 and u_2 are solutions of (A) for $\lambda = \sqrt{\nu}$ and $\lambda = -\sqrt{\nu}$ respectively.

Example 7. In certain cases the integral equation *

$$(\alpha) \qquad u(s) = \lambda \int K(s, t) u(t)\, dt + \mu \int H(s, t) u(t)\, dt,$$

where $K(s, t)$ and $H(s, t)$ are symmetric kernels continuous on S, λ and μ are parameters, and u the function to be determined, may be reduced to the class of integral equations considered in § 3.

Let λ_1 be that characteristic number of $K(s, t)$ which is smallest in absolute value. Then for any value of λ_0 such that

$$|\lambda_0| < |\lambda_1|,$$

there exists at least one value of μ for which the integral equation has a continuous solution u. For the maximum of

$$\left| \iint K(s, t) f(s) f(t)\, ds\, dt \right|,$$

* Max Mason, *Randwertaufgaben bei gewöhnlichen Differentialgleichungen, Dissertation*, Göttingen, 1903, p. 5.

for all continuous functions f such that

$$\int f^2 = 1,$$

is equal to the reciprocal of $|\lambda_1|$, and therefore the functional transformation

$$f(s) - \lambda_0 \int K(s, t) f(t)\, dt$$

satisfies all the conditions of § 1. Since λ_0 is not a characteristic number, it can easily be shown that the inverse transformation,

$$T(f) = f(s) - \lambda_0 \int k(s, t) f(t)\, dt,$$

where $k(s, t)$ is the reciprocal of $K(s, t)$, also satisfies the conditions of § 1. Operating with $T(f)$ on the integral equation (α), where $\lambda = \lambda_0$, we obtain

$$(\beta) \qquad u(s) = \mu \int T_s H(s, t) u(t)\, dt,$$

which is of the type of the equation (17), § 3. From the equation (β) we can pass back to the equation (α), and therefore the statement is proved.

Let $H(s, t)$ be a positive semi-definite kernel, and let λ be any number not equal to a characteristic number of $K(s, t)$; then the functional transformation

$$f(s) - \lambda \int k(s, t) f(t)\, dt$$

possesses all the properties imposed on the functional transformation $\bar{T}(f)$ of Example 6. Operating with this on the equation (α), we obtain an equation of the type discussed in Example 6.

Let λ take on a value equal to a characteristic number $\bar{\lambda}$ of $K(s, t)$ and let $H(s, t)$ be a positive semi-definite kernel which is orthogonal to all the characteristic functions of $K(s, t)$ corresponding to $\bar{\lambda}$. The transformation

$$f(s) - \bar{\lambda} \int K(s, t) f(t)\, dt$$

defines an inverse transformation $\bar{T}(f)$ which has the property (b), § 1, and which is applicable to every function that is orthogonal to all the characteristic functions of $K(s, t)$ corresponding to $\bar{\lambda}$. Operating with this $\bar{T}(f)$ on the equation (α), we obtain an equation of the same form as the one considered in Example 6, and it is not difficult to see that we again have the existence of values of μ besides the obvious one, $\mu = 0$.

The following is an example of the integral equation (α) in which the kernel $H(s, t)$ is not positive-definite, and to the value $\lambda = -1$ there corresponds no *real* value of μ.

$$u(s) = \lambda \int \psi(s)\psi(t)u(t)dt + \mu \int [h_1(s)h_1(t) - 2h_2(s)h_2(t)]u(t)dt,$$

where

$$\int \psi^2 = \int h_1^2 = \int h_2^2 = 1,$$

$$\int h_1 h_2 = 0, \qquad \int h_1 \psi = 0, \qquad \int h_2 \psi = 0.$$

March, 1910.

ON THE UNIFORM CONVERGENCE OF THE DEVELOPMENTS IN BESSEL FUNCTIONS*

BY

CHARLES N. MOORE

In many problems of mathematical physics we are led to the question of whether or not it is possible to represent an arbitrary function by a series of the form

$$A_1' J_\nu(\lambda_1 x) + A_2' J_\nu(\lambda_2 x) + \cdots, \tag{1}$$

where the A's are constants, $J_\nu(x)$ represents a Bessel function of order ν, and the λ's are the roots of the equation

$$l\lambda J_\nu'(\lambda) + hJ_\nu(\lambda) = 0, \tag{2}$$

in which h and l are any constants not both zero. In most cases we find that we have a solution of our problem if the A's can be so determined that:

(a) The series (1) converges uniformly to $f(x)$ throughout any closed interval lying in the interval $0 \leqq x \leqq 1$ and not including a point of discontinuity of $f(x)$, or, in some instances, the origin or the point $x = 1$, and

(b) The sum of the first n terms of the series remains finite for all values of n and all values of x lying in the interval $0 \leqq x \leqq 1$.

The ordinary formal method of determining the coefficients of an expansion of an arbitrary function in terms of normal functions, gives us the following value for A_n'

$$A_n' = \frac{\int_0^1 xf(x)J_\nu(\lambda_n x)\,dx}{\int_0^1 x\left[J_\nu(\lambda_n x)\right]^2 dx}, \tag{3}$$

where λ_n is the nth positive root of equation (2). This determination of the A's is found on investigation to give the desired expansion of $f(x)$ except in the case

$$l\nu + h = 0, \tag{4}$$

when an extra term is required in order that the series may have the proper value.

* Presented to the Society, October 30, 1909 and February 25, 1911.

That the case in which (4) is satisfied is really not an exceptional case at all, but only *seems* so on account of the notation employed, has been pointed out in a previous paper by the writer.* Moreover, as was also brought out in the previous paper, the appearance of an exceptional case can be avoided by introducing the notation

$$F_\nu(\lambda, x) \equiv \left(\frac{x}{2}\right)^\nu \sum_{i=0}^{\infty} \frac{(-1)^i(\lambda x)^{2i}}{2^{2i} i!\, \Gamma(\nu + i + 1)} \equiv \frac{1}{\lambda^\nu} J_\nu(\lambda x), \tag{5}$$

and considering the series

$$\sum_{n=1}^{\infty} F_\nu(\lambda_n, x) \frac{\int_0^1 x f(x) F_\nu(\lambda_n, x)\, dx}{\int_0^1 x\, [F_\nu(\lambda_n, x)]^2 dx} = \sum_{n=1}^{\infty} A_n F_\nu(\lambda_n, x), \tag{6}$$

where the λ's are the successive roots, positive or zero, of the equation

$$\left[l \frac{\partial}{\partial x} F_\nu(\lambda, x) + h F_\nu(\lambda, x) \right]_{x=1} = 0, \dagger \tag{7}$$

arranged in increasing order of magnitude. This series is identical with the series (1) where the λ's are the successive positive roots of (2), except in the case where (4) is satisfied, and in that case only differs from it by having an extra term at the beginning. This extra term is finite and continuous, and hence in all questions involving merely convergence or the remaining finite of the sum of n terms of the series, we may deal with series (1), and it is only necessary to consider series (6) when we come to the question of the value of the development.

It has been shown by several writers‡ that the series (6) will, under very broad restrictions upon $f(x)$, converge uniformly to $f(x)$ throughout any closed interval, lying in the interval $0 < x < 1$ and not including a point of discontinuity of $f(x)$. The question of whether or not, under certain restrictions upon $f(x)$, the series (6) will converge to $f(0)$ at the origin,§ and will converge uniformly to $f(x)$ in an interval including the origin and in an inter-

*Cf. these Transactions, vol. 10 (1909), p. 420. That the exceptional case was such only on account of the notation employed, and that it would cease to be exceptional if we adopted the proper notation, was pointed out to me by Professor BÔCHER, to whose valuable suggestions I owe many ideas that have been of great use in this as well as in the preceding investigation.

†The *positive* roots of this equation are obviously identical with the *positive* roots of equation (2).

‡Cf. DINI, *Serie di Fourier* (Pisa, 1880), pp. 246–269; KNESER, Mathematische Annalen, vol. 63 (1906–07), p. 505; HOBSON, Proceedings of the London Mathematical Society, ser. 2, vol. 7 (1909), p. 359.

§It is only in the case $\nu = 0$ that this question is anything but a trivial one, since for $\nu > 0$ every term of the series vanishes when $x = 0$ and the value of the series is obviously zero for this value of x.

val including the point $x = 1$ has not been settled in the papers referred to; neither has it been shown that, under suitable restrictions upon $f(x)$, the sum of n terms of the series will remain finite for all values of n and all values of x in the interval $0 \leqq x \leqq 1$. HILBERT has shown* for the particular case where $\nu = 0$ and the l of equation (7) is zero that if $f(x)$ vanishes for $x = 1$ and satisfies the rather stringent conditions of being continuous, together with its first and second derivatives, throughout the interval $0 \leqq x \leqq 1$, the series (6) will converge uniformly to $f(x)$ throughout the interval $0 \leqq x \leqq 1$.

For most of the applications to mathematical physics it is essential to know something about the behavior of the series (6) in the neighborhood of the origin, of the point $x = 1$, and of points of discontinuity of $f(x)$, when such exist, in order to be sure that we really have a solution of the physical problem we are discussing. In the majority of these cases, conditions like HILBERT's are too binding, and it is necessary to remove the restriction that the function should have no discontinuity; it is also necessary, even in the particular case that HILBERT treats, to determine the behavior of the series in the neighborhood of the point $x = 1$ when $f(1) \neq 0$.

The object of the present paper is to show that in case $f(x)$ is made up of a number of pieces in each of which $f(x)$ and its derivative $f'(x)$ are continuous, while $f''(x)$ exists, and is finite and integrable, the series (6) will possess the desired properties. These conditions on $f(x)$ are usually met with in the applications; in a later paper the writer proposes to discuss the uniform summability of the series in the neighborhood of $x = 0$ and $x = 1$, thus showing the existence of a solution of the physical problem under extremely broad conditions.†

LEMMA 1. *If in the interval $0 \leqq x \leqq 1$, the function $f(x)$ is continuous, save at a finite number of points at which it has a finite jump, if in each interval of continuity it has a first derivative that is continuous, save at a finite number of points at which it has a finite jump, and if finally in each interval of continuity of the first derivative the function has a second derivative that is finite and integrable, the coefficient A_n' of the general term of the series* (1) *may be written in the form*

$$(8)\qquad \sum_{m=1}^{m=k} M_m \frac{\cos\left[(n\pi + q)c_m - \frac{2\nu+3}{4}\pi\right]}{\sqrt{n}} + Mf(1)\frac{\cos\left[(n\pi+q) - \frac{2\nu+3}{4}\pi\right]}{\sqrt{n}} + \frac{\nu K f(0)}{\lambda_n} + r_n,$$

*Göttinger Nachrichten (1904), *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen* (*Zweite Mitteilung*), p. 229. Previously KNESER had discussed a special case slightly different from the one treated by HILBERT, using similar restrictions upon $f(x)$. (Cf. Archiv der Mathematik und Physik, 3rd ser., vol. 7 (1904), pp. 123–133.)

† Cf. the introduction to the paper in these Transactions referred to above.

where the M's and K are constants, the c's are the points of discontinuity of $f(x)$, and r_n is the general term of an absolutely convergent series.

We have *

$$(9 \qquad \int_0^1 x\,[J_\nu(\lambda_n x)\,]^2 dx = \frac{A^2}{2\lambda_n} + \frac{\psi_1(\lambda_n)}{\lambda_n^2},$$

where A is a constant, and $\psi_1(\lambda_n)$ *is used to represent any function of* λ_n *that remains finite when* λ_n *increases indefinitely.* Hence it follows from (3) that if we subtract from A_n' the term

$$(10) \qquad \frac{2\lambda_n}{A^2}\int_0^1 x f(x) J_\nu(\lambda_n x)\,dx,$$

we shall have left the term

$$(11) \qquad \frac{-\dfrac{2\psi_1(\lambda_n)}{A^2\lambda_n}\displaystyle\int_0^1 x f(x) J_\nu(\lambda_n x)\,dx}{\dfrac{A^2}{2\lambda_n} + \dfrac{\psi_1(\lambda_n)}{\lambda_n^2}}.$$

We will show first that (11) is the general term of an absolutely convergent series. Then, that by subtracting from (10) quantities that are the general terms of absolutely convergent series, we can reduce it to the sum of the first three terms in (8). Our lemma will thus be established.

Let us use $\phi_m(x)$ to represent a function that is continuous in the interval $0 \leqq x \leqq c_{m+1}$, that has in this interval a first derivative that is continuous, save at a finite number of points at which it has a finite jump, and a second derivative that is finite and integrable, and furthermore is such that

$$(12) \qquad \phi_m(x) = f(x) \quad (c_m \leqq x \leqq c_{m+1};\ m = 0, 1, 2, \cdots, k),$$

where for the sake of uniformity we have set

$$c_0 = 0, \qquad c_{k+1} = 1.$$

Then we may write

$$(13) \qquad \int_0^1 x f(x) J_\nu(\lambda_n x)\,dx = \sum_{m=0}^{m=k} \int_0^{c_{m+1}} x\,\phi_m(x) J_\nu(\lambda_n x)\,dx - \sum_{m=1}^{m=k} \int_0^{c_m} x\,\phi_m(x) J_\nu(\lambda_n x)\,dx.$$

But if c is a constant lying in the interval $0 < x < 1$, and $\phi(x)$ is a function that is continuous in the interval $0 \leqq x \leqq c$ and has a first derivative that is continuous in this interval save at a finite number of points, we have,† if we set $\lambda_n x = y$ and integrate by parts, .

* Cf. equation (46) of the writer's article in these Transactions, just referred to.

† Cf. BYERLY, *An Elementary Treatise on Fourier's Series, etc.*, p. 223, formula (4).

$$\int_0^c x\phi(x)J_\nu(\lambda_n x)\,dx = \frac{1}{\lambda_n^2}\int_0^{c\lambda_n} y^{\nu+1}J_\nu(y)\frac{1}{y^\nu}\phi\left(\frac{y}{\lambda_n}\right)dy$$

$$= \frac{1}{\lambda_n^2}\left[y^{\nu+1}J_{\nu+1}(y)\frac{1}{y^\nu}\phi\left(\frac{y}{\lambda_n}\right)\right]_0^{c\lambda_n}$$

$$-\frac{1}{\lambda_n^2}\int_0^{c\lambda_n} y^{\nu+1}J_{\nu+1}(y)\left\{-\frac{\nu}{y^{\nu+1}}\phi\left(\frac{y}{\lambda_n}\right)+\frac{1}{\lambda_n y^\nu}\phi'\left(\frac{y}{\lambda_n}\right)\right\}dy$$

$$(14)\qquad = \frac{1}{\lambda_n}cJ_{\nu+1}(c\lambda_n)\phi(c)+\frac{\nu}{\lambda_n^2}\int_0^{c\lambda_n} J_{\nu+1}(y)\phi\left(\frac{y}{\lambda_n}\right)dy$$

$$-\frac{1}{\lambda_n^3}\int_0^{c\lambda_n} yJ_{\nu+1}(y)\phi'\left(\frac{y}{\lambda_n}\right)dy$$

$$= \frac{c\phi(c)J_{\nu+1}(c\lambda_n)}{\lambda_n}+\frac{\nu}{\lambda_n}\int_0^c J_{\nu+1}(\lambda_n x)\phi(x)\,dx$$

$$-\frac{1}{\lambda_n}\int_0^c xJ_{\nu+1}(\lambda_n x)\phi'(x)\,dx.$$

Moreover, we have from the asymptotic expansion * of $J_{\nu+1}(x)$

$$(15)\qquad J_{\nu+1}(\lambda_n x) = \frac{\psi(\lambda_n x)}{\lambda_n^{\frac{1}{2}}x^{\frac{1}{2}}},$$

where *we use* $\psi(x)$ *to represent any function of* x *that is continuous for all values of* $x > 0$, *remains finite for all such values of* x, *and approaches a finite limit when* x *approaches zero.*

From (14) and (15) we have

$$(16)\qquad \left|\int_0^c x\phi(x)J_\nu(\lambda_n x)\,dx\right| < \frac{K_1}{\lambda_n^{\frac{3}{2}}},$$

where K_1 is a positive constant. Combining (16) and (13), we obtain

$$(17)\qquad \left|\int_0^1 xf(x)J_\nu(\lambda_n x)\,dx\right| < \frac{K_2}{\lambda_n^{\frac{3}{2}}},$$

where K_2 is a positive constant. But for values of n so great that

$$\left|\frac{\psi_1(\lambda_n)}{\lambda_n^2}\right| < \frac{A^2}{4\lambda_n},$$

the quantity (11) is less in absolute value than

$$\frac{8\,|\psi_1(\lambda_n)|}{A^4}\cdot\left|\int_0^1 xf(x)J_\nu(\lambda_n x)\,dx\right|,$$

* Cf. LIPSCHITZ, Crelle's Journal, vol. 56 (1859), pp. 193–196.

and hence, in view of (17), is less in absolute value than $K_3/\lambda_n^{\frac{3}{2}}$, where K_3 is a positive constant. Consequently, since a positive constant C exists such that*

$$\lambda_n > Cn, \tag{18}$$

it follows that (11) is the general term of an absolutely convergent series.

It remains to consider the term (10) which was subtracted from A_n' to produce (11). We must first, however, discuss the last two terms on the right hand side of equation (14).

The asymptotic expansion of $J_{\nu+1}(x)$ gives

$$J_{\nu+1}(\lambda_n x) = \sqrt{\frac{2}{\pi\lambda_n x}} \cos\left(\lambda_n x - \frac{2\nu+3}{4}\pi\right) + \frac{\psi(\lambda_n x)}{\lambda_n^{\frac{3}{2}} x^{\frac{3}{2}}}. \tag{19}$$

Substituting this value in the integral of the third term on the right hand side of (14), we obtain

$$\begin{aligned}\int_0^c x\phi'(x) J_{\nu+1}(\lambda_n x)\,dx = \frac{A}{\sqrt{\lambda_n}} \int_0^c \sqrt{x}\phi'(x)\cos(\lambda_n x - \alpha)\,dx \\ + \frac{1}{\lambda_n^{\frac{3}{2}}} \int_0^c \frac{\phi'(x)}{\sqrt{x}} \psi(\lambda_n x)\,dx,\end{aligned} \tag{20}$$

where for simplicity we have used A and α in place of $\sqrt{2/\pi}$ and $(2\nu+3)\pi/4$.

If $\phi(x)$ is a function whose first derivative is continuous in the interval $0 \leqq x \leqq c$, except at a finite number of points at which it has a finite jump, and whose second derivative is finite and integrable there, we have from an integration by parts

$$\begin{aligned}\int_0^c \sqrt{x}\phi'(x)\cos(\lambda_n x - \alpha)\,dx = \frac{1}{\lambda_n} \sum_{i=1}^{i=r+1} [\sqrt{k_i}\phi'(k_i - 0)\sin(\lambda_n k_i - \alpha) \\ - \sqrt{k_{i-1}}\phi'(k_{i-1} + 0)\sin(\lambda_n k_{i-1} - \alpha)] \\ - \frac{1}{\lambda_n}\int_0^c \left[\sqrt{x}\phi''(x) + \frac{\phi'(x)}{2\sqrt{x}}\right] \sin(\lambda_n x - \alpha)\,dx,\end{aligned} \tag{21}$$

where $k_1, k_2, \cdots, k_r$ are the points of discontinuity of $\phi'(x)$, and for the sake of uniformity we have set

$$k_0 = 0, \qquad k_{r+1} = c.$$

Combining (20) and (21), we have

$$\int_0^c x\phi'(x) J_{\nu+1}(\lambda_n x)\,dx = \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{3}{2}}}. \tag{22}$$

If we set

$$F(x) = \frac{\phi(x) - \phi(0)}{x}, \tag{23}$$

* Cf. equation (23) of the writer's article in these Transactions, referred to above.

we obtain for the integral in the second term on the right hand side of (14)

$$(24)\quad \int_0^c J_{\nu+1}(\lambda_n x)\phi(x)\,dx = \phi(0)\int_0^c J_{\nu+1}(\lambda_n x)\,dx + \int_0^c x F(x) J_{\nu+1}(\lambda_n x)\,dx.$$

If we substitute in the second term on the right hand side of (24) the value of $J_{\nu+1}(\lambda_n x)$ given by (19), we get

$$(25)\quad \int_0^c x F(x) J_{\nu+1}(\lambda_n x)\,dx = \frac{A}{\sqrt{\lambda_n}}\int_0^c \sqrt{x}\cos(\lambda_n x - \alpha) F(x)\,dx + \frac{1}{\lambda_n^{\frac{3}{2}}}\int_0^c \frac{\psi(\lambda_n x)}{\sqrt{x}} F(x)\,dx,$$

where again, for the sake of simplicity, we have introduced A and α in place of $\sqrt{2}/\pi$ and $(2\nu+3)\pi/4$. By means of an integration by parts, the integral in the first term on the right hand side of (25) reduces to the form

$$(26)\quad \frac{\sin(\lambda_n c - \alpha)}{\lambda_n}\sqrt{c}\,F(c) - \frac{1}{\lambda_n}\int_0^c \left[\sqrt{x}\,F'(x) + \frac{F(x)}{2\sqrt{x}}\right]\sin(\lambda_n x - \alpha)\,dx.$$

Substituting this expression in (25), we obtain for the second term on the right hand side of (24)

$$(27)\quad \int_0^c x F(x) J_{\nu+1}(\lambda_n x)\,dx = \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{3}{2}}}.$$

If we set $\lambda_n x = y$, we get for the integral in the first term on the right hand side of (24)

$$(28)\quad \int_0^c J_{\nu+1}(\lambda_n x)\,dx = \frac{1}{\lambda_n}\int_0^{c\lambda_n} J_{\nu+1}(y)\,dy,$$

where the integral on the right hand side may be written

$$(29)\quad \int_0^\infty J_{\nu+1}(y)\,dy - \int_{c\lambda_n}^\infty J_{\nu+1}(y)\,dy,$$

since these integrals converge.

Making use of (19), we obtain for the second term in (29)

$$(30)\quad \int_{c\lambda_n}^\infty J_{\nu+1}(y)\,dy = A\int_{c\lambda_n}^\infty \frac{\cos(y-\alpha)}{\sqrt{y}}\,dy + \int_{c\lambda_n}^\infty \frac{\psi(y)}{y^{\frac{3}{2}}}\,dy,$$

where this reduction is legitimate since the two integrals on the right hand side of (30) both converge. For the second of these integrals we have

$$(31)\quad \left|\int_{c\lambda_n}^\infty \frac{\psi(y)}{y^{\frac{3}{2}}}\,dy\right| < N\int_{c\lambda_n}^\infty \frac{dy}{y^{\frac{3}{2}}} = \frac{N}{c^{\frac{1}{2}}\lambda_n^{\frac{1}{2}}},$$

where N is a positive constant. For the first integral on the right hand side of (30) we have from an integration by parts

$$(32)\quad \int_{c\lambda_n}^{\infty} \frac{\cos(y-\alpha)}{\sqrt{y}}\,dy = \left[\frac{\sin(y-\alpha)}{\sqrt{y}}\right]_{c\lambda_n}^{\infty} + \frac{1}{2}\int_{c\lambda_n}^{\infty} \frac{\sin(y-\alpha)}{y^{\frac{3}{2}}}\,dy = \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{1}{2}}}.$$

Combining (31) and (32) with (30), we obtain

$$(33)\quad \int_{c\lambda_n}^{\infty} J_{\nu+1}(y)\,dy = \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{1}{2}}}.$$

Substituting (33) in (29), and remembering that (29) is only another way of writing the integral on the right hand side of (28), we get

$$(34)\quad \int_0^c J_{\nu+1}(\lambda_n x)\,dx = \frac{K'}{\lambda_n} + \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{3}{2}}},$$

where for the sake of simplicity we have used K' in place of the first term in (29).

Combining (24), (27), and (34), we obtain

$$(35)\quad \int_0^c J_{\nu+1}(\lambda_n x)\phi(x)\,dx = \frac{K'\phi(0)}{\lambda_n} + \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{3}{2}}}.$$

From (14), (22), and (35), we get

$$(36)\quad \int_0^c x\phi(x)J_\nu(\lambda_n x)\,dx = \frac{c\phi(c)J_{\nu+1}(c\lambda_n)}{\lambda_n} + \frac{\nu K'\phi(0)}{\lambda_n^2} + \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{5}{2}}}.$$

Finally from (13) and (36), we obtain

$$(37)\quad \begin{aligned}\int_0^1 xf(x)J_\nu(\lambda_n x)\,dx = &\sum_{m=0}^{m=k} \frac{c_{m+1}\phi_m(c_{m+1})J_{\nu+1}(c_{m+1}\lambda_n)}{\lambda_n} \\ &- \sum_{m=1}^{m=k} \frac{c_m\phi_m(c_m)J_{\nu+1}(c_m\lambda_n)}{\lambda_n} + \frac{\nu K'f(0)}{\lambda_n^2} + \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{5}{2}}}.\end{aligned}$$

It follows from (37) that by subtracting from (10) a quantity that is the general term of an absolutely convergent series, we may reduce it to the form

$$(38)\quad \frac{2}{A^2}\sum_{m=0}^{m=k} c_{m+1}\phi_m(c_{m+1})J_{\nu+1}(c_{m+1}\lambda_n) - \frac{2}{A^2}\sum_{m=1}^{m=k} c_m\phi_m(c_m)J_{\nu+1}(c_m\lambda_n) + \frac{\nu Kf(0)}{\lambda_n},$$

where we have replaced $2K'/A^2$ by K. If, now, we can show that by subtracting from (38) a quantity that is the general term of an absolutely convergent series we may reduce it to the sum of the first three terms in (8), our lemma will be proved.

We have [loc. cit., equation (58)]

$$(39)\quad \cos\left(\lambda_n c - \frac{(2\nu+3)\pi}{4}\right) = \cos\left[(n\pi+q)c - \frac{(2\nu+3)\pi}{4}\right] + \frac{\psi_1(\lambda_n)}{n}.$$

Combining (39) with (19) and (18), we obtain

$$(40)\qquad J_{\nu+1}(\lambda_n c) = \sqrt{\frac{2}{\pi\lambda_n c}}\cos\left[(n\pi+q)c - \frac{(2\nu+3)\pi}{4}\right] + \frac{\psi_1(\lambda_n)}{n^{\frac{3}{2}}}.$$

But since [loc. cit. equation (57)]

$$(41)\qquad \lambda_n = n\pi + q + \frac{\psi_1(\lambda_n)}{n},$$

we have by (18)

$$(42)\qquad \frac{1}{\sqrt{\lambda_n}} - \frac{1}{\sqrt{n\pi}} = \frac{n\pi - \lambda_n}{\sqrt{\lambda_n}\, n\pi(\sqrt{\lambda_n} + \sqrt{n\pi})} = \frac{\psi_1(n)}{n^{\frac{3}{2}}}.$$

Combining (40) and (42), we obtain

$$(43)\qquad J_{\nu+1}(\lambda_n c) = \frac{\sqrt{2}\cos\left[(n\pi+q)c - \frac{2\nu+3}{4}\pi\right]}{\pi\sqrt{nc}} + \frac{\psi_1(n)}{n^{\frac{3}{2}}}.$$

It follows at once from (43) that by subtracting from (38) a quantity that is the general term of an absolutely convergent series, we can reduce it to the sum of the first three terms in (8). Our lemma is therefore established.

Before beginning the proof of the next lemma, we shall state a few formulæ connected with Bessel functions, which will be useful in the course of that proof.

We have [loc. cit. equation (58)]

$$(44)\qquad \cos(\lambda_n x - \alpha) = \cos(n\pi x + qx - \alpha) + \frac{\psi(x, \lambda_n)}{n},$$

where λ_n is the nth positive root of equation (2) and

$$(45)\qquad |\psi(x, \lambda_n)| < K_1 \qquad (0 \leqq x \leqq 1;\ n = 1, 2, 3, \cdots),$$

K_1 being a positive constant.

From the asymptotic expansion * we have

$$(46)\qquad J_\nu(\lambda_n x) = \frac{A}{\sqrt{\lambda_n x}}\cos\left(\lambda_n x - \frac{(2\nu+1)\pi}{4}\right) + \frac{\psi(\lambda_n x)}{(\lambda_n x)^{\frac{3}{2}}},$$

where

$$(47)\qquad |\psi(x)| < K_2, \qquad |\psi'(x)| < K_3 \qquad (x > 0),$$

K_2 and K_3 being positive constants.

* Cf. LIPSCHITZ, loc. cit. That the second inequality in (47) holds may be seen by differentiating the expression of the form (47) for $J_\nu(x)$ and comparing the result with the value of $J'_\nu(x)$ obtained by combining the equation

$$J'_\nu(x) = -J_{\nu+1}(x) + \frac{\nu}{x}J_\nu(x)$$

with the expression of the form (46) for $J_\nu(x)$ and the expression of the form (19) for $J_{\nu+1}(x)$.

LEMMA 2. *The quantity*

$$\frac{\cos\left[(n\pi+q)c-\frac{2\nu+3}{4}\pi\right]J_\nu(\lambda_n x)}{\sqrt{n}}, \tag{48}$$

where c is any positive constant $\leqq 1$, is the general term of a series that is uniformly convergent in any interval

$$0 \leqq x \leqq x_0 < c. \tag{49}$$

The quantity (48) may be written in the form

$$\cos\left(qc-\frac{2\nu+3}{4}\pi\right)\frac{\cos n\pi c J_\nu(\lambda_n x)}{\sqrt{n}} - \sin\left(qc-\frac{2\nu+3}{4}\pi\right)\frac{\sin n\pi c J_\nu(\lambda_n x)}{\sqrt{n}}, \tag{50}$$

and hence if we can show that each of the quantities

$$\frac{\cos n\pi c J_\nu(\lambda_n x)}{\sqrt{n}}, \qquad \frac{\sin n\pi c J_\nu(\lambda_n x)}{\sqrt{n}}, \tag{51}$$

is the general term of a series that is uniformly convergent in the interval (49), our lemma will be proved.

Let us consider the series whose general term is the first expression in (51); the proof for the series whose general term is the second expression in (51) is entirely analogous. Since $0 < c < 2$, the series that we are discussing converges for $x = 0$;* it remains to be shown that it converges uniformly in the interval $0 < x \leqq x_0 < c$.

Given δ, positive and arbitrarily small, let us choose m such that †

$$\left|\sum_{n=p}^{n=q}\frac{\sin}{\cos}\frac{(n\pi z)}{\sqrt{n}}\right| < \eta = \frac{\delta}{1+C_2+C_3} \qquad \left(\begin{matrix} c-x_0 \leqq z \leqq c+x_0 \\ q>p\geqq m\end{matrix}\right), \tag{52}$$

$$\sum_{n=p}^{n=q}\frac{1}{n^{\frac{3}{2}}} < \eta \qquad (q>p\geqq m), \tag{53}$$

where

$$C_2=\frac{A}{\sqrt{k}}(K_1+4), \qquad C_3=\frac{2^{\frac{3}{2}}N^{\frac{1}{2}}(3CK_3k+2K_2N)+6K_2C^{\frac{3}{2}}}{3C^{\frac{3}{2}}k^{\frac{3}{2}}}, \tag{54}$$

the symbols A, C, K_1, K_2, K_3 representing the same quantities as in equations

* For $\nu > 0$ all the terms of the series are zero, and its convergence is obvious; for $\nu = 0$, cf. SCHLÖMILCH, *Algebraische Analysis*, 3d ed., 1862, § 31, also BÔCHER, Annals of Mathematics, ser. 2, vol. 7 (1906), pp. 109–111.

† That the inequality (52) holds for sufficiently large values of m is evident from a reference to page 111 of the article by BÔCHER, quoted in the previous footnote.

(46), (18), (45), and (47); k is the smallest positive root of $J'_\nu(x) = 0$, and N is a positive constant such that *

$$\lambda_n < Nn. \tag{55}$$

Let s be the greatest integer such that $\lambda_s x < k$, and μ and ν any two integers such that

$$m \leqq \mu \leqq \nu.$$

Three cases must be considered:

$$(A)\quad \mu \leqq s < \nu, \qquad (B)\quad \nu \leqq s, \qquad (C)\quad \mu > s.$$

Beginning with (A), we write

$$\sum_{n=\mu}^{n=\nu} \frac{\cos n\pi c\, J_\nu(\lambda_n x)}{\sqrt{n}} = \sum_{n=\mu}^{n=s} + \sum_{n=s+1}^{n=\nu} = R_1 + R_2. \tag{56}$$

From (52) and a lemma due to ABEL, † we have

$$|R_1| < \eta \qquad (0 < x \leqq x_0) \tag{57}$$

For R_2 we have from (46) ‡

$$R_2 = A \sum_{s+1}^{\nu} \frac{\cos n\pi c \cos(\lambda_n x - \alpha)}{\sqrt{n\lambda_n x}} + \sum_{s+1}^{\nu} \frac{\cos n\pi c\, \psi(\lambda_n x)}{\sqrt{n\lambda_n^3 x^3}}. \tag{58}$$

The general term of the first series on the right hand side of (58) may, in view of (44), be written in the form

$$\frac{\cos n\pi c \cos(n\pi x + qx - \alpha)}{\sqrt{n\lambda_n x}} + \frac{\cos n\pi c\, \psi(x, \lambda_n)}{n\sqrt{n\lambda_n x}}. \tag{59}$$

From (45), (53), and the fact that s was so chosen that $\lambda_{s+1} x \geqq k$, we have for the series whose general term is the second term in (59)

$$\left| \sum_{n=s+1}^{n=\nu} \frac{\cos n\pi c\, \psi(x, \lambda_n)}{n\sqrt{n\lambda_n x}} \right| < \sum_{n=s+1}^{n=\nu} \frac{K_1}{\sqrt{k} n^{\frac{3}{2}}} < \frac{K_1}{\sqrt{k}} \eta \qquad (0 < x \leqq x_0). \tag{60}$$

The first term in (59) reduces to the form

$$\begin{aligned} &\frac{\cos n\pi c \cos n\pi x}{\sqrt{n\lambda_n x}} \cos(qx - \alpha) - \frac{\cos n\pi c \sin n\pi x}{\sqrt{n\lambda_n x}} \sin(qx - \alpha) \\ &= \frac{\cos(qx - \alpha)}{2} \left[\frac{\cos n\pi(c + x) + \cos n\pi(c - x)}{\sqrt{n\lambda_n x}} \right] \\ &\quad - \frac{\sin(qx - \alpha)}{2} \left[\frac{\sin n\pi(c + x) - \sin n\pi(c - x)}{\sqrt{n\lambda_n x}} \right]. \end{aligned} \tag{61}$$

* That such a positive constant exists, is easily inferred from equation (41).

† Cf. TANNERY, *Fonctions d'une variable*, § 70. Abel's lemma applies by virtue of the fact that $J_\nu(\lambda_n x)$ steadily decreases or steadily increases with increasing n as long as $n \leqq s$.

‡ For the sake of convenience in writing we have replaced $(2\nu + 1)\pi/4$ by α.

But we have from (52)

$$(62)\qquad \left|\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi(c+x)}{\sqrt{n}}\right|<\eta,\qquad \left|\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi(c-x)}{\sqrt{n}}\right|<\eta,$$
$$\left|\sum_{n=s+1}^{n=\nu}\frac{\sin n\pi(c+x)}{\sqrt{n}}\right|<\eta,\qquad \left|\sum_{n=s+1}^{n=\nu}\frac{\sin n\pi(c-x)}{\sqrt{n}}\right|<\eta,$$

when x varies in the interval $0<x\leqq x_0$. Hence, from the lemma of Abel referred to above,

$$(63)\qquad \left|\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi(c+x)}{\sqrt{n\lambda_n x}}\right|=\frac{1}{\sqrt{\lambda_{s+1}x}}\left|\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi(c+x)}{\sqrt{n}}\cdot\frac{\sqrt{\lambda_{s+1}}}{\sqrt{\lambda_n}}\right|<\frac{\eta}{\sqrt{k}}\qquad(0<x\leqq x_0).$$

By the use of the other inequalities in (62) and Abel's lemma, we can obtain inequalities analogous to (63) for each of the sums

$$\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi(c-x)}{\sqrt{n\lambda_n x}},\qquad \sum_{n=s+1}^{n=\nu}\frac{\sin n\pi(c+x)}{\sqrt{n\lambda_n x}},\qquad \sum_{n=s+1}^{n=\nu}\frac{\sin n\pi(c-x)}{\sqrt{n\lambda_n x}}.$$

Combining these inequalities and (63), and taking note of the fact that (61) is another way of writing the first term in (59), we obtain

$$(64)\qquad \left|\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi c\cos(n\pi x+qx-\alpha)}{\sqrt{n\lambda_n x}}\right|<\frac{4\eta}{\sqrt{k}}\qquad(0<x\leqq x_0).$$

Since (59) is another way of writing the general term of the first series on the right hand side of (58), we have from (60), (64), and (59)

$$(65)\qquad \left|A\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi c\cos(\lambda_n x-\alpha)}{\sqrt{n\lambda_n x}}\right|<\frac{A(K_1+4)\eta}{\sqrt{k}}=C_2\eta\qquad(0<x\leqq x_0).$$

We consider next the series that forms the second term on the right hand side of (58). If we set

$$(66)\qquad \epsilon_n=-\sum_{r=n}^{r=\infty}\frac{\cos r\pi c}{\sqrt{r}},$$

we have for this series*

$$(67)\qquad \sum_{n=s+1}^{n=\nu}\frac{\cos n\pi c}{\sqrt{n}}\cdot\frac{\psi(\lambda_n x)}{(\lambda_n x)^{\frac{3}{2}}}=-\epsilon_{s+1}\frac{\psi(\lambda_{s+1}x)}{(\lambda_{s+1}x)^{\frac{3}{2}}}$$
$$+\sum_{n=s+1}^{n=\nu}\left(\frac{\psi(\lambda_n x)}{(\lambda_n x)^{\frac{3}{2}}}-\frac{\psi(\lambda_{n+1}x)}{(\lambda_{n+1}x)^{\frac{3}{2}}}\right)\epsilon_{n+1}+\epsilon_{\nu+1}\frac{\psi(\lambda_{\nu+1}x)}{(\lambda_{\nu+1}x)^{\frac{3}{2}}}.$$

But from the Law of the Mean

$$(68)\qquad \frac{\psi(\lambda_n x)}{(\lambda_n x)^{\frac{3}{2}}}-\frac{\psi(\lambda_{n+1}x)}{(\lambda_{n+1}x)^{\frac{3}{2}}}=\frac{\psi'(\xi_n x)}{\xi_n^{\frac{3}{2}}x^{\frac{1}{2}}}-\frac{3\psi(\xi_n x)}{2\xi_n^{\frac{5}{2}}x^{\frac{3}{2}}}\qquad(\lambda_n<\xi_n<\lambda_{n+1}).$$

*Cf. equation (26) of the author's article in these Transactions, vol. 8 (1907), p. 299.

Moreover, in view of (47), (18), and (55), we have

$$\left|\sum_{n=s+1}^{n=\nu}\frac{\psi'(\xi_n x)}{\xi_n^{\frac{3}{2}}x^{\frac{1}{2}}}\right| < \sum_{n=s+1}^{n=\nu}\frac{K_3}{\lambda_n^{\frac{3}{2}}x^{\frac{1}{2}}} < \frac{K_3}{C^{\frac{3}{2}}x^{\frac{1}{2}}}\sum_{n=s+1}^{n=\nu}\frac{1}{n^{\frac{3}{2}}} < \frac{K_3}{C^{\frac{3}{2}}x^{\frac{1}{2}}}\int_s^\nu \frac{dx}{x^{\frac{3}{2}}} < \frac{2K_3}{C^{\frac{3}{2}}(sx)^{\frac{1}{2}}}$$

(69)

$$= \frac{2K_3}{C^{\frac{3}{2}}[(s+1)x]^{\frac{1}{2}}}\cdot\left(\frac{s+1}{s}\right)^{\frac{1}{2}} < \frac{2^{\frac{3}{2}}K_3N^{\frac{1}{2}}}{C^{\frac{3}{2}}(\lambda_{s+1}x)^{\frac{1}{2}}} < \frac{2^{\frac{3}{2}}K_3N^{\frac{1}{2}}}{C^{\frac{3}{2}}k^{\frac{1}{2}}} \qquad (x>0),$$

and

$$\left|\sum_{n=s+1}^{n=\nu}\frac{\psi(\xi_n x)}{\xi_n^{\frac{5}{2}}x^{\frac{3}{2}}}\right| < \sum_{s+1}^{\nu}\frac{K_2}{\lambda_n^{\frac{5}{2}}x^{\frac{3}{2}}} < \frac{K_2}{C^{\frac{5}{2}}x^{\frac{3}{2}}}\sum_{s+1}^{\nu}\frac{1}{n^{\frac{5}{2}}} < \frac{K_2}{C^{\frac{5}{2}}x^{\frac{3}{2}}}\int_s^\nu \frac{dx}{x^{\frac{5}{2}}}$$

(70)

$$< \frac{2K_2}{3C^{\frac{5}{2}}(sx)^{\frac{3}{2}}} < \frac{2^{\frac{3}{2}}K_2N^{\frac{3}{2}}}{3C^{\frac{5}{2}}k^{\frac{3}{2}}} \qquad (x>0).$$

From (66) and (52) we have

$$|\epsilon_n| \leqq \eta \qquad (n \geqq m). \tag{71}$$

Combining (67) with (68), (69), (70), (71), and (47), we obtain

$$\left|\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi c}{\sqrt{n}}\cdot\frac{\psi(\lambda_n x)}{(\lambda_n x)^{\frac{3}{2}}}\right| < \frac{2^{\frac{3}{2}}N^{\frac{1}{2}}(3CK_3k+2K_2N)+6K_2C^{\frac{5}{2}}}{3C^{\frac{5}{2}}k^{\frac{3}{2}}}\eta = C_3\eta \tag{72}$$

$$(x>0).$$

From (58), (65), and (72) we have

$$|R_2| < (C_2+C_3)\eta \qquad (0<x\leqq x_0), \tag{73}$$

which, combined with (56) and (57), gives

$$\left|\sum_{n=\mu}^{n=\nu}\frac{\cos n\pi c J_\nu(\lambda_n x)}{\sqrt{n}}\right| < (1+C_2+C_3)\eta = \delta \qquad (0<x\leqq x_0). \tag{74}$$

By a discussion precisely analogous to that of R_1 in Case (A), we get for Case (B)

$$\left|\sum_{n=\mu}^{n=\nu}\frac{\cos n\pi c J_\nu(\lambda_n x)}{\sqrt{n}}\right| < \eta < \delta \qquad (0<x\leqq x_0), \tag{75}$$

and by a discussion precisely analogous to that of R_2 in Case (A), we get for Case (C)

$$\left|\sum_{n=\mu}^{n=\nu}\frac{\cos n\pi c J_\nu(\lambda_n x)}{\sqrt{n}}\right| < (C_2+C_3)\eta < \delta \qquad (0<x\leqq x_0). \tag{76}$$

Combining (74), (75), and (76), we see that the series whose general term is the first expression in (51) converges uniformly for all values of x lying in the interval $0<x\leqq x_0$. Since the series is also convergent for $x=0$, it follows that it is uniformly convergent in the interval $0\leqq x\leqq x_0$. As we have pointed out before, the discussion of the series whose general term is the second expression in (51), is entirely analogous, and hence our lemma is to be regarded as completely established.

LEMMA 3. *If the quantity* (48) *is written in the form*

$$(77)\quad -\frac{A}{2\sqrt{\pi x}}\left[\begin{aligned}&\cos\{q(x+c)-\nu\pi\}\frac{\cos n\pi(x+c)}{n}-\sin\{q(x+c)-\nu\pi\}\frac{\sin n\pi(x+c)}{n}\\&+\cos q(x-c)\frac{\sin n\pi(x-c)}{n}+\sin q(x-c)\frac{\cos n\pi(x-c)}{n}\end{aligned}\right]$$
$$+v_n(x),$$

$v_n(x)$ will be the general term of a series that is uniformly convergent in any interval

$$(78)\qquad 0<x_0\leqq x\leqq 1.$$

By means of (46), (48) reduces to the form

$$(79)\quad A\frac{\cos\left[(n\pi+q)c-\frac{2\nu+3}{4}\pi\right]\cos\left(\lambda_n x-\frac{2\nu+1}{4}\pi\right)}{\sqrt{n\lambda_n x}}+\frac{\cos\left[(n\pi+q)c-\frac{2\nu+3}{4}\pi\right]\psi(\lambda_n x)}{\sqrt{n}(\lambda_n x)^{\frac{3}{2}}}.$$

The second term of (79) is obviously the general term of a series that is uniformly convergent in the interval (78); if then, by subtracting from the first term a number of quantities that have the same property, we can reduce it to an expression which is equal to the first term in (77), our lemma will be established.

From (44) we see that the first term in (79) can be written in the form

$$(80)\quad A\frac{\cos\left[(n\pi+q)c-\frac{2\nu+3}{4}\pi\right]\cos\left[(n\pi+q)x-\frac{2\nu+1}{4}\pi\right]}{\sqrt{n\lambda_n x}}+A\frac{\cos\left[(n\pi+q)c-\frac{2\nu+3}{4}\pi\right]\psi(x,\lambda_n)}{n\sqrt{n\lambda_n x}}.$$

Since the second term of (80) is obviously the general term of a series that is uniformly convergent in the interval (78), it only remains to consider the first term. By means of (42) it can be reduced to the form

$$(81)\quad A\frac{\cos\left[(n\pi+q)c-\frac{2\nu+3}{4}\pi\right]\cos\left[(n\pi+q)x-\frac{2\nu+1}{4}\pi\right]}{n\sqrt{\pi x}}+A\frac{\cos\left[(n\pi+q)c-\frac{2\nu+3}{4}\pi\right]\psi_1(n)}{n^{\frac{3}{2}}\sqrt{nx}}.$$

Here again the second term is obviously the general term of a series that is uniformly convergent in the interval (78); if we can show further that the first term is equal to the first term of (77), our lemma will be proved.

By means of a well known trigonometric formula the first term of (81) may be reduced to the form

$$A\frac{\cos\left[(n\pi+q)(x+c)-(\nu+1)\pi\right]+\cos\left[(n\pi+q)(x-c)+\frac{\pi}{2}\right]}{2n\sqrt{\pi x}}$$

$$=-A\frac{\cos\left[(n\pi+q)(x+c)-\nu\pi\right]+\sin(n\pi+q)(x-c)}{2n\sqrt{\pi x}} \tag{82}$$

$$=-\frac{A}{2\sqrt{\pi x}}\left[\cos\{q(x+c)-\nu\pi\}\frac{\cos n\pi(x+c)}{n}-\sin\{q(x+c)-\nu\pi\}\frac{\sin n\pi(x+c)}{n}\right.$$

$$\left.+\cos q(x-c)\frac{\sin n\pi(x-c)}{n}+\sin q(x-c)\frac{\cos n\pi(x-c)}{n}\right].$$

The right hand side of (82) is identical with the first term of (77), and consequently our lemma is completely established.

Lemma 4. *The quantity* (48) *is the general term of a series for which the sum of the first n terms remains finite for all values of n and all values of x lying in the interval* $0\leqq x\leqq 1$.

That the sum of the first n terms of this series remains finite for all values of n and all values of x lying in the interval $0\leqq x\leqq x_0<c$, is an immediate consequence of Lemma 2. It remains to be shown that this sum remains finite for all values of n and all values of x lying in the interval (78).

We know from Lemma 3 that when the expression (48) is written in the form (77), $v_n(x)$ is the general term of a series that is uniformly convergent in the interval (78) and for which therefore the sum of the first n terms remains finite for all values of n and all values of x in the interval (78). It only remains to be shown then that the first term of (77) is the general term of a series for which the sum of the first n terms remains finite for all values of n and all values of x lying in the interval (78). We will prove this by showing that each term of the quantity in brackets in (77) is the general term of such a series.

The series

$$\sum_{n=1}^{\infty}\frac{\sin n\pi z}{n} \tag{83}$$

is the Fourier's development of the periodic function

$$\phi(z)=\frac{\pi-z}{2} \qquad 0<z<2\pi,$$

$$\phi(z+2\pi)=\phi(z) \qquad z\neq 0,\ \pm 2\pi,\ \pm 4\pi,\ \cdots$$

and consequently, from a well known property of the Fourier's developments, the sum of the first n terms of this series remains finite for all values of n and all values of z. Hence the series whose general terms are the second and third terms of the expression in brackets in (77) have the property we desire to establish, and it only remains to examine the series whose general terms are the first and fourth terms of this expression.

If we represent by $s_n(z)$ the sum of the first n terms of the series

$$\sum_{n=1}^{\infty} \frac{\cos nz}{n}, \tag{84}$$

we have * for $\sin(z/2) \neq 0$

$$s_n(z) = \frac{-\sin\frac{z}{2} + \sum_{m=1}^{m=n-1}\left(\frac{1}{m} - \frac{1}{m+1}\right)\sin\frac{2m+1}{2}z + \frac{1}{n}\sin\frac{2n+1}{2}z}{2\sin\frac{z}{2}}.$$

Consequently

$$|s_n(z)| < \frac{1 + \sum_{m=1}^{m=n-1}\left(\frac{1}{m} - \frac{1}{m+1}\right) + \frac{1}{n}}{2\left|\sin\frac{z}{2}\right|} = \frac{1}{\left|\sin\frac{z}{2}\right|}, \tag{85}$$

and hence we have for the sum of the first n terms of the series whose general term is the fourth term of the expression in brackets in (77)

$$|s_n(x)| < \frac{|\sin q(x-c)|}{|\sin \frac{1}{2}\pi(x-c)|}.$$

It follows from the last inequality that the sum of the first n terms of this series remains finite for all values of n and all values of x lying in the interval (78).

We have finally to consider the series whose general term is the first term of the expression in brackets in (77). In view of (85) we have for the first n terms of this series

$$|s_n(x)| < \frac{|\cos\{q(x+c) - \nu\pi\}|}{|\sin\frac{1}{2}\pi(x+c)|}. \tag{86}$$

In order to assure ourselves that $s_n(x)$ remains finite for all values of n and *all* values of x lying in the interval (78) we must examine the values that q may take on. We have †

$$q = \frac{\pi}{2} + \frac{2\nu+1}{4}\pi + k_1\pi \quad \text{or} \quad q = \frac{2\nu+1}{4}\pi + k_2\pi, \tag{87}$$

* Cf. the writer's paper, these Transactions, vol. 10 (1909), p. 408, equation (66).

† Loc. cit., p. 415. The α in these values of q was used, for the sake of simplicity, in place of the quantity $(2\nu+1)\pi/4$.

where k_1 and k_2 are positive or negative integers or zero. Consequently, even for $c = 1$, the expression on the right hand side of (86) remains finite for all values of x lying in the interval (78), and hence the same is true of $s_n(x)$.

We have now shown that each term of the expression in brackets in (77) is the general term of a series for which the sum of the first n terms remains finite for all values of n and all values of x in the interval (78). Hence the whole first term of (77) is the general term of such a series and our lemma is proved.

LEMMA 5. *The expression* (48) *is, for the case* $c < 1$, *the general term of a series that is uniformly convergent in any interval*

$$c < x_1 \leqq x \leqq 1; \tag{88}$$

for the case $c = 1$, *it is the general term of a series that is uniformly convergent in any interval*

$$0 < x_1 \leqq x \leqq 1, \tag{89}$$

provided the l *of equation* (2) *is not zero.*

Let us consider first the case where $c < 1$. We know from Lemma 3 that when (48) is written in the form (77), $v_n(x)$ is the general term of a series that is uniformly convergent in the interval (88). It remains to be shown that the first term of (77) is also the general term of a series that is uniformly convergent in this interval.

The series

$$\sum_{n=1}^{\infty} \frac{\sin nz}{n}, \qquad \sum_{n=1}^{\infty} \frac{\cos nz}{n}$$

converge uniformly * throughout the interval

$$\delta_1 \leqq z \leqq 2\pi - \delta_2,$$

where δ_1 and δ_2 are fixed positive quantities, which may be taken as small as we please. Hence each term of the quantity in brackets in (77) is the general term of a series that is uniformly convergent in the interval (88). Therefore the whole first term is the general term of such a series, and our lemma is proved for the case where $c < 1$.

We turn now to the case $c = 1$, $l \neq 0$. As in the previous case, we may, by the use of Lemma 3, reduce the proof of our lemma to the proof of the fact that the first term of (77) is the general term of a series that is uniformly convergent in the interval in question. We will now establish this fact by showing that for the values that c and l have in the case we are considering, this term is identically zero. Thus our lemma will be completely proved.

*Cf. BÔCHER, loc. cit., p. 111.

Since the l of equation (2) is not zero, we have for q [cf. the footnote to (87)]

$$q = k_2\pi + \frac{2\nu+1}{4}\pi. \tag{90}$$

Moreover, we saw in the course of the proof of Lemma 3 that the first term in (81) was identical with the first term in (77). If we substitute in the first term of (81) the value $c = 1$ and the value of q given by (90), we find that the first factor in the numerator of this term vanishes identically. Hence the first term of (77) vanishes for the values of l and c in question and, as pointed out before, our lemma is proved.

LEMMA 6. *The expression*

$$\frac{1}{\lambda_n} J_\nu(\lambda_n x), \tag{91}$$

where $\nu > 0$, is the general term of a series for which the sum of the first n terms remains finite for all values of n and all values of x lying in the interval

$$0 \leqq x \leqq 1. \tag{92}$$

The theorem is obvious for $x = 0$, since all the terms of the series vanish in that case. Hence it only remains to establish it for the interval

$$0 < x \leqq 1. \tag{93}$$

Let α_1 and α_2 be the smallest positive roots of

$$\frac{d}{dx}\left\{\frac{J_\nu(x)}{x}\right\} = 0, \tag{94}$$

and $J_\nu(x)$ respectively, and let k be the smaller of the two quantities α_1, α_2. Now choose a positive integer q such that $\lambda_q > k$ and

$$\frac{\lambda_{m+1}}{\lambda_{m-1}} < 2 \qquad (m \geqq q). \tag{95}$$

From (15) we have

$$|J_\nu(\lambda_n x)| < \frac{N}{\lambda_n^{\frac{1}{2}} x^{\frac{1}{2}}}, \tag{96}$$

where N is a positive constant. Hence for values of x lying in the interval

$$\frac{k}{\lambda_q} \leqq x \leqq 1, \tag{97}$$

the series whose general term is (91) is uniformly convergent, and consequently the sum of the first n terms of this series remains finite for all values of n and all values of x lying in this interval. If we can show further that the sum of

the first n terms of the series remains finite for all values of n and all values of x lying in the interval

$$0 < x < \frac{k}{\lambda_q}, \tag{98}$$

our lemma will be completely established.

Let us consider, then, any value of x lying in the interval (98), and let p be the largest positive integer such that $\lambda_p x < k$. There will be two cases

$$(A) \quad p \leqq n, \qquad (B) \quad p > n.$$

Consider first Case (A) and let us write

$$\sum_{m=1}^{m=n} \frac{J_\nu(\lambda_m x)}{\lambda_m} = \sum_{m=1}^{m=p-1} + \sum_{m=p}^{m=n} = S_1 + S_2. \tag{99}$$

We have from (96)

$$|S_2| < \sum_{m=p}^{m=n} \frac{N}{\lambda_n^{\frac{3}{2}} x^{\frac{1}{2}}} < \frac{N}{\gamma x^{\frac{1}{2}}} \sum_{m=p}^{m=n} \frac{1}{\lambda_m^{\frac{3}{2}}} (\lambda_m - \lambda_{m-1}), \tag{100}$$

where γ is the lower limit of the distances between two successive positive roots of equation (2).* But since $\lambda_q x < k$ for values of x in the interval (98), and since p is the *greatest* integer such that $\lambda_p x < k$, it follows that $p \geqq q$, and we have from (95)

$$\sum_{m=p}^{m=n} \frac{1}{\lambda_m^{\frac{3}{2}}} (\lambda_m - \lambda_{m-1}) < \int_{\lambda_{p-1}}^{\lambda_n} \frac{dx}{x^{\frac{3}{2}}} < \frac{2}{\sqrt{\lambda_{p-1}}} = \frac{2}{\sqrt{\lambda_{p+1}}} \sqrt{\frac{\lambda_{p+1}}{\lambda_{p-1}}} < \frac{2^{\frac{3}{2}}}{\sqrt{\lambda_{p+1}}}. \tag{101}$$

Combining (100) and (101), we obtain, since $\lambda_{p+1} x \geqq k$,

$$|S_2| < \frac{N \cdot 2^{\frac{3}{2}}}{\gamma \sqrt{\lambda_{p+1}}\, x} \leqq \frac{2^{\frac{3}{2}} N}{\gamma \sqrt{k}} \qquad \left(0 < x < \frac{k}{\lambda_q}\right). \tag{102}$$

The discussion of S_1 is slightly different in the two cases $\nu > 1$ and $0 < \nu \leqq 1$. Consider first the case $\nu > 1$. We have

$$S_1 = \sum_{m=1}^{m=p-1} \frac{J_\nu(\lambda_m x)}{\lambda_m x} x \leqq \frac{1}{\gamma} \sum_{m=1}^{m=p-1} \frac{J_\nu(\lambda_m x)}{\lambda_m x} (\lambda_{m+1} - \lambda_m) x. \tag{103}$$

But since, for $\nu > 1$, the function $J_\nu(y)/y$ is positive and steadily increases in the interval $0 < y < k$, where k has the same meaning as above, we have

$$\sum_{m=1}^{m=p-1} \frac{J_\nu(\lambda_m x)}{\lambda_m x} (\lambda_{m+1} x - \lambda_m x) < \int_0^{\lambda_p x} \frac{J_\nu(y)}{y}\, dy < \int_0^k \frac{J_\nu(y)}{y}\, dy. \tag{104}$$

Combining (103) and (104), we obtain

$$0 < S_1 < \frac{1}{\gamma} \int_0^k \frac{J_\nu(y)}{y}\, dy \qquad \left(\nu > 1;\ 0 < x < \frac{k}{\lambda_q}\right). \tag{105}$$

* That such a lower limit exists and is different from zero is easily inferred from the fact that the limit of the distance between the nth and the $(n+1)$th roots of (2) exists and is equal to π.

Taking up next the case $0 < \nu \leqq 1$, we have

$$S_1 = \sum_{m=1}^{m=p-1} \frac{J_\nu(\lambda_m x)}{\lambda_m x} x < \frac{1}{\gamma_1} \sum_{m=1}^{m=p-1} \frac{J_\nu(\lambda_m x)}{\lambda_m x} (\lambda_m - \lambda_{m-1}) x, \tag{106}$$

where we have set $\lambda_0 = 0$ for the sake of uniformity, and have used γ_1 to represent the smaller of the two quantities γ and λ_1. But since, for $0 < \nu \leqq 1$, the function $J_\nu(y)/y$ is positive and steadily decreases in the interval $0 < y < k$, where k has the same meaning as above, we have

$$\sum_{m=1}^{m=p-1} \frac{J_\nu(\lambda_m x)}{\lambda_m x} (\lambda_m x - \lambda_{m-1} x) < \int_0^{\lambda_{p-1} x} \frac{J_\nu(y)}{y} dy < \int_0^k \frac{J_\nu(y)}{y} dy, \tag{107}$$

and hence, combining (107) with (106),

$$0 < S_1 < \frac{1}{\gamma_1} \int_0^k \frac{J_\nu(y)}{y} dy \qquad \left(0 < \nu \leqq 1;\ 0 < x < \frac{k}{\lambda_q}\right). \tag{108}$$

From (105) and (108) we have

$$0 < S_1 < K \qquad \left(\nu > 0;\ 0 < x < \frac{k}{\lambda_q}\right), \tag{109}$$

where K is a positive constant. Combining (99), (102), and (109), we obtain

$$\left| \sum_{m=1}^{m=n} \frac{J_\nu(\lambda_m x)}{\lambda_m} \right| < K + \frac{2^{\frac{3}{2}} N}{\gamma \sqrt{k}} \qquad \left(0 < x < \frac{k}{\lambda_q}\right), \tag{110}$$

and hence in Case (A) the sum of the first n terms of the series whose general term is (91) remains finite for all values of n and all values of x in the interval (98). The treatment of Case (B) is entirely analogous to the treatment of S_1 in Case (A), and consequently we may infer that the sum of the first n terms of the series remains finite for all values of n and all values of x in the interval (98). Hence, as was pointed out before, our lemma is proved.

THEOREM I. *If $f(x)$ satisfies the conditions of Lemma* 1, *the series* (6) *will, provided $\nu = 0$ or $f(0) = 0$, converge uniformly to $f(x)$ throughout any interval*

$$0 \leqq x \leqq x_0 < c_1, \tag{111}$$

where c_1 is the smallest value of x for which $f(x)$ has a discontinuity; provided the l of equation (7) *is not zero or $f(1)$ is zero, it will converge uniformly to $f(x)$ throughout any interval*

$$c_k < x_1 \leqq x \leqq 1, \tag{112}$$

where c_k is the largest value of x for which $f(x)$ has a discontinuity; finally, the sum of n terms of the series will remain finite for all values of n and all values of x lying in the interval

$$0 \leqq x \leqq 1. \tag{113}$$

In view of Lemma 1, the general term of the series (1), and hence the general term of the series (6),* may be written in the form

$$\sum_{m=1}^{m=k} M_m \frac{\cos\left[(n\pi+q)c_m - \frac{2\nu+3}{4}\pi\right]}{\sqrt{n}} J_\nu(\lambda_n x)$$

$$+ Mf(1)\frac{\cos\left[(n\pi+q) - \frac{2\nu+3}{4}\pi\right]}{\sqrt{n}} J_\nu(\lambda_n x) \tag{114}$$

$$+ \frac{\nu K f(0)}{\lambda_n} J_\nu(\lambda_n x) + r_n J_\nu(\lambda_n x),$$

where the M's and K are constants, the c's are points of discontinuity of $f(x)$, and r_n is the general term of an absolutely convergent series.

If either $\nu = 0$ or $f(0) = 0$, the third term in (114) drops out, and it only remains to consider the other three terms. Since the Bessel function of any order remains finite for all values of the argument, it is obvious that the last term in (114) is the general term of a series that is uniformly convergent in the interval (111). It follows from Lemma 2 that the first and second terms of (114) are also the general terms of series that are uniformly convergent in this interval. Hence the whole expression (114) is the general term of such a series, and since (114) is only another way of writing the general term of (6), it follows that (6) is uniformly convergent in the interval (111). Since this is the case and since the terms of (6) are continuous in this interval, it must define there a continuous function of x, $\chi(x)$. However, for values of x lying in the interval $0 < x \leqq x_0$, (6) converges to $f(x)$,† i. e.,

$$\chi(x) = f(x) \qquad (0 < x \leqq x_0).$$

Both sides of the last equation have a limit as x approaches zero. Equating these limits, we get

$$\chi(0) = f(0).$$

Hence (1) converges uniformly to $f(x)$ throughout the interval (111), provided $\nu = 0$ or $f(0) = 0$, and the first part of our theorem is proved.

We wish next to show that, provided the l of equation (7) is not zero, or $f(1)$ is zero, the series (6) will converge uniformly to $f(x)$ throughout the interval (112).

Consider first the case $l \neq 0$. We saw in the previous discussion that the general term of (6) could be written in the form (114). It is obvious that the

* In the special case where (4) holds, (114) is the $(n+1)$th term of (6), but that of course makes no difference in a discussion of convergence.

† This, as we have stated earlier in this paper, has been established in previous discussions of the series (6). Cf. the articles by DINI, KNESER, and HOBSON quoted above.

third and fourth terms of (114) are the general terms of series that are uniformly convergent in the interval (112); it follows from Lemma 5 that the first term of (114) and, since the l of equation (7) is not zero, the second term of (114) are the general terms of series that are uniformly convergent in this interval. Hence the whole expression (114) is the general term of such a series, and since (114) is only another way of writing the general term of (6), it follows that (6) is uniformly convergent in the interval (112), provided the l of equation (7) is not zero.

Considering next the case of $f(1)=0$, we see that the second term of (114) is zero for this case, and is therefore the general term of a series that is uniformly convergent in the interval (112), whether the l of equation (7) is or is not zero. Hence, in this case also, (6) is uniformly convergent in the interval (112).

That the series (6) has the value $f(x)$ in this interval, under the conditions prescribed, was proved by HOBSON in the paper already quoted.

We come finally to the proof of the third part of our theorem, namely that the sum of n terms of the series (6) remains finite for all values of n and all values of x lying in the interval (113). Since r_n is the general term of an absolutely convergent series, the fourth term of (114) is obviously the general term of a series that is absolutely and uniformly convergent in the interval (113), and for which therefore the sum of n terms remains finite for all values of n and all values of x lying in this interval. It follows at once from Lemma 4 that the first and second terms of (114) are also the general terms of such series, and from Lemma 6 that the third term is the general term of such a series. Hence the whole expression (114), or its equivalent the general term of (6), is the general term of such a series.

Our theorem is, therefore, completely established.

The preceding theorem establishes the uniform convergence of the series (6) to $f(x)$ throughout the interval (111) only when $\nu=0$ or $f(0)=0$. That (6) could not possibly converge uniformly to $f(x)$ in the interval (111) in case neither of these restrictions was fulfilled is easy to see, since when $\nu>0$ the value of (6) for $x=0$ is always zero. *However, in the case* $\nu>0$, $f(0)\neq 0$, *it is possible to obtain a development of* $f(x)$ *that is uniformly convergent in the interval* (111), *in the form of a constant term plus a series of the type* (6). To do this, we have only to develop the function $\phi(x)=f(x)-f(0)$ in a series of the form (6), and precede this development by the constant term $f(0)$.

The reasoning in Theorem I is also *immediately* applicable to proving that *under the given conditions on* $f(x)$ *the series* (6) *will converge uniformly in any closed sub-interval of* $0<x<1$ *that does not include a point of discontinuity of* $f(x)$. This uniform convergence has been previously established,*

* By KNESER for developments in terms of J_0; by HOBSON for developments in terms of $J_\nu(\nu\geqq 0)$. Cf. the references given above.

but it seems worth while to point out the fact that it can be obtained independently from the results of the present paper.

THEOREM II. *If $f(x)$ satisfies the conditions of Lemma* 1, *the series*

$$A_1 e^{-\lambda_1 z} F_0(\lambda_1, x) + A_2 e^{-\lambda_2 z} F_0(\lambda_2, x) + \cdots, \tag{115}$$

where the A's are the coefficients of the series (6) *for the case $\nu = 0$, and the λ's are the roots, positive or zero, of equation* (7) *arranged in increasing order of magnitude, will converge absolutely in the region*

$$\begin{pmatrix} 0 \leqq x \leqq 1 \\ z > 0 \end{pmatrix}; \tag{116}$$

its value $u(z, x)$ will be continuous there, will approach $f(x)$ as we approach any point of the interval $0 \leqq x < 1$ lying on the line $z = 0$, at which $f(x)$ is continuous, will approach $f(1)$ as we approach the point $x = 1$, $z = 0$, if the l of equation (7) *is not zero and $f(x)$ is continuous at $x = 1$, and finally will remain finite when we approach any point of the interval $0 \leqq x \leqq 1$ lying on the line $z = 0$.*

We obtain at once from the definition of A_n and Lemma 1

$$|A_n| = |A_n'| < K_1 \qquad (n = 1, 2, \cdots),$$

where K_1 is a positive constant. Then since $F_0(\lambda, x)$ remains finite for all values of x and λ, we have for the general term of (115), if we make use of (18),

$$|A_n e^{-\lambda_n z} F_0(\lambda_n, x)| < \frac{K_1 |F_0(\lambda_n, x)|}{e^{\lambda_n z}} < \frac{K}{e^{\lambda_n z_0}} < \frac{K}{(\lambda_n z_0)^2} < \frac{K}{C^2 z_0^2 n^2} \quad (z \geqq z_0 > 0),$$

where K is a positive constant. Hence (115) converges absolutely and uniformly for all values of x and z lying in the region

$$\begin{pmatrix} 0 \leqq x \leqq 1 \\ z \geqq z_0 > 0 \end{pmatrix}. \tag{117}$$

Since also its terms are continuous in this region, it represents a function that is continuous there or, since z_0 may be taken as any constant > 0, is continuous in the region (116).

At any point of the interval $0 \leqq x < 1$ at which $f(x)$ is continuous, and at the point $x = 1$ when the l of equation (7) is not zero and $f(x)$ is continuous there, the series (6) converges to $f(x)$. Hence, for values of x in the interval $0 \leqq x < 1$ at which $f(x)$ is continuous, and for $x = 1$ in the case $l \neq 0$ and $f(x)$ is continuous at $x = 1$, $u(z, x)$ will approach $f(x)$ as z approaches $+ 0$, since the convergence factors

$$e^{-\lambda_1 z}, \ e^{-\lambda_2 z}, \ \cdots$$

satisfy the conditions of a theorem due to Bromwich.* Furthermore, (6) converges uniformly to $f(x)$ throughout any closed interval, in which $f(x)$ is continuous, that lies in the interval $0 \leqq x \leqq 1$ and does not include the point $x = 1$ for the case where the l of equation (7) is zero. Consequently $u(z, x)$ will approach $f(x)$ *uniformly* for all values of x lying in such an interval as z approaches $+0$, as can be shown by a slight modification of the reasoning in Bromwich's theorem. Hence $u(z, x)$ will approach $f(x)$ as we approach any point of the interval $0 \leqq x \leqq 1$ lying on the line $z = 0$ at which $f(x)$ is continuous, † along any path lying in the region (116).

It remains finally to be shown that $u(z, x)$ remains finite as we approach any point of the interval $0 \leqq x \leqq 1$ lying on the line $z = 0$.

If we represent by $s_n(x)$ the sum of the first n terms of the series (6), we may write the series (115) in the form

$$\sum_{n=1}^{\infty} [\, s_n(x) - s_{n-1}(x) \,]\, e^{-\lambda_n z}, \tag{118}$$

where for the sake of uniformity we have taken $s_0(x) = 0$. We know from Theorem I that $s_n(x)$ remains finite for all values of n and all values of x lying in the interval (113). Consequently, since for $z > 0$

$$\lim_{n=\infty} e^{-\lambda_n z} = 0,$$

the series obtained by dropping the parentheses in the series (118) is one for which the limit of the general term is zero. Hence, since no parenthesis contains more than a fixed number of terms, we may drop the parentheses without changing the value of the series. Rearranging the terms in such a manner that

* Mathematische Annalen, vol. 65 (1908), p. 358. BROMWICH's conditions are

$$\left.\begin{array}{l} \sum\limits_{n=\mu}^{n=\nu} n^k |\, \Delta^{k+1} v_n \,| < K \\ \lim\limits_{n=\infty} n^k v_n = 0 \end{array}\right\} a > 0,$$

$$\lim_{a=+0} v_n = 1,$$

where μ and ν are two integers, $v_1, v_2, v_3, \cdots$ are the convergence factors, $\Delta^{k+1} v_n$ represents their $(k+1)$th differences, and K is a positive constant. When the convergence factors are introduced into a convergent series, as in the present instance, the k of this theorem must be taken equal to zero.

A simple proof of the special case here dealt with of BROMWICH's theorem can be obtained from the expression (119), which is equal to the series (115), by throwing it into the form

$$f(x)\, e^{-\lambda_1 z} + \sum_{n=1}^{n=\infty} \varepsilon_n(x) [\, e^{-\lambda_n z} - e^{-\lambda_{n+1} z} \,],$$

where

$$\varepsilon_n(x) = s_n(x) - f(x).$$

† We must except, of course, the point $x = 1$ for the case where the l of equation (7) is equal to zero.

no term is carried over more than a fixed number of terms and reinserting parentheses does not change the value of the series, and hence (118) may be written in the form

$$\sum_{n=1}^{\infty} s_n(x)\left[e^{-\lambda_n z} - e^{-\lambda_{n+1} z}\right]. \tag{119}$$

But since

$$|s_n(x)| < M \qquad (n=1, 2, \cdots; 0 \leqq x \leqq 1),$$

where M is a positive constant, we have, in view of the fact that (119) converges to $u(z, x)$,

$$|u(z, x)| \leqq \sum_{n=1}^{\infty} |s_n(x)| \cdot \left[e^{-\lambda_n z} - e^{-\lambda_{n+1} z}\right] < M \sum_{n=1}^{\infty} \left[e^{-\lambda_n z} - e^{-\lambda_{n+1} z}\right] = M e^{-\lambda_1 z},$$

and consequently $u(z, x)$ remains finite when we approach any point of the interval $0 \leqq x \leqq 1$ lying on the line $z = 0$.

Our theorem is, therefore, completely established.

A single example will show clearly the manner in which the above results can be applied to problems in mathematical physics. Let us consider the problem of determining the steady flow of heat in a semi-infinite circular cylinder whose base is kept at a temperature

$$v = f(r),$$

where r is the distance from the axis of the cylinder.

This problem reduces to the determination of a function $v(r, z)$ which is finite throughout the region $z \geqq 0$, $0 \leqq r \leqq 1$, and is continuous throughout this region except when $z = 0$ and r has a value for which $f(r)$ is discontinuous; which satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{\partial^2 v}{\partial z^2} = 0 \tag{120}$$

in the region

$$\begin{pmatrix} z > 0 \\ 0 < r < 1 \end{pmatrix}; \tag{121}$$

and which also satisfies the conditions

$$l\frac{\partial v}{\partial r} + hv = 0 \qquad (z > 0, r = 1) \tag{122}$$

and

$$v = f(r) \qquad (z = 0).* \tag{123}$$

It can be shown easily that the function

$$Ae^{-\lambda z} F_0(\lambda, r), \tag{124}$$

where A is a constant and λ is a root of equation (7), satisfies (120) and (122).

* Cf. CARSLAW, *Introduction to the Theory of Fourier's Series and Integrals, etc.*, Chap. XV, p. 321.

It remains to construct from functions of the form (124) a continuous function which will also satisfy (120) and (122) and in addition will approach the boundary value $f(r)$, where $f(r)$ is continuous, and which will remain finite where that is not the case.

Consider the series

$$A_1 e^{-\lambda_1 z} F_0(\lambda_1, r) + A_2 e^{-\lambda_2 z} F_0(\lambda_2, r) + \cdots, \tag{125}$$

where the λ's are the roots, positive or zero, of equation (7), arranged in increasing order of magnitude, and the A's are the coefficients of the series (6) for $\nu = 0$ corresponding to the given function $f(r)$. It follows from Theorem II that if $f(r)$ satisfies the conditions imposed on $f(x)$ in Lemma 1, this series will converge in the region

$$\begin{pmatrix} z > 0 \\ 0 \leqq r \leqq 1 \end{pmatrix} \tag{126}$$

and define there a continuous function $u(r, z)$; that this function satisfies (120) and (122) can be shown in a manner similar to that in which its convergence and continuity were established. Finally it follows from Theorem II that $u(r, z)$ will approach $f(r)$ when we approach a point on the line $z = 0$ at which $f(r)$ is continuous,* and that it will remain finite when we approach any other point of this line. Thus $u(r, z)$ forms a complete solution of the physical problem.†

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* We must except here the point $r = 1$ for the case in which the l of equation (7) is zero. The necessity for this exception is now apparent from the physical point of view as well as the mathematical one, for the case $l = 0$ corresponds to the case in which the convex surface of the cylinder is kept at temperature zero, and hence we could not approach the temperature at the circumference of the base of the cylinder as we approached this circumference along *any* path except in the very special case that this temperature was zero.

† In the discussion of other problems of mathematical physics that involve developments in Bessel functions of order zero, we may encounter convergence factors slightly different from those of Theorem II. However, in most cases the necessary modifications in Theorem II will be only those of form, since the convergence factors will satisfy the same general theorem that was used in the proof of Theorem II.

DETERMINATION OF THE ORDINARY AND MODULAR TERNARY LINEAR GROUPS*

BY

HOWARD H. MITCHELL

INTRODUCTION.

It is proposed in this paper to determine the finite groups of collineations in three homogeneous variables, a collineation being of the form

$$\rho x_i' = \sum_{j=1}^{3} a_{ij} x_j \qquad (i = 1, 2, 3).$$

The determination will be made both in the case where the coefficients a_{ij} are ordinary numbers and in the case where they are marks of a finite Galois field, $GF(p^n)$, p being an odd prime and n a positive integer. The variables x_i are therefore regarded as homogeneous coördinates of the points either of the ordinary plane or of a modular plane,† $PG(2, p^n)$.

The finite groups of collineations in the ordinary plane have been already determined.‡ The treatment of the problem in this paper is however different from any thus far given. It is based almost entirely on geometrical methods and it is hoped may prove of interest. The discussion for this case is contained in §§ 2–10.

Considerable work has been done on the ternary modular groups. For the case where the coefficients of the transformations are in the $GF(p)$, the question was considered by BURNSIDE.§ For the same case a complete determination of those groups which contain operators of period p was made by DICKSON.‖

* Presented to the Society, under different titles, December 30, 1909, April 30 and September 7, 1910.

† O. VEBLEN and W. H. BUSSEY, *Finite Projective Geometries*, these Transactions, vol. 7 (1906), pp. 241–259.

‡ For a bibliography of this subject consult WIMAN, *Endliche Gruppen linearer Substitutionen*, *Encykopädie der Mathematischen Wissenschaften*, Bd. I, pp. 528–530. The first accurate solution of the problem is due to BLICHFELDT, *On the Order of Linear Homogeneous Groups*, these Transactions, vol. 4 (1903), pp. 387–397, and vol. 5 (1904), pp. 310–325; also *The Finite, Discontinuous, Primitive Groups of Collineations in Three Variables*, Mathematische Annalen, vol. 63 (1907), pp. 552–572.

§ Proceedings of the London Mathematical Society, vol. 26 (1895), pp. 58–106.

‖ American Journal of Mathematics, vol. 27 (1905), pp. 189–202.

For the case where the coefficients are in the $GF(p^2)$, a discussion of those groups whose orders are divisible by p^6, p^5 and p^4 was given by R. L. BÖRGER.*

As a result of the determination of the ternary modular groups in this paper, the subgroups of the two systems of simple groups, $LF(3, p^k)$ and $HO(3, p^{2k})$, are found in the cases where p is an odd prime.† The $LF(3, p^k)$ is the group of all ternary transformations the coefficients of which lie in the $GF(p^k)$ and which have for determinant a cube in that field. It is identical with the group of all collineations of the modular plane, $PG(2, p^k)$, if $p^k - 1$ is not divisible by 3, and is a self-conjugate subgroup of that group of index 3 if $p^k - 1$ is divisible by 3. Its order is

$$\frac{1}{\mu}(p^{2k} + p^k + 1)(p^k + 1)p^{3k}(p^k - 1)^2,$$

where μ is the greatest common divisor of 3 and $p^k - 1$. The $HO(3, p^{2k})$ is a subgroup of the $LF(3, p^{2k})$ having an invariant of the form

$$x_1^{p^k+1} + x_2^{p^k+1} + x_3^{p^k+1} = 0.$$

Its order is

$$\frac{1}{\nu}(p^{2k} - p^k + 1)(p^k + 1)^2 p^{3k}(p^k - 1),$$

where ν is the greatest common divisor of 3 and $p^k + 1$.

The binary modular groups have been fully determined. For the case where the coefficients are in the $GF(p)$ the determination was made by GIERSTER.‡ In the general case the problem has been solved by E. H. MOORE§ and by WIMAN.‖ A treatment based on these two papers is given by DICKSON (*Linear Groups*, Chap. XII). A new treatment will be given in this paper.

The writer wishes to acknowledge his indebtedness to Professor O. VEBLEN of Princeton University for constant and valuable aid in the preparation of the paper.

§1. BINARY GROUPS.

We consider transformations in two homogeneous variables of the form

$$\rho x_i' = \sum_{j=1}^{2} a_{ij} x_j \qquad (i = 1, 2).$$

* *Ibid.*, vol. 32 (1910), pp. 289–298.

† See DICKSON, *Linear Groups*, pp. 75–78, 126–144.

‡ Mathematische Annalen, vol. 18 (1881).

§ *The Subgroups of the Generalized Finite Modular Group, Decennial Publications of the University of Chicago*, vol. 9 (1904), pp. 141–190.

‖ *Bestimmung aller Untergruppen einer doppelt unendlichen Reihe von einfachen Gruppen*, Bihang till K. Svenska Vet.-Akad. Handlingar, vol. 25, part 1, no. 2.

These transformations may be regarded as permuting the points $(x_1 x_2)$ of a line (either ordinary or modular). Any such transformation leaves invariant either one or two points on the line. In the case of the ordinary line any transformation of finite period leaves invariant two points. The groups containing such transformations only are well known. They are cyclic groups of order d, dihedral groups of order $2d$, tetrahedral groups of order 12, octahedral groups of order 24, and icosahedral groups of order 60. No further discussion will be given of these groups.

We therefore consider those groups on the modular line which contain transformations leaving invariant a single point. Any such transformation is additive and of period p.* Any group containing these additive transformations will contain at least one additive group, which leaves invariant a point and contains all the additive transformations which have that point for fixed point. The order of any such additive group is a power of p, say p^m. If the fixed point be (10), the additive group is represented by $[x_1 + \lambda x_2,\ x_2]$, where λ takes all the values in an additive field $\alpha_1\lambda_1 + \alpha_2\lambda_2 + \cdots + \alpha_m\lambda_m$, the α's being integers $0, 1, 2, \cdots, p-1$, and each λ_j a mark not included in the additive field $\alpha_1\lambda_1 + \cdots + \alpha_{j-1}\lambda_{j-1}$. There will be conjugate with it $1 + fp^m$ additive groups, f being 0 or a positive integer. All additive transformations must therefore lie in these conjugate additive groups.

The additive group will be self-conjugate under a maximum metacyclic group of order $d_1 p^m$, where d_1 is a factor of $p^m - 1$ (in special cases we may have $d_1 = 1$). Such a metacyclic group contains p^m conjugate cyclic groups, each of which leaves invariant two points and is of period d_1. If (01) be the other fixed point of one of these cyclic groups, the metacyclic group may be generated by the additive group together with $[\eta x_1,\ x_2]$, where η is of period d_1 and belongs to the multiplier field of the additive field. If Ω denotes the order of the whole group there will be $\Omega/d_1 p^m$ additive groups containing $(p^m - 1)\Omega/d_1 p^m$ additive transformations. There will also be in the group Ω/d_1 or $\Omega/2d_1$ cyclic groups of order d_1 containing $(d_1 - 1)\Omega/d_1$ or $(d_1 - 1)\Omega/2d_1$ transformations, according as one of those cyclic groups is self-conjugate under itself only or under a dihedral group interchanging its two fixed points. Any other maximal cyclic group of order d_i, which does not lie in a metacyclic group, will be self-conjugate either under itself only or under a dihedral group of order $2d_i$ interchanging its two fixed points. Conjugate with such a cyclic group there will be Ω/d_i or $\Omega/2d_i$ cyclic groups containing $(d_i - 1)\Omega/d_i$ or $(d_i - 1)\Omega/2d_i$ transformations.

The order of the group will be equal to the number of transformations which it contains. Hence Ω must satisfy a Diophantine equation of the following

* Throughout the paper p denotes the modulus. The discussion of the binary groups applies also to the case $p = 2$.

form:*

$$\Omega = 1 + (p^m - 1)\frac{\Omega}{d_1 p^m} + \sum_{i=1}^{r}(d_i - 1)\frac{\Omega}{f_i d_i} \qquad (f_i = 1, 2).$$

The coefficient of Ω on the right of the above equation must be less than unity. Also, since the group contains a subgroup of order $d_1 p^m$, the coefficient of Ω on the right must be equal to or greater than $(d_1 p^m - 1)/d_1 p^m$. If $d_1 = 1$, we find therefore that either the final sum is absent, or it contains but one term, in which case $f_2 = 2$. If $d_1 > 1$, either $r = 1$, $f_1 = 1$, or $r = 2$, $f_1 = f_2 = 2$.

If $d_1 = 1$ and the final sum is absent, we obtain $\Omega = p^m$, i. e., a single additive group.

If $d_1 = 1$ and there is one term in the final sum, we have either $p^m = 2$, $\Omega = 2d_2$, which represents a dihedral group, or $p^m = 3$, $d_2 = 2$, $\Omega = 12$, which represents a tetrahedral group.

If $d_1 > 1$, $r = 1$, $f_1 = 1$, we obtain $\Omega = d_1 p^m$, which represents a single metacyclic group.

If $d_1 > 1$, $r = 2$, $f_1 = f_2 = 2$, we shall show that d_1 and d_2 cannot contain any common factor except 2. The cyclic groups of order d_1, when transformed by one of their number, will be permuted in cycles of period d_1 or $d_1/2$. Hence the total number of conjugate groups of order d_1 must be of the form $1 + fd_1/2$. If we suppose the groups of order d_1 to be transformed by one of the groups of order d_2, it is evident that the number of the former must also be of the form $f'd_2/2$. Hence d_1 and d_2 can contain no common factor except 2.

The order, Ω, must be the least common multiple of p^m, $2d_1$, and $2d_2$. For it must be divisible by the least common multiple, since the group contains subgroups of those orders. Moreover if we collect the three terms on the right and write the equation

$$\Omega = 1 + \frac{I}{M}\Omega,$$

where M denotes the least common multiple and I is an integer, it is clear that Ω cannot be greater than M.

Hence, if p is odd, Ω must have one of the two values, $d_1 d_2 p^m$, and $2d_1 d_2 p^m$, according as d_1 and d_2 are both divisible by 2 or have no common factor. For $p = 2$, $\Omega = d_1 d_2 p^m$.

Putting $\Omega = d_1 d_2 p^m$ in the equation, we obtain

$$(d_2 - d_1)p^m - 2(d_2 - 1) = 0.$$

Hence

$$d_2 = fp^m + 1, \qquad d_1 = f(p^m - 2) + 1,$$

* This type of Diophantine equation originated in C. JORDAN's attempt to determine all finite collineation groups in the ordinary plane, Journal für die reine und angewandte Mathematik, vol. 84 (1878), p. 89.

where f denotes an integer. But d_1 is a divisor of $p^m - 1$. Hence

$$f = 1,\ d_1 = p^m - 1,\ d_2 = p^m + 1,\ \Omega = (p^m + 1)p^m(p^m - 1).$$

Putting $\Omega = 2d_1 d_2 p^m$ in the equation, we obtain

$$(d_2 - d_1)p^m - (2d_2 - 1) = 0.$$

Hence

$$d_2 = \tfrac{1}{2}(fp^m + 1), \qquad d_1 = \tfrac{1}{2}(fp^m + 1) - f,$$

where f denotes an odd integer. But d_1 is a divisor of $p^m - 1$. Hence

$$f = 1,\ d_1 = \frac{p^m - 1}{2},\ d_2 = \frac{p^m + 1}{2},\ \Omega = \tfrac{1}{2}(p^m + 1)p^m(p^m - 1);$$

or $f = 3,\ p^m = 3,\ d_1 = 2,\ d_2 = 5,\ \Omega = 60$.

We consider the possible group of order $(p^m + 1)p^m(p^m - 1)$. Each additive group of order p^m is self-conjugate under a metacyclic group of order $(p^m - 1)p^m$. Hence there are $p^m + 1$ such additive groups. The $p^m + 1$ fixed points of these additive groups must therefore be permuted among themselves. The two fixed points of each cyclic group of order $p^m - 1$ are interchanged by transformations of the group, i. e., they are among the $p^m + 1$ points. The additive group leaving fixed one of these points is transitive on the rest.

Since three points may be sent by a transformation into any other three, we may choose three of the $p^m + 1$ points as $(01), (10), (11)$. The following three transformations will then be in the group: $[\eta x_1, x_2]$, $[x_1 + x_2, x_2]$, $[x_1, x_1 + x_2]$, where the period of η is $p^m - 1$. The powers of η together with 0 form the Galois field, $GF(p^m)$. The three transformations generate a group of order $(p^m + 1)p^m(p^m - 1)$, which is the group of all transformations with coefficients in the $GF(p^m)$.

We may construct the group of order

$$\tfrac{1}{2}(p^m + 1)p^m(p^m - 1)$$

by replacing η by η^2. It is the group of all transformations with coefficients in the $GF(p^m)$ and having for determinant a square in that field.

The group of order 60 is an icosahedral group. For it contains $\Omega/2 \cdot 2 = 15$ involutions, each of which lies in one four-group. Hence the fifteen involutions arrange themselves in five four-groups. Each involution leaves fixed the four-group in which it lies and permutes the other four in pairs. The group must then be a G_{60}^5, i. e., an icosahedral group. Such a group may be generated by the three operators, E_1: $[x_1 + x_2, x_2]$, E_2: $[ix_1, -x_1 - ix_2]$, E_3: $[-ix_1, ix_2]$ $(i^2 = -1)$, which satisfy the generational relations:*

$$E_1^3 = E_2^2 = E_3^2 = (E_1E_2)^3 = (E_1E_3)^2 = (E_2E_3)^3 = I.$$

* E. H. Moore, Proceedings of the London Mathematical Society, vol. 27 (1897), pp. 357–366; Dickson, *Linear Groups*, p. 289.

§ 2. Canonical Forms of Transformations in the Plane.*

The transformations in the plane† may be classified according to the five types of invariant figures. Those of type I leave fixed a triangle; those of type II leave fixed two points and two lines; those of type III leave fixed a lineal element; those of type IV leave fixed all the points of a line and all the lines through a point off that line; those of type V leave fixed all the points of a line and all the lines through a point on that line (Fig. 1).

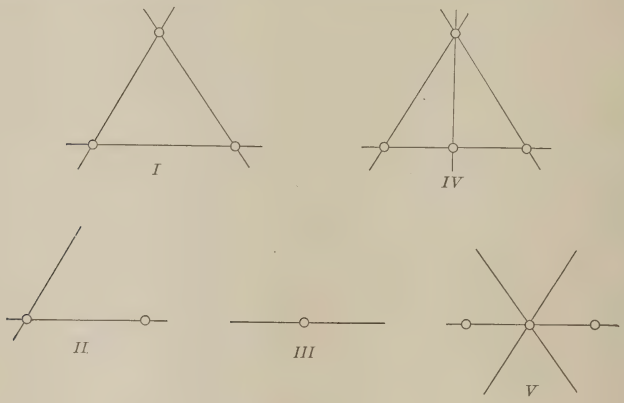

Fig. 1.

A transformation of type I may be written in canonical form $[\alpha x_1, \beta x_2, \gamma x_3]$. This transformation is of finite period in the ordinary plane provided the ratios of the three quantities, α, β, γ, are roots of unity (not unity itself).

A transformation of type II may be written $[\alpha x_1, x_2 + x_3, x_3]$, where $\alpha \neq 0, 1$. The period of this transformation is infinite in the ordinary plane and in the modular plane contains p as a factor.

A transformation of type III may be written $[x_1 + x_2, x_2 + x_3, x_3]$. The period is infinite in the ordinary plane and equal to p in the modular plane (since $p > 2$).

A transformation of type IV may be written $[\alpha x_1, x_2, x_3]$, where $\alpha \neq 0, 1$. In the ordinary plane it is of finite period if α is a root of unity. We will refer to a transformation of this type as an *homology*. In particular, if the period is 2, i. e., if $\alpha = -1$, we will refer to it as a *reflection*.

A transformation of type V may be written $[x_1, x_2 + x_3, x_3]$. In the ordinary plane it is of infinite period and in the modular plane of period p. We

* Cf. Veblen and Young, *Projective Geometry;* Dickson, *Linear Groups*, chaps. X, XI.

† For the modular plane the discussion will be limited to the case where p is an odd prime.

shall refer to a transformation of this type as an *elation*. Any power d of the transformation of type II such that $\alpha^d = 1$ is an elation.

§ 3. General Theorems.

Theorem 1. *A group leaving fixed a line and represented on the points of that line by a four-group contains a reflection with that line as axis; dually, a group leaving fixed a point and represented on the lines through the point by a four-group contains a reflection with that point as center.*

We shall prove the first half of this theorem. We choose the fixed line as $x_3 = 0$ and the fixed pair of points of one involution of the four-group as (100) and (010). If then we choose as (110) one of the fixed points of another of the three involutions, the other fixed point will be (1–10). Then any two transformations which are represented on the fixed line by these two involutions are of the form

$$[\alpha x_1 + \gamma_1 x_3,\ -\alpha x_2 + \gamma_2 x_3,\ x_3], \qquad [\beta x_2 + \delta_1 x_3,\ \beta x_1 + \delta_2 x_3,\ x_3].$$

Their product is $[\alpha\beta x_2 + \epsilon_1 x_3,\ -\alpha\beta x_1 + \epsilon_2 x_3,\ x_3]$. This is a transformation which is represented on the fixed line by the third involution of the four-group.

The square of any one of these three transformations will leave fixed all the points on $x_3 = 0$ and hence will be either an homology or an elation with that line as axis or else the identity. One at least of them, however, must be an homology of even period, since if the periods of α^2 and β^2 are both odd the period of $-\alpha^2\beta^2$ will be even. An homology of even period will contain as a power a reflection.

The dual theorem may be proved in a similar manner.

Theorem 2. *A group permuting cyclically the vertices of a triangle and represented on each side of the triangle by a cyclic group of order d is of order $3dd'$, where d' is a factor of d which is divisible by all the prime factors of d of the form $3f - 1$ and all the factors 3 with the exception of at most one.*

We choose the fixed triangle as the triangle of reference. A transformation which is of period d on $x_3 = 0$ may be written $[x_1, \omega x_2, \omega^e x_3]$, where ω is of period d, the coefficient of x_3 being a power of ω since the period of the transformation on $x_1 = 0$ and $x_2 = 0$ must be a factor of d. If we transform this transformation by a transformation permuting cyclically the vertices of the fixed triangle, we obtain $[x_1, \omega^{-e} x_2, \omega^{1-e} x_3]$. The eth power of the first transformation by the second is the homology $[x_1, x_2, \omega^{e^2-e+1} x_3]$.

From the theory of quadratic forms it follows that $e^2 - e + 1$ cannot be divisible by any primes of the form $3f - 1$ or by 3 to a higher power than the first. The period of the homology is then divisible by all the prime factors of the form $3f - 1$ and all the factors 3 with the exception of at most one.

Theorem 3. *A group making all six permutations on the vertices of a tri-*

angle and represented on each side of the triangle by a dihedral group of order $2d$ *is of order* $2d^2$ *or* $6d^2$ *if* d *is divisible by* 3 *and of order* $6d^2$ *if* d *is not divisible by* 3.

We choose the fixed triangle as the triangle of reference. The group will contain a transformation of period d on $x_3 = 0$ of the form, $[x_1, \omega x_2, \omega^e x_3]$, where ω is of period d. A transformation interchanging $x_1 = 0$ and $x_2 = 0$ and leaving $x_3 = 0$ fixed transforms this into $[x_1, \omega^{-1}x_2, \omega^{e-1}x_3]$. As a product we obtain the homology (or the identity), $[x_1, x_2, \omega^{2e-1}x_3]$. Similarly there is in the group the homology, $[x_1, x_2, \omega^{2-e}x_3]$. The product of the first homology by the square of the second gives $[x_1, x_2, \omega^3 x_3]$. This homology is of period $d/3$ or d according as d is or is not divisible by 3.

In particular if there are no homologies with centers at the vertices of the triangle and having for axes the opposite sides, we must have $d = 3$. If there are only reflections, we must have $d = 2, 6$.

§ 4. Multiplicative Groups containing only Homologies and Transformations of Type I; General Properties.

Since homologies and transformations of type I are called multiplicative, groups containing transformations of those types only will be referred to as *multiplicative* groups. Any group in the ordinary plane is thus multiplicative.

Theorem 4. *In a multiplicative group a transformation which leaves fixed the center of an homology must leave fixed its axis and vice-versa.*

The product of an homology and the transformed of its inverse by a transformation leaving fixed its center but not its axis is an elation.

Theorem 5. *No multiplicative group can contain two homologies such that the line joining their centers passes through the intersection of their axes.*

The product of two homologies such that the line joining the centers passes through the intersection of their axes is of type II or III.

§ 5. Multiplicative Groups containing Homologies of higher Period than 3.

Theorem 6. *No multiplicative group which does not leave invariant a point, line, or triangle can contain homologies of period greater than* 5.

A group which does not leave invariant a point, line, or triangle and which contains homologies of period greater than 5 will contain two such homologies which are not commutative. These two homologies will leave invariant the point of intersection of their axes and the line joining their centers. But they cannot generate on the line joining their centers a group which is cyclic, dihedral, tetrahedral, octahedral, or icosahedral. Hence no group can contain homologies of period greater than 5.

Theorem 7. *No multiplicative group which does not leave invariant a point, line, or triangle can contain homologies of period* 5.

Consider two homologies of period 5 which are not commutative. They must generate the icosahedral group on the line joining their centers. Since there are in the icosahedral group involutions interchanging the fixed points of the C_5 there will be two homologies of period 5 represented on the fixed line by the same C_5. As a product we may then obtain an homology of period 5 having the fixed line for axis. But there will also be a reflection having that line for axis, since the icosahedral group contains a four-group as a subgroup (Theorem 1). There must then be an homology of period 10 having that line for axis, which is impossible (Theorem 6).

Theorem 8. *No multiplicative group which does not leave fixed a point, line, or triangle can contain homologies of period* 4.

Consider two homologies of period 4 which are not commutative. They must generate the octahedral group on the line joining their centers. The centers and axes of the two homologies must then harmonically separate each other. We choose them as (010) and $x_2 = 0$, (01–1) and $x_2 - x_3 = 0$. Since in a G_{24} there are involutions interchanging the two fixed points of a C_4 there is an homology with center (001) and axis $x_3 = 0$ and consequently an homology with center (100) and axis $x_1 = 0$. Hence there must be an axis joining any two centers. Consequently, since but six axes can pass through (100), there can be no homologies the centers of which do not lie on one of those six axes. Since we assume no line remains invariant under the group there must be other centers on each of the six axes. If a center on $x_2 = 0$ be chosen as (10–1), the corresponding axis must harmonically separate (100) and $x_1 = 0$, and will therefore be $x_1 - x_3 = 0$. But (10–1) and $x_1 - x_3 = 0$ do not harmonically separate (01–1) and $x_2 - x_3 = 0$. The two homologies having them for centers and axes cannot then generate an octahedral group on $x_1 + x_2 + x_3 = 0$. No group containing homologies of period 4 is therefore possible.

§6. Multiplicative Groups containing Homologies of Period 3.

Theorem 9. *The Hessian group G_{216} is the only multiplicative group containing homologies of period* 3, *which does not leave invariant a point, line, or triangle.*

A group which contains homologies of period 3 and which does not leave invariant a point, line, or triangle, will contain two such homologies which are not commutative. We choose the center and axis of the first as (100) and $x_1 = 0$, and the center of the second as (111). If the point of intersection of the two axes be chosen as (01–1), the axis of the second will be some line of the pencil, $Ax_1 + x_2 + x_3 = 0$. The group of the points on the line, $x_2 - x_3 = 0$, which joins the two centers, must be the tetrahedral group G_{12}. For if it were the octahedral or icosahedral group there would be an homology of period 3 having $x_2 - x_3 = 0$ for axis, since in either of these groups there are involutions

interchanging the fixed points of the C_3. But since there must be a reflection having that line for axis (Theorem 1), there would then be an homology of period 6 having that line for axis, which is impossible (Theorem 6). In order that the two homologies shall generate the tetrahedral group on $x_2 - x_3 = 0$, the axis of the second must be $x_1 + x_2 + x_3 = 0$. The two homologies are then

$$[\omega x_1,\ x_2,\ x_3],$$

$$x_1' = (\omega + 2)x_1 + (\omega - 1)x_2 + (\omega - 1)x_3,$$

$$x_2' = (\omega - 1)x_1 + (\omega + 2)x_2 + (\omega - 1)x_3, \quad (\omega^2 + \omega + 1 = 0).$$

$$x_3' = (\omega - 1)x_1 + (\omega - 1)x_2 + (\omega + 2)x_3$$

There are then on $x_2 - x_3 = 0$ the four centers (100), (111), (ω11), ($\omega^2$11), and there are four axes $x_1 = 0$, $x_1 + x_2 + x_3 = 0$, $\omega^2 x_1 + x_2 + x_3 = 0$, $\omega x_1 + x_2 + x_3 = 0$, passing through (01–1). The group is of order 24, having (2, 1) isomorphism with the tetrahedral group. The transformations which are represented by C_2 on $x_2 - x_3 = 0$ are C_4 in the plane. Together these three C_4 form the quaternion G_8.

In any group containing this group of order 24 every other homology of period 3 must be commutative with one of the above four. For consider one which is not. It cannot generate with the reflection with center (01–1) and axis

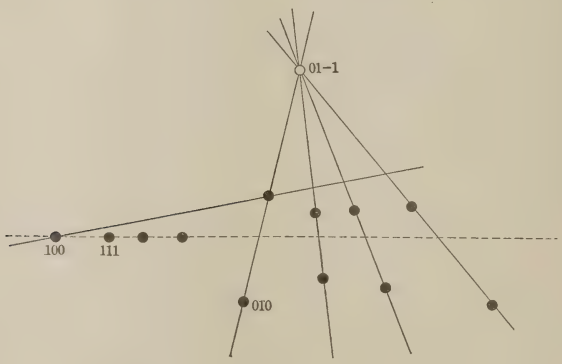

FIG. 2.

$x_2 - x_3 = 0$ a tetrahedral group on the line joining their centers, since we have seen that the transformations in the plane which are represented by C_2 in the tetrahedral group are C_4 in the plane. The group on the line joining the centers must then be dihedral. But this will involve an homology of period 3 having for axis the line joining the centers. But since the group of the lines through

(01–1) must be the tetrahedral group, there can be but four axes through it. There can be therefore no such homology.

There can be but two homologies of period 3 commutative with each of the four homologies in the group of order 24, since if the group of the points on one of the four axes is not dihedral, there will be a reflection having that line for axis (Theorem 1) and hence an homology of period 6 having that line for axis. If a center on $x_1 = 0$ be chosen as (010), the corresponding axis must pass through (100) and with the center harmonically separate (01–1) and $x_2 - x_3 = 0$, which are the center and axis of the reflection (Fig. 2). The homology is then $[x_1, \omega x_2, x_3]$.

There are then twelve homologies of period 3 in the group which form by threes four triangles. The group leaving one of the triangles point-wise invariant is of order 9, all six permutations are made on its vertices, and the group is transitive on the triangles. Its order is therefore $9 \cdot 6 \cdot 4 = 216$. It is the well-known Hessian group, permuting among themselves the nine inflexional points of a cubic curve. In this case they are (01–1), (01–ω), (01–ω^2), (10–1), (10–ω), (10–ω^2), (1–10), (1–ω0), (1–$\omega^2$0). It is the largest group which leaves invariant the Abelian G_9 generated by $[x_1, \omega x_2, \omega^2 x_3]$, $[x_2, x_3, x_1]$. From the unique choice of coördinates it follows that there is a single conjugate set of G_{216} under the whole collineation group of the plane.

§ 7. Multiplicative Groups which contain Reflections; General Properties.

Having made an exhaustive study of multiplicative groups which contain homologies of higher period than 2, we will assume in the remaining discussion of those groups that the only homologies which are present are reflections. We suppose the existence of a group which contains reflections and consider a particular reflection in the group. We denote its center and axis by A and a respectively. There will be a group commutative with it which is (2, 1) isomorphic with the group of the points on a. There will be points conjugate with A under the group, some of which will not lie on a, since we assume that no triangle is left invariant. The points conjugate with A will lie on lines through A which will form one or more conjugate sets. Such a line may be an axis of a reflection, in which case its center will lie on a (Theorem 5). There will then be a center at its intersection with a, which may or may not be conjugate with A.

Consider a line through A which contains centers of reflections and which is not itself an axis of a reflection. There cannot then be a reflection with center at its intersection with a. The group of the points on this line cannot contain a four-group as a subgroup and hence will be dihedral, containing a cyclic base of odd order. Since a dihedral group may be generated by two reflections, the axes of all the reflections, whose centers lie on the line, will pass through a point.

All the reflections with centers on the line will be conjugate. We shall refer to a line of this character as an o-line.

There will be a cyclic group of odd order, d, having the o-line for one of the sides of its fixed triangle. This cyclic group will be self-conjugate in general only under the dihedral group of order $2d$. If however $d = 3$, it may be self-conjugate under a group of order 18, under which all six permutations are made on the vertices of its fixed triangle (Theorem 3). If therefore $d \neq 3$, there will be conjugate with this cyclic group $\Omega/2d$ cyclic groups, where Ω denotes the order of the whole group. In these groups there will be $(d-1)\Omega/2d$ transformations excluding the identity. If $d = 3$, there will be conjugate with the cyclic group either $\Omega/2\cdot 3$ or $\Omega/6\cdot 3$ cyclic groups containing

$$(3-1)\frac{\Omega}{2\cdot 3} \quad \text{or} \quad (3-1)\frac{\Omega}{6\cdot 3}$$

transformations other than the identity. If two o-lines are not conjugate, the two cyclic groups which leave them fixed will not be conjugate. Corresponding then to each conjugate set of o-lines on which the group is dihedral there will be a conjugate set of cyclic groups of type I.

Any two o-lines through the center of a reflection which are conjugate under the whole group are conjugate also under the group commutative with that reflection. For if two o-lines, b and b', through A are conjugate, there are transformations which send b to b' and a center on b into any center on b'. In particular there are transformations which leave A fixed and send b to b'. Hence if the group commutative with a reflection be of order g, there will be $\frac{1}{2}g$ o-lines in each conjugate set through its center.

Theorem 10. *In a multiplicative group which does not contain homologies of period 3 but which contains a C_3 self-conjugate under a G_9 permuting cyclically the vertices of its fixed triangle, any two of the C_3 which are conjugate under the whole group are conjugate also under the subgroup leaving the G_9 invariant.*

We choose two of the C_3 in the G_9 as those generated by $[x_1, \omega x_2, \omega^2 x_3]$, $[x_2, x_3, x_1]$, where $\omega^2 + \omega + 1 = 0$. The four fixed triangles of the C_3 in the G_9 then form the Hessian configuration. All the G_9 in which one of these C_3 lies form a single conjugate set. For if the group leaving its fixed triangle point-wise invariant is of order d, it will lie in $d/3$ groups G_9. But the group leaving the triangle point-wise invariant which also leaves one of the G_9 invariant is of order 3, since we have assumed no homologies of period 3 to be present. Hence the $d/3$ groups G_9 are all conjugate.

If there are transformations in the group which transform one C_3 into another C_3 in the same G_9, there will then be transformations which transform the first C_3 into the second, and a G_9 in which the first lies into any G_9 in

which the second lies. In particular the two C_3 will be conjugate under the group which leaves invariant the G_9 in which both lie.

§ 8. Multiplicative Groups containing no Four-Groups.

Theorem 11. *The only multiplicative groups, which do not leave invariant a point, line, or triangle, and which contain reflections but no four-groups, are the two Hessian groups G_{36} and G_{72}.*

If a group contains reflections but no four-groups, all the centers of the reflections will lie on o-lines through one of them. Consequently all reflections in the group will be conjugate. Commutative with each reflection there will be a certain group, whose order we denote by g. No transformation except the identity can be commutative with more than one of the reflections. If Ω denotes the order of the whole group, there will then be Ω/g reflections and $(g-1)\Omega/g$ transformations other than the identity in the groups commutative with the reflections. We suppose there are r conjugate sets of o-lines on which the centers of the reflections lie. The order of the group will then be equal to the product of the order of the group commutative with a single reflection times the number of reflections. Hence

$$\Omega = g\left\{1 + \frac{g}{2}\sum_{i=1}^{r}(d_i - 1)\right\}.$$

The order of the group will also be equal to the number of transformations which it contains, as follows:

$$\Omega = 1 + (g-1)\frac{\Omega}{g} + \sum_{i=1}^{r}(d_i - 1)\frac{\Omega}{f_i d_i} + \cdots,$$

where, if $d_i \neq 3, f_i = 2$; if $d_i = 3, f_i = 2, 6$.

We find that there are the following four solutions:

$$g = 2,\ r = 1,\ f_1 = 2,\ \Omega = 2d.$$

$$g = 2,\ r = 4,\ d_i = 3,\ f_i = 6,\ \Omega = 18.$$

$$g = 4,\ r = 2,\ d_i = 3,\ f_i = 6,\ \Omega = 36.$$

$$g = 8,\ r = 1,\ d_1 = 3,\ f_1 = 6,\ \Omega = 72.$$

The first solution gives simply a single dihedral group. In each of the other three cases there are four C_3, each of which is invariant under a G_{18}. If two C_3 be $[x_1, \omega x_2, \omega^2 x_3]$, $[x_2, x_3, x_1]$, where $\omega^2 + \omega + 1 = 0$, the four C_3 are then determined. If these groups exist they must then be subgroups of the Hessian G_{216}. They do exist, since the G_{216} is readily shown to be isomorphic with a tetrahedral group permuting the four triangles.

§ 9. Multiplicative Groups containing Four-Groups.

Theorem 12. *If a multiplicative group contains four-groups, every dihedral group which it contains will be contained by a group leaving invariant a conic.*

We choose two reflections which generate the dihedral group as those with centers (001) and (011), and axes $x_3 = 0$ and $x_2 + \lambda x_3 = 0$ respectively. The invariant family of conics is then $Ax_1^2 + x_2^2 + \lambda x_3^2 = 0$, where A is the parameter (Fig. 3).

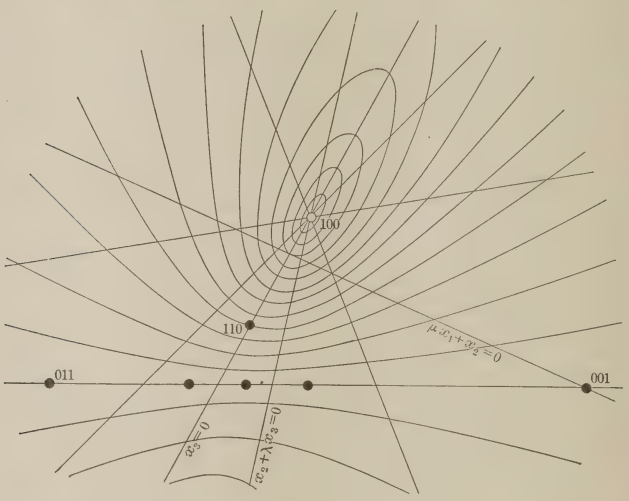

Fig. 3.

If there is a single reflection which does not lie in a four-group, the centers of all the other reflections will lie on o-lines through its center. Hence all reflections will be conjugate and no reflection can lie in a four-group. If four-groups do appear therefore, every reflection must lie in a four-group. Consider for example the reflection with center (001) and axis $x_3 = 0$, and suppose that the only reflections with which it is commutative have centers at (100) and (010). Since no triangle is supposed left invariant, (100) will be conjugate with points not on $x_1 = 0$ or $x_2 = 0$. These points must then lie on o-lines through (001), and hence must be conjugate with (001). But if (100) is a center of a reflection, more than two reflections will be commutative with that reflection. Hence more than two reflections in such a case must be commutative with the reflection with center (001).

There will be then in any case reflections commutative with that with center (001), whose centers are not (100) or (010). If the center of such a reflection

is (110) and its axis is $\mu x_1 + x_2 = 0$, the reflection will leave invariant in common with the dihedral group the conic $\mu x_1^2 + x_2^2 + \lambda x_3^2 = 0$.

Theorem 13. *A group which does not leave invariant a point, line, or triangle, and which contains four-groups, will contain either an octahedral group G_{24} or an icosahedral group G_{60}.*

Any group which does not leave invariant a point, line, or triangle, and which contains reflections, will contain two reflections which are not commutative and hence a dihedral group. It will then contain a group leaving invariant a conic (Theorem 12). The groups which leave a conic invariant are the regular solid groups. The only two which contain dihedral groups as subgroups are the octahedral group G_{24}, and the icosahedral group G_{60}.

Theorem 14. *The G_{168}^7 is the only multiplicative group which contains an octahedral group G_{24}, but not an icosahedral group G_{60}.*

If we choose the four points which are permuted by the octahedral group as (111), (–111), (1–11), (11–1) (Fig. 4), it may be generated by the following transformations:

$$S\colon\ [-x_1, x_3, x_2],$$

$$T\colon\ [x_2, x_1, x_3],$$

$$U\colon\ [x_1, x_3, x_2].$$

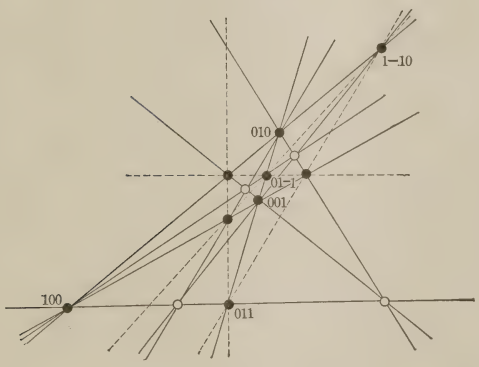

FIG. 4.

These generators satisfy the relations:

$$S^2 = T^2 = U^2 = (ST)^3 = (SU)^2 = (TU)^3 = I.$$

The G_{24} contains two sets of reflections, three in one set conjugate with SU and six in the other set conjugate with U. Any group which contains the G_{24} must contain reflections not in the G_{24}, since the latter can be invariant under

no larger group. Every reflection not commutative with one of the three reflections conjugate with SU must generate with the four-group containing those three reflections an octahedral group, since it will leave invariant a conic common to that four-group. There can be no dihedral group of order greater than 8 (Theorem 12). Hence each of the reflections conjugate with SU can be commutative with but four reflections. The invariant four-group of the G_{24} generated by S, T, and U, i. e., the four-group with fixed triangle $(100)(010)(001)$, can therefore lie in but three other octahedral groups, i. e., groups having for fixed triangles $(100)(011)(01\text{–}1)$, $(010)(101)(10\text{–}1)$, $(001)(110)(1\text{–}10)$. Hence every reflection in the group containing the G_{24} must be commutative with a reflection of the G_{24}. No more reflections can be commutative with a reflection which is conjugate under the G_{24} with SU, and but two more reflections can be commutative with each of the six reflections which are conjugate with U under that group. The only group which may contain the G_{24} is therefore a group containing $3 + 6 + 6 \cdot 2 = 21$ reflections. The centers of these twenty-one reflections must lie on the four axes and on four o-lines through a center of one of them. They will all be conjugate, and the order of the group will be $8 \cdot 21 = 168$.

A reflection commutative with U must generate with S a dihedral group of order 8 (Theorem 12). Such a reflection is

$$V: \begin{aligned} x_1' &= -\lambda^2 x_2 - \lambda^2 x_3, \\ x_2' &= -2x_1 + \lambda x_2 - \lambda x_3, \\ x_3' &= -2x_1 - \lambda x_2 + \lambda x_3. \end{aligned}$$

This reflection must generate a G_{24} with the dihedral G_6 generated by T and U, and hence either its product by T must be of period 3 or the product of VU by T must be of period 3. Without loss of generality we may take $(TV)^3 = I$. The condition for this is $\lambda^2 - \lambda + 2 = 0$, $\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{-7}$. In this case we have the generational relations:

$$V^2 = (VU)^2 = (VT)^3 = I, \qquad VSVS = U.$$

A group of order 168 is then generated by S, T, U, and V. We suppose the operators of the G_{24} written down in a horizontal row, and form a multiplication table (multiplying on the left) of seven rows, of which the multipliers are I, V, TV, UTV, STV, $SUTV$, $TSUTV$. If these seven rows be numbered from 1 to 7 in the order in which their multipliers are written, they are permuted as follows by the four generators when applied as left-hand multipliers:

$$\begin{aligned} S&: (1)(2)(35)(46)(7), \\ T&: (1)(23)(4)(5)(67), \\ U&: (1)(2)(34)(56)(7), \\ V&: (12)(3)(4)(56)(7). \end{aligned}$$

The 168 operators are thus permuted among themselves. The rows must all be distinct, for if two coincide, all must coincide. This is impossible, since V does not lie in the octahedral group. Hence a group of order 168 is generated. It is representable on seven letters, e. g., on the seven rows, and is simple.

Theorem 15. *The G^{6}_{360} is the only multiplicative group containing an icosahedral group G_{60}.*

We choose (100), (010), (001) as the three centers of one of the four-groups of a G_{60}. Since a four-group in a G_{60} is self-conjugate under a tetrahedral group, there will be four C_3 permuting cyclically these three centers. Each of the C_3 must be self-conjugate under a G_6. We choose as (111) the fixed point of a C_3 through which pass the three axes of reflections. Two of the transformations of the tetrahedral group are then

$$E_1\colon [x_2, x_3, x_1], \qquad E_2\colon [x_1, -x_2, -x_3].$$

A reflection generating with E_1 a dihedral G_6 takes the form

$$E_3\colon \begin{aligned} x_1' &= (\alpha^2+\alpha)x_1 - \alpha x_2 + (\alpha+1)x_3, \\ x_2' &= -\alpha x_1 + (\alpha+1)x_2 + (\alpha^2+\alpha)x_3, \\ x_3' &= (\alpha+1)x_1 + (\alpha^2+\alpha)x_2 - \alpha x_3. \end{aligned}$$

The centers of the fifteen reflections in the group with the exception of (010) and (001) must lie on o-lines through (100) which contain three or five centers (Fig. 5). At least one of the three reflections in the dihedral group containing

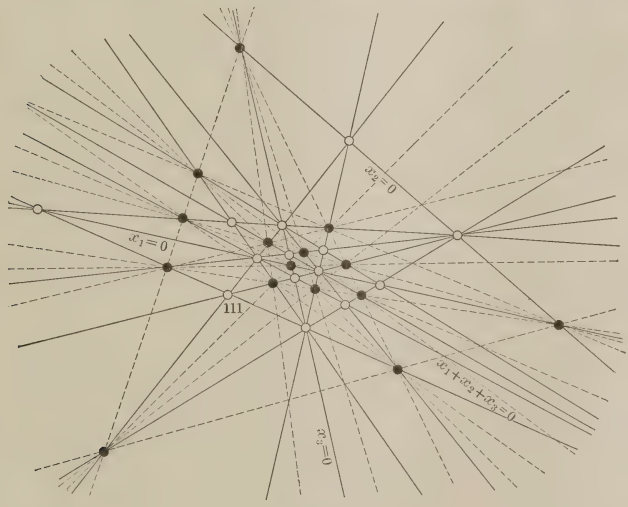

FIG. 5.

E_1 must be such that its product by E_2 is of period 3. In order that $(E_2E_3)^3 = I$, we must have $\alpha^2 + \alpha - 1 = 0$, $\alpha = (-1 \pm \sqrt{5})/2$. In this case the three transformations satisfy the following relations:

$$E_1^3 = E_2^2 = E_3^2 = (E_1E_2)^3 = (E_1E_3)^2 = (E_2E_3)^3 = I.$$

They therefore generate the icosahedral group G_{60}.*

There is a single conjugate set of G_{60} under the whole collineation group, since the two G_{60} which correspond to the two values of α are conjugate under the transformation $[x_1, x_3, x_2]$.

We inquire first whether any group can contain the G_{60} and have only two reflections commutative with each reflection. The reflections in such a group must arrange themselves by threes in four-groups, the vertices of the fixed triangles of these four-groups being permuted cyclically. Every reflection not in the four-group with fixed triangle (100)(010)(001) must generate with that four-group a G_{60}. Corresponding then to each pair of o-lines through one center which contain 3 centers there will be a pair of o-lines which contain 5 centers. Since there is no C_4 commutative with a reflection, the group cannot contain as a subgroup either the Hessian G_{36} or the G_{72}. Consequently since there can be but one conjugate set of o-lines containing 3 centers, the vertices of the fixed triangles of the C_3 cannot be permuted cyclically (Theorem 10).

If we denote by Ω the order of the whole group and by g the order of the group commutative with a four-group, Ω must then satisfy a Diophantine equation of the form:

$$\Omega = 1 + (g-1)\frac{\Omega}{3g} + (5-1)\frac{\Omega}{2\cdot 5} + (3-1)\frac{\Omega}{2\cdot 3} + \cdots.$$

In order that the coefficient of Ω on the right shall be less than unity, we must have $g = 4$, $\Omega = 60$. Hence no group can contain the G_{60} if only two reflections are commutative with a reflection.

We suppose next that a reflection is commutative with more than two reflections. Since the group cannot contain a dihedral group of order greater than 10 (Theorem 12), the centers and axes of two reflections which are commutative with the same reflection, but not with each other, must separate each other harmonically. There can then be but four reflections commutative with any reflection. A four-group can then be commutative only with itself. A reflection commutative with E_2 must then generate a G_{24} with the G_{12} generated by E_1 and E_2. Such a G_{24} is an octahedral group and will leave invariant a conic. This conic cannot be $x_1^2 + x_2^2 + x_3^2 = 0$, since that is left invariant by a G_{60}. It must then be $x_1^2 + \omega x_2^2 + \omega^2 x_3^2 = 0$, where ω is a cube root of unity. There must be at least 45 reflections in the group, since a reflection not in the G_{60}

* E. H. Moore, Proceedings of the London Mathematical Society, vol. 28 (1897), pp. 357-366; Dickson, *Linear Groups*, p. 289.

must be conjugate with at least 30 reflections under that group. We may show that the maximum number must be 45. There are three reflections in the invariant four-group of the G_{12} generated by E_1 and E_2, and two more reflections commutative with each of those three. Every reflection not commutative with any one of the three reflections of the four-group will leave a conic invariant in common with the four-group, and hence must generate with the four-group either a G_{24} or a G_{60}. The four-group can lie in but three octahedral groups other than the one under which it remains invariant, i. e., octahedral groups whose invariant four-groups contain a reflection in common with the above four-group. Each of the three G_{24} will contain four reflections not commutative with any one of three reflections of that four-group. Since the four-group can be commutative only with itself, every G_{60} in which it lies must contain the G_{12} generated by E_1 and E_2. Since the conic $x_1^2 + \omega x_2^2 + \omega^2 x_3^2 = 0$ is left invariant by a G_{24}, there can be but two such G_{60}, i. e., those having for fixed conics $x_1^2 + x_2^2 + x_3^2 = 0$, $x_1^2 + \omega^2 x_2^2 + \omega x_3^2 = 0$. There can then be but $3 + 3\cdot 2 + 3\cdot 4 + 2\cdot 12 = 45$ reflections. There will be four o-lines through the center of a reflection, on which lie 5 centers. Hence the group commutative with a reflection must be of order 8 and the whole group of order $8\cdot 45 = 360$. One of the reflections commutative with E_2 is

$$E_4\colon\ [x_1,\ \omega^2 x_3,\ \omega x_2]\,.$$

This reflection satisfies with the three generators of the G_{60} the following relations:

$$E_4^2 = (E_1 E_4)^2 = (E_2 E_4)^2 = (E_3 E_4)^3 = I.$$

It therefore generates with the G_{60} a G_{360}^6.* Two G_{360}^6 corresponding to the choice of ω contain the G_{60}. The G_{360} however contains two sets of G_{60} and a transformation carrying a G_{60} not conjugate with the G_{60} generated by E_1, E_2, E_3 into the latter transforms one G_{360} into the other. There is therefore a single conjugate set of G_{360} in the plane.

§ 10. Groups containing only Transformations of Type I.

A complete discussion of multiplicative groups which contain homologies has been given. Any group which contains only transformations of type I must be of odd order, since a transformation of period 2 is a reflection. Any cyclic group contained by such a group can then be self-conjugate only under itself or under a group permuting cyclically the vertices of its fixed triangle. If Ω denotes the order of the whole group, it must then satisfy a Diophantine equation of the form

$$\Omega = 1 + \sum_{i=1}^{r} (d_i - 1)\,\frac{\Omega}{f_i d_i} \qquad (f_i = 1, 3)\,.$$

* Moore, loc. cit., Dickson, loc. cit.

If the order of the group is divisible by 3, we may take d_1 to be divisible by 3. If then $f_1 = 3$, there must be four conjugate sets of C_3 (Theorem 10). Hence $r \geqq 4$, and each of the first four d's is divisible by 3. This can happen only if $r = 4$, $d_2 = d_3 = d_4 = 3$, $\Omega = 3d_1$. If on the other hand $f_1 = 1$, we find $r = 1$, $\Omega = d_1$, or $r = 2$, $d_1 = 3$, $\Omega = 3d_2$.

If the order of the group is not divisible by 3, we have $r = 1$, $\Omega = d_1$. We have therefore

Theorem 16. *No multiplicative group which does not leave invariant a point, line or triangle can contain only transformations of type I.*

The discussion of the groups in the ordinary plane is therefore complete.

§ 11. Groups containing Transformations of Type III, but not Elations.

We now consider the possibility of groups containing transformations of type III, but not elations, and consequently no transformations of type II. There can be no homologies present which have a common center but different axes, or vice-versa, so that Theorem 4 holds. Theorem 5 does not hold, but instead we have

Theorem 17. *A group containing transformations of type III but not elations cannot contain two homologies such that the line joining their centers passes through the intersection of their axes unless they are both of period* 2.

The product of two homologies such that the line joining their centers passes through the intersection of their axes will leave invariant two points on that line unless one transformation on that line is conjugate with the inverse of the other. Unless the homologies are both of period 2 we may choose their powers so that this is not the case. The product is therefore of type II in the plane.

Theorems 6, 7, 8, 9 are proved independently of the assumption of the non-existence of transformations of type III in the group. The Hessian group G_{216} however contains no transformations of type III. Hence we have

Theorem 18. *There is no group which contains transformations of type III but not elations, which does not leave invariant a point, line, or triangle, and which contains homologies of higher period than* 2.

If transformations of type III appear, there may be a new kind of o-lines through the center of a reflection, i. e., o-lines on which the group of the points is metacyclic, containing an additive base. There cannot be more than a single conjugate set of such o-lines, since their number must be of the form $1 + fp^m$, where p^m denotes the order of the additive group on the line. If the order of the group commutative with a reflection is g, and the order of the subgroup of that group which leaves fixed an o-line of this sort is h, the number of such o-lines through the center of the reflection will be g/h.

Theorem 10 holds in this case also. In the proof of Theorem 11 however it

is necessary to consider the additional case where o-lines of the sort described above appear.

If there are no reflections commutative with a reflection, and if there is a set of o-lines, each of which is left invariant by a metacyclic group of order hp^m, any group which exists (of order Ω) will contain $(g-1)\Omega/g$ transformations in the groups commutative with the reflections and $(p^m-1)\Omega/hp^m$ transformations of type III. Hence Ω must satisfy an equation of the form:

$$\Omega = 1 + (g-1)\frac{\Omega}{g} + (p^m-1)\frac{\Omega}{hp^m} + \cdots.$$

In order that the coefficient of Ω on the right shall be less than unity, no more terms representing transformations can appear, and also $h = g$. We then have $\Omega = gp^m$, which represents the single metacyclic group.

If we combine this result with Theorem 11, and observe that the G_{36} and G_{72} contain no transformations of type III, we have

Theorem 19. *No group exists which contains transformations of type III but not elations, which does not leave invariant a point, line, or triangle, and which contains reflections but no four-groups.*

Theorem 12 holds in this case without modification. We prove also

Theorem 20. *If a group containing four-groups contains a metacyclic group of order $2p^m$ (the operators of period p being of type III), that metacyclic group will be contained by a group leaving invariant a conic.*

A metacyclic group of order $2p^m$ will leave invariant a one-parameter family of conics. For consider a single transformation of type III, such as

$$[x_1 + x_2 + x_3,\ x_2 + 2x_3,\ x_3].$$

This leaves invariant the family $x_2^2 + Ax_3^2 - 4x_1x_3 = 0$ (Fig. 6). The above

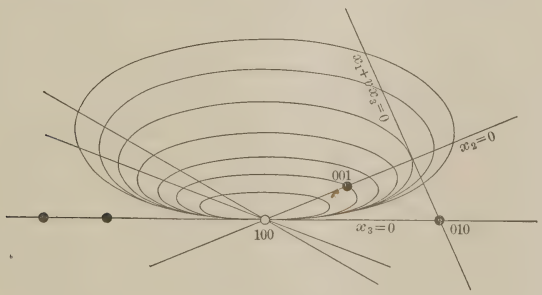

FIG. 6.

transformation is invariant under any transformation of type III leaving fixed (100) and $x_3 = 0$ and the same family of conics. Hence we may choose the

center of a reflection in the metacyclic group arbitrarily as (010). The axis will be a line, $x_2 + \mu x_3 = 0$, passing through (100). This reflection will transform the above transformation into a transformation which is its inverse on $x_3 = 0$. The transformed transformation is

$$[x_1 - x_2 + (1 - 2\mu)x_3,\ x_2 - 2x_3,\ x_3].$$

The product of the two is an elation unless $\mu = 0$. The reflection must therefore leave fixed the conics. Hence every reflection in the metacyclic group leaves fixed the conics. Since the metacyclic group of order $2p^m$ may be generated by the p^m reflections which it contains, it leaves fixed the conics.

Since we have assumed the existence of four-groups, the reflection with center (010) and axis $x_2 = 0$ will lie in a four-group. Since the metacyclic group is invariant under any elation with center (100) and axis $x_3 = 0$, we may choose the center of a reflection in that four-group arbitrarily as (001). Its axis will be a line, $x_1 + \nu x_3 = 0$, passing through (010). This reflection leaves invariant in common with the metacyclic group the conic $x_2^2 - 2\nu x_3^2 - 4x_1x_3 = 0$.

Theorem 21. *If a group containing four-groups contains transformations of type III, it will contain a group leaving invariant a conic, which is either an octahedral group G_{24}, an icosahedral group G_{60}, a group of order $(p^m + 1)p^m(p^m - 1)/2$, or a group of order $(p^m + 1)p^m(p^m - 1)$.*

Any group which contains reflections will contain two reflections which are not commutative, provided no point, line or triangle is left invariant. Hence it will contain either a dihedral group or a metacyclic group and therefore (either by Theorem 12 or by Theorem 20) a group leaving invariant a conic. The group leaving a conic invariant is simply isomorphic with the group of the points on a line.* Hence this group must be either an octahedral group G_{24}, an icosahedral group G_{60}, a group of order $(p^m + 1)p^m(p^m - 1)/2$, or a group of order $(p^m + 1)p^m(p^m - 1)$. (See § 1.)

Theorem 22. *A $G^7_{168}(p = 3)$ is the only group containing transformations of type III such that its largest subgroup leaving invariant a conic is a G_{24}.*

The proof of this theorem is the same as that of Theorem 14; except that, having obtained the group of order 168, we have yet to determine when it contains transformations of type III. This will be the case only if $p = 3, 7$. For $p = 7$ the G_{168} leaves invariant a single conic. Hence the theorem is as stated.

Theorem 23. *A $G^6_{360}(p = 5)$ and a $G_{720}(p = 5)$ are the only groups containing transformations of type III such that their largest subgroup leaving invariant a conic is a G_{60}.*

The proof is the same as that of Theorem 15 with slight modifications. We make use of Theorem 20 as well as of Theorem 12 in showing that the only

* Veblen and Young, *Projective Geometry*, Chapter VIII, Theorem 15.

o-lines which exist contain 3 or 5 centers. We show in the same way that there can be but 45 reflections in the group. It was shown under Theorem 15 that the group commutative with a reflection was of order 8, since there must be through its center four o-lines containing 5 centers each. For $p = 5$ however it is possible that there may also be a group of order 16 permuting these four o-lines. A transformation of period 8 transforming the $G_{360}(p = 5)$ into itself is given by

$$[x_1,\ (2\omega + 1)x_2 + (\omega - 1)x_3,\ -(\omega + 2)x_2 + (2\omega + 1)x_3].$$

A group of order 720 therefore exists.

The G_{360} also contains transformations of type III if $p = 3$. In this case however it leaves invariant a single conic. Hence the theorem is as stated.

§ 12. Groups containing Transformations of Type III which contain larger Subgroups leaving a Conic Invariant.

Theorem 24. *There exists no group which does not contain elations or homologies of higher period than* 2, *and which contains a set of reflections each of which is commutative with a group* (2, 1) *isomorphic with either the tetrahedral, octahedral, or icosahedral group.*

If a group commutative with a reflection is (2, 1) isomorphic with either the tetrahedral or icosahedral group, it can contain no four-group. For all the involutions in either a G_{12} or a G_{60} lie in four-groups (on the axis), in any one of which all three involutions are conjugate. If two of the involutions are performed by reflections, the third will be performed by a C_4. But this is impossible, since the three involutions are conjugate. Hence no one of the involutions on the line can be performed by a reflection.

If the group commutative with a reflection is (2, 1) isomorphic with an octahedral group, none of the involutions of the included tetrahedral group can be performed by reflections. The six involutions not in the tetrahedral group however may be performed by reflections. In this case there will be twelve reflections commutative with the given reflection. The group commutative with the given reflection will contain four conjugate dihedral groups of order 12 and three conjugate dihedral groups of order 8.

Each of these dihedral groups will be contained by a group leaving invariant a conic (Theorem 12). Hence some of the twelve reflections commutative with the given reflection will be conjugate with that reflection, and hence all will be. There will then be twelve reflections commutative with every reflection. A dihedral group of order 8 will therefore lie in a group leaving invariant a conic which contains two conjugate sets of reflections. Such a group will be of order $(p^m + 1)p^m(p^m - 1)$, where $2(p^m - 1) = 8$, $p^m = 5$. No group is then possible unless $p = 5$.

Consider now a dihedral G_{12}. Each reflection commutative with one of the six reflections of this G_{12} (other than the invariant reflection) must generate a G_{120} with the G_{12}. Since a G_{120} contains 25 reflections and the G_{12} contains 7, the reflections commutative with these six reflections which are not in the dihedral group must be grouped in sets of 18. But there are 60 such reflections and hence this is impossible. No group can then exist for $p = 5$.

Theorem 25. *A G^7_{2520} is the only group such that its largest subgroup leaving invariant a conic is a G_{120} $(p = 5)$, provided that no elations are present.*

A G_{120} $(p = 5)$ is generated by

$$E_1\colon [x_2, x_3, x_1], \qquad E_2\colon [x_1, -x_2, -x_3], \qquad E_5\colon [x_1, x_3, x_2],$$

$$E_3\colon x_1' = x_1 - 2x_2 - 2x_3, \qquad x_2' = -2x_1 - 2x_2 + x_3, \qquad x_3' = -2x_1 + x_2 - 2x_3.$$

The invariant conic is $x_1^2 + x_2^2 + x_3^2 = 0$. The G_{120} contains two conjugate sets of reflections. Each reflection conjugate with E_2 is commutative with a dihedral group of order 8; each reflection conjugate with E_5 is commutative with a dihedral group of order 12. (Fig. 7.)

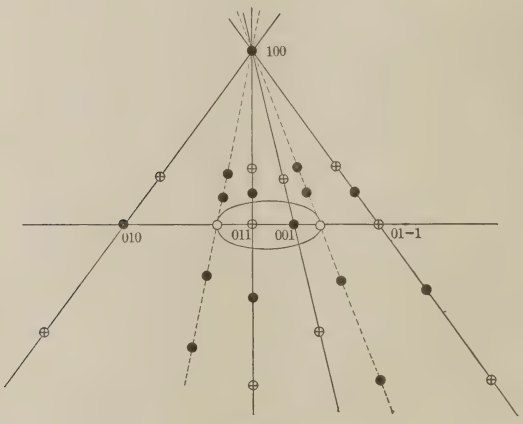

FIG. 7.

A group which contains the G_{120} will contain reflections not in the G_{120}. Since the G_{120} is supposed the largest subgroup leaving invariant a conic, no dihedral group of order greater than 12 can appear (Theorem 12). It may be shown readily that under any group containing the G_{120} E_2 must be invariant under a larger group. The only possible larger group which will not introduce a dihedral group of order greater than 12 and which will allow the group of the

points on $x_1 = 0$ and $x_1 - x_3 = 0$ to be dihedral is a G_{24} having for fixed triangle the fixed triangle of the G_{12} generated by E_1 and E_2. This G_{24} contains the C_3:

$$[x_1,\ \omega x_2,\ \omega^2 x_3] \qquad (\omega^2+\omega+1=0).$$

The fixed triangles of all the C_3 in the group will then be invariant under a G_9. The four-group which is invariant under the G_{12} generated by E_1 and E_2 will then lie in three G_{120}. Each of those G_{120} will contain 22 reflections not in that four-group. The four-group will also lie in nine G_{24} under which it is not invariant, i. e., G_{24} whose invariant four-groups have each a reflection in common with the above four-group. Each of these nine G_{24} will contain 4 reflections not in any one of the three G_{120}. There will then be $3+3\cdot 22+9\cdot 4=105$ reflections. These will all be conjugate and the order of the group must be $24\cdot 105 = 2520$.

As a product of the above C_3 and E_5 we obtain

$$E_4\colon\ [x_1,\ \omega^2 x_3,\ \omega x_2].$$

The five generators are found to satisfy the relations:

$$E_1^3 = E_2^2 = E_3^2 = E_4^2 = E_5^2 = I, \qquad (E_i E_{i+1})^3 = I, \qquad (E_i E_j)^2 = I \quad (j > i+1).$$

A G^7_{2520} is then generated.*

Theorem 26. *No group which does not contain elations can contain a group of order* $(p^m+1)p^m(p^m-1)/2$ *or* $(p^m+1)p^m(p^m-1)$ *leaving a conic invariant, provided* $p^m > 5$, *and provided it contains no larger group leaving a conic invariant.*

The group commutative with a reflection in a group of order

$$\tfrac{1}{2}(p^m+1)p^m(p^m-1) \quad \text{or} \quad (p^m+1)p^m(p^m-1)$$

leaving a conic invariant is dihedral and hence the group of the points on the axis of the reflection is dihedral. There is but one pair of points on the axis which are interchanged provided the group of the points on the axis is larger than the four-group, i. e., provided the group commutative with the reflection is of order greater than 8. For the group of order $(p^m+1)p^m(p^m-1)/2$ this will be true if $p^m \pm 1 > 8$, $p^m > 9$; for the group of order $(p^m+1)p^m(p^m-1)$ it will be true if $2(p^m \pm 1) > 8$, $p^m > 5$. If then $p^m > 5$ and if the group is not the $G_{168}(p^m = 7)$ or the $G_{360}(p^m = 3^2)$, there will be but one pair of points on the axis which are interchanged by the group.

Excluding for the moment these two cases, we may show that there can be no more reflections commutative with a reflection. For there certainly cannot

* MOORE, l. c.; DICKSON, l. c.

be more than two such reflections, i. e., the two leaving the two points pointwise invariant. (If a reflection interchanges the points, it will leave fixed the conic.) But if we add two reflections we must add more than two, since any such reflection must generate together with a dihedral group containing the reflection with which it is commutative (other than the dihedral group which leaves the latter reflection invariant) a group of order $(p^m+1)p^m(p^m-1)/2$ or $(p^m+1)p^m(p^m-1)$ leaving invariant a conic. This group can have in common with the other group which leaves fixed a conic only the reflections of the dihedral group. Hence there will be more than two reflections added which are commutative with the latter reflection.

If the group is the $G_{168}(p^m=7)$ or the $G_{360}(p^m=3^2)$, there can be no more reflections commutative with a reflection. For since the group is supposed the largest group present which leaves invariant a conic, there can be no dihedral group present of order greater than 10 (Theorem 12). Hence there can be only four reflections commutative with any reflection in the group.

Suppose now that there is a reflection not commutative with any reflection in the group. It will leave invariant in common with any four-group of that group a conic. It must generate with that four-group some group leaving invariant a conic which can contain no reflections commutative with any of the three reflections of the four-group except those three reflections themselves. Such a group must be an icosahedral group G_{60}. Every reflection not in the original group leaving fixed a conic must then generate with this four-group a G_{60}. But there cannot be any transformations commutative with the four-group other than those of the four-group itself without involving more reflections commutative with the reflections of the four-group. Hence there cannot be more than four C_3 permuting cyclically the vertices of the fixed triangle of the four-group, i. e., all the G_{60} must contain the same tetrahedral group. A tetrahedral group however leaves fixed but three conics, one of which is the original conic. There can then be but two such G_{60}, each of which contains twelve reflections other than those of the four-group. There cannot then be more than 24 more reflections, which is impossible.

§ 13. Groups containing only Transformations of Types III and I.

Theorem 27. *There exists no group which contains only transformations of types* III *and* I *and which does not leave invariant a point, line, or triangle.*

Any group which contains only transformations of types III and I must be of odd order, since a transformation of period 2 is a reflection.

An additive group of order p^m containing transformations of type III will be self-conjugate under a metacyclic group of order d_1p^m, containing p^m conjugate cyclic groups of order d_1 which contain transformations of type I. (In partic-

ular d_1 may be unity.) Any cyclic group of order d_i containing transformations of type I will be self-conjugate either under itself only or under a group of order $3d_i$ permuting cyclically the vertices of its fixed triangle.

If we denote the order of the whole group by Ω, and enumerate the transformations which the group must contain, we are led to the following Diophantine equation:

$$\Omega = 1 + (p^m - 1)\frac{\Omega}{d_1 p^m} + \sum_{i=1}^{r}(d_i - 1)\frac{\Omega}{f_i d_i} \qquad (f_i = 1, 3).$$

If $f_1 = 3$, we may show that $d_1 = 3$. Let the fixed point and line of an additive group be (100) and $x_3 = 0$ and let the vertices of the fixed triangle of a cyclic group of order d_1 in the metacyclic group, under which the additive group is self-conjugate, be (100), (010), (001). A transformation of period d_1 in that cyclic group will be given by

$$[x_1, \omega x_2, \omega^e x_3], \tag{1}$$

where the period of ω is d_1. A transformation of type III in the additive group will be given by

$$\begin{aligned} x_1' &= x_1 + \alpha x_2 + \beta x_3, \\ x_2' &= \quad x_2 + \gamma x_3, \\ x_3' &= \quad\quad x_3 \end{aligned} \qquad (\alpha \neq 0,\ \gamma \neq 0). \tag{2}$$

If we transform (2) by (1), we obtain

$$\begin{aligned} x_1' &= x_1 + \alpha\omega x_2 + \beta\omega^e x_3, \\ x_2' &= \quad x_2 + \gamma\omega^{e-1} x_3, \\ x_3' &= \quad\quad x_3. \end{aligned} \tag{3}$$

If we transform (2) by (3), we obtain

$$\begin{aligned} x_1' &= x_1 + \alpha x_2 + (\beta + \alpha\gamma\omega^{e-1} - \alpha\gamma\omega) x_3, \\ x_2' &= \quad x_2 \quad + \quad \gamma x_3, \\ x_3' &= \quad\quad\quad x_3. \end{aligned} \tag{4}$$

The transformation (4) is the same as (2) on the line $x_3 = 0$. Hence (4) must be identical with (2). Hence $\omega^{e-1} = \omega$, $e \equiv 2 \pmod{d_1}$. But if the vertices of the fixed triangle of (1) are permuted cyclically, we must have $e^2 - e + 1 \equiv 0 \pmod{d_1}$. [See the discussion under Theorem 2.] Hence $2^2 - 2 + 1 \equiv 0 \pmod{d_1}$; $d_1 = 3$.'

But if $f_1 = 3$, $d_1 = 3$, there must be four conjugate sets of C_3 (Theorem 10).

Hence $r \geqq 4$. The coefficient of Ω on the right is then greater than unity, and hence there can be no solution.

If $f_1 = 1$, $r = 1$, $\Omega = d_1 p^m$. This represents a single metacyclic group.

If $d_1 = 1$, either $r = 2$, $f_2 = 3$, $p^m = 3$, $\Omega = 3d_2$, or else $r = 1$, $\Omega = p^m$. The first represents a group permuting cyclically the vertices of a triangle, and the second a single additive group.

§ 14. Groups containing Elations.

Theorem 28. *The only groups which do not leave invariant a point or a line and which contain elations are: the $HO(3, p^{2k})$; groups containing the $HO(3, p^{2k})$ as self-conjugate subgroups of index 3, if $p^k + 1$ is divisible by 3; the $LF(3, p^k)$; groups containing the $LF(3, p^k)$ as self-conjugate subgroups of index 3, if $p^k - 1$ is divisible by 3.*

Any group which contains elations will contain a group of largest order consisting wholly of elations with a common axis and center. We denote the order of such a group by p^k. We choose the center and axis of one such group as (010) and $x_3 = 0$. Since we assume that no point or line remains invariant under the group, there will be groups of elations whose centers do not lie on $x_3 = 0$ and whose axes do not pass through (010). We choose the center and axis of such a group as (001) and $x_2 = 0$. By hypothesis then the order of the group consisting of all the elations with center (001) and axis $x_2 = 0$ will be less than or equal to p^k. The group generated by these two groups of elations will leave invariant (100) and $x_1 = 0$. The group of the points on $x_1 = 0$ cannot contain an additive group of higher order than p^k, and hence (by § 1) must be of order $(p^k + 1)p^k(p^k - 1)/2$ or $60(p^k = 3)$. (The group of order $(p^k + 1)p^k(p^k - 1)$ on the line cannot be generated by additive transformations only.)

Under either of these two groups, (010) and $x_3 = 0$ will be conjugate with (001) and $x_2 = 0$. Hence (010) and $x_3 = 0$ will be conjugate with the center and axis of any group of elations, since in any conjugate set of such groups there will be groups whose centers do not lie on $x_3 = 0$ and whose axes do not pass through (010).

If the group of the points on $x_1 = 0$ is of order $(p^k + 1)p^k(p^k - 1)/2$, we may choose it as the group generated by the two groups of elations

$$(E): [x_1, x_2 + \lambda x_3, x_3], \qquad [x_1, x_2, \lambda x_2 + x_3],$$

where λ takes all values in the $GF(p^k)$. This group contains an invariant reflection with center (100) and axis $x_1 = 0$ (Theorem 1). The group contains a set of cyclic groups of period $(p^k - 1)/2$ on $x_1 = 0$. One such group is generated by $[x_1, \eta x_2, \eta^{-1} x_3]$, where η is of period $p^k - 1$. This cyclic group is of

period $p^k - 1$ on $x_2 = 0$ and $x_3 = 0$. The group generated by the two groups of elations (E) also contains a set of cyclic groups of period $(p^k + 1)/2$ on $x_1 = 0$. Such a cyclic group which leaves fixed $(01I)$ and $(01{-}I)$, where I is a square-root of a not-square in the $GF(p^k)$, may be generated by a transformation of the form

$$x_1' = x_1, \qquad Ix_2' + x_3' = J(Ix_2 + x_3), \qquad Ix_2' - x_3' = J^{-1}(Ix_2 - x_3),$$

where J is a mark of period $p^k + 1$. This cyclic group is of period $p^k + 1$ on $Ix_2 + x_3 = 0$ and $Ix_2 - x_3 = 0$.

A group of order $60\,(p^k = 3)$ on $x_1 = 0$ is generated by the two groups of elations:

$$[x_1,\ x_2 + x_3,\ x_3], \qquad [x_1,\ x_2,\ ix_2 + x_3],$$

where $i^2 = -1$. The group contains a set of cyclic groups of period 2 on $x_1 = 0$. One such group is generated by $[x_1, -ix_2, ix_3]$. It is of period 4 on $x_2 = 0$ and $x_3 = 0$. The group in the plane is of order 120, as it contains the invariant reflection $[x_1, -x_2, -x_3]$ (Theorem 1).

Since we assume that no point or line remains invariant, there will be groups of elations whose centers do not lie on $x_1 = 0$ and whose axes do not pass through (100). We consider one such group. It must generate together with the reflection with center (100) and axis $x_1 = 0$ a group on the line joining their centers which does not contain an additive group of higher order than p^k. If $p^k > 3$, this group on the line must be of order $(p^k + 1)p^k(p^k - 1)/2$ or $(p^k + 1)p^k(p^k - 1)$. If $p^k = 3$, it must be of order 12, 24, or 60. In any case there will be a reflection having this line for axis, and having for center the point of intersection of the axis of the generating group of elations with $x_1 = 0$ (Theorem 1). The involution on $x_1 = 0$ performed by this reflection must leave invariant the group of the points on $x_1 = 0$.

In the special case where the group of the points on $x_1 = 0$ is of order $60\,(p^k = 3)$, the involution performed by this reflection on $x_1 = 0$ must belong to the group of order 60. Suppose for example that the center of the reflection is (010) and its axis $x_2 = 0$. The group of the points on $x_2 = 0$ contains a C_4 generated by $[x_1, -ix_2, ix_3]$. It must then be of order 24. But the fixed points of this C_4 will be interchanged by the group. The transformed transformation is $[ix_1, -ix_2, x_3]$. This is of period 4 on $x_1 = 0$. The group of the points on $x_1 = 0$ is now no longer of order 60 and hence it must contain an additive group of higher order than 3. We therefore exclude this case from the discussion entirely.

In the general case, in order that the reflection whose center lies on $x_1 = 0$ and whose axis passes through (100) shall permute among themselves the $p^k + 1$ centers of elations on $x_1 = 0$, either (i) its center must lie at some point conju-

gate with $(01I)$ and its axis pass through the conjugate point with respect to the $GF(p^k)$, or (ii) its center will coincide with one of the centers of elations and its axis will pass through one of the other centers.

We consider the case (i). Consider in particular the reflection with center $(01I)$ and axis $Ix_2 + x_3 = 0$. The axes of the elations with centers on $Ix_2 + x_3 = 0$ must all pass through $(01I)$. The group of the points on $Ix_2 + x_3 = 0$ contains a cyclic group of order $p^k + 1$ with fixed points (100) and $(01{-}I)$. It must then be of order $(p^k + 1)p^k(p^k - 1)$. Since there must be transformations leaving fixed $(01I)$ and interchanging (100) and $(01{-}I)$, the group of the points on $x_1 = 0$ must be of the same order. The axes of all elations whose centers lie on $x_1 = 0$ must then pass through (100). Hence a unique correspondence must be established between the centers and axes of elations and between the centers and axes of reflections. There can then be no centers of elations on any of the $p^k + 1$ axes of elations through (100) except those on $x_1 = 0$.

All the centers of elations except those on $x_1 = 0$ must then lie on the $p^k(p^k - 1)$ lines through (100) whose coördinates are in the $GF(p^{2k})$, but not in the $GF(p^k)$. Since there will be $p^k + 1$ on each of these lines, the total number of centers of elations will be

$$p^k + 1 + (p^k + 1)p^k(p^k - 1) = p^{3k} + 1.$$

All the centers not on $x_1 = 0$ will be conjugate under the group generated by the two groups of elations (E). We choose a center on $Ix_2 + x_3 = 0$ arbritrarily as $(-11{-}I)$. The $p^{3k} + 1$ centers will then be the points $(\epsilon, \alpha I + \beta, \gamma I + \delta)$, where $\epsilon, \alpha, \beta, \gamma, \delta$ are in the $GF(p^k)$ and $\alpha\delta - \beta\gamma = \epsilon^2$. (Fig. 8

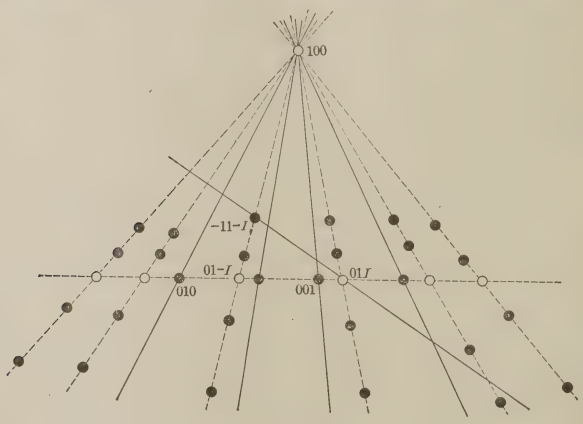

FIG. 8.

represents the configuration for $p^k = 3$.) The coefficients of the transformations permuting these points will lie in the $GF(p^{2k})$. We first seek to determine those groups which contain only transformations having for determinant a cube in the $GF(p^{2k})$.

The group will be transitive on the $p^{2k}(p^{2k} - p^k + 1)$ points whose coördinates lie in the $GF(p^{2k})$, but which are not centers of elations. The group of the points on any one of the axes of elations is metacyclic, containing p^{2k} involutions with a common fixed point. (The p^{3k} other axes of elations must meet that axis in p^{2k} points in sets of p^k each.) Hence (100) is conjugate with all the points on the axes of elations through it whose coördinates lie in the $GF(p^{2k})$ with the exception of the $p^k + 1$ centers of elations on $x_1 = 0$. It is also conjugate with all the points on the axes of reflections through it whose coördinates lie in the $GF(p^{2k})$, but which are not centers of elations.

Since all six permutations are made on the vertices of the triangle $(100)(01I)(01{-}I)$, there will be an homology of period $p^k + 1$ or $(p^k + 1)/2$ with center at each vertex (Theorem 3). But if $p^k + 1$ is divisible by 3, the determinant of an homology of period $p^k + 1$ will not be a cube in the $GF(p^{2k})$. Hence the period of the homology will be $(p^k + 1)/\nu$. (See Introduction.) The group leaving invariant (100) and $x_1 = 0$ is then of order

$$\frac{1}{\nu}(p^k + 1)^2 p^k (p^k - 1).$$

The whole group is of order

$$\frac{1}{\nu}(p^{2k} - p^k + 1)(p^k + 1)^2 p^{3k}(p^k - 1).$$

In order that the group of elations with center $(-11{-}I)$ shall permute among themselves the $p^{3k} + 1$ centers, we find that it must be given by

$$x_1' = \tfrac{1}{2}(2 + I\lambda)x_1 + \tfrac{1}{4}I\lambda x_2 - \tfrac{1}{4}\lambda x_3,$$

$$x_2' = -\tfrac{1}{2}I\lambda x_1 + \tfrac{1}{4}(4 - I\lambda)x_2 + \tfrac{1}{4}\lambda x_3,$$

$$x_3' = \tfrac{1}{2}I^2\lambda x_1 + \tfrac{1}{4}I^2\lambda x_2 + \tfrac{1}{4}(4 - I\lambda)x_3,$$

where λ takes all values in the $GF(p^k)$.

We may identify the group determined with the hyperorthogonal group,* $HO(3, p^{2k})$, by observing that it leaves invariant the function

$$x_1^{p^k+1} + \tfrac{1}{4}\left(x_2 + \frac{1}{I}x_3\right)^{p^k+1} - \tfrac{1}{4}\left(x_2 - \frac{1}{I}x_3\right)^{p^k+1}.$$

*Cf. L. E. Dickson, Mathematische Annalen, vol. 52 (1899), pp. 561–581; also *Linear Groups*, Chap. V.

The locus of the points which are such that their coördinates satisfy this function equated to zero are the $p^{3k}+1$ centers of elations.

If p^k+1 is divisible by 3, a group exists containing the $HO(3, p^{2k})$ as a self-conjugate subgroup of index 3. It may be generated by the $HO(3, p^{2k})$ together with an homology of period p^k+1.

As follows from the unique choice of coördinates, there is a single conjugate set of $HO(3, p^{2k})$ under the whole collineation group.

We turn our attention again to the two groups of elations (E), and consider case (ii). If there is a group of elations with center on one of the axes through (100), and axis passing through the intersection of another one of those axes with $x_1=0$, the group on that axis must be of order $(p^k+1)p^k(p^k-1)$, since it contains a cyclic group of order p^k-1. Hence (100) will be conjugate with each of the p^k+1 centers of elations on $x_1=0$, since the fixed points of this cyclic group will be interchanged. Hence there will be at least p^k+1 axes of elations passing through each center. There can then be no centers not on any one of the p^k+1 axes through (100), since in that case we found that a polar configuration was determined. There will be p^k+1 centers on each of

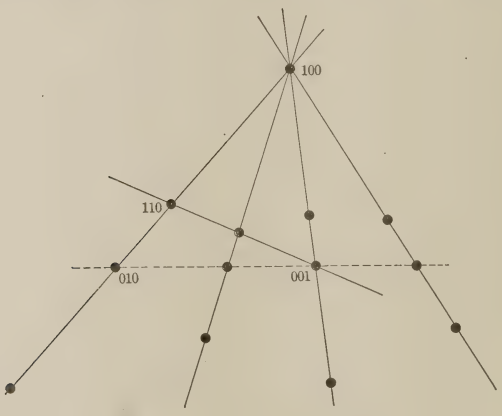

FIG. 9.

those p^k+1 axes, i. e., $p^{2k}+p^k+1$ in all. (Fig. 9 represents the configuration for $p^k=3$.)

If one of the centers on $x_3=0$ be (110), the $p^{2k}+p^k+1$ centers will be the points $(\alpha\beta\gamma)$, where α, β, γ lie in the $GF(p^k)$. The coefficients of the transformations in the group will then lie in the $GF(p^k)$. We consider first the groups containing only transformations the determinants of which are cubes in the $GF(p^k)$.

A group of elations with center (110) and axis passing through (001) will be given by

$$[(1+\lambda)x_1 - \lambda x_2,\ \lambda x_1 + (1-\lambda)x_2,\ x_3],$$

where λ takes all values in the $GF(p^k)$. Under the group generated by this group of elations together with the reflection with center (100) and axis $x_1 = 0$, (001) will remain invariant and there will be transformations interchanging (100) and (010). All six permutations will then be made on the vertices of the triangle (100)(010)(001). Hence there will be an homology of period $p^k - 1$ or $(p^k - 1)/2$ with center at each of the vertices (Theorem 3). But since if $p^k - 1$ is divisible by 3 the determinant of an homology of that period will not be a cube in the $GF(p^k)$, the period of the homology will be $(p^k - 1)/\mu$. (See Introduction.) The group of the points on any axis will be of order $(p^k + 1)p^k(p^k - 1)$, and consequently the group leaving (100) and $x_1 = 0$ fixed will be of order $(p^k + 1)p^k(p^k - 1)^2/\mu$. The group leaving fixed only (100) will be of order $(p^k + 1)p^{3k}(p^k - 1)^2/\mu$, and the whole group of order $(p^{2k} + p^k + 1)(p^k + 1)p^{3k}(p^k - 1)^2/\mu$. This is the $LF(3, p^k)$.*

If $p^k - 1$ is divisible by 3, a group exists containing the $LF(3, p^k)$ as a self-conjugate subgroup of index 3. It may be generated by the $LF(3, p^k)$ together with an homology of period $p^k - 1$.

§ 15. Subgroups of the Simple Groups $LF(3, p^k)$.

The subgroups of the $LF(3, p^k)$ are those groups which have been determined in this paper such that the coefficients of the transformations which they contain are in the $GF(p^k)$, and the determinant of any one of those transformations is a cube in that field, together with those subgroups which leave invariant a point, line, or triangle. There is a single conjugate set of each one of the groups under the whole collineation group of the plane. We suppose that the plane in which we are working is the $PG(2, p^k)$. If then the $LF(3, p^k)$ is the whole collineation group of the $PG(2, p^k)$, i. e., if $p^k - 1$ is not divisible by 3, there will be a single conjugate set of each subgroup; if $p^k - 1$ is divisible by 3, but a subgroup is invariant under a group of three times its order which is not a subgroup of the $LF(3, p^k)$, there will be also a single conjugate set of those subgroups; if however $p^k - 1$ is divisible by 3 and a subgroup is not invariant under a group which is not a subgroup of the $LF(3, p^k)$, then there will be three conjugate sets of those subgroups.

The subgroups are as follows:

1. Groups of order $(p^k + 1)p^{3k}(p^k - 1)^2/\mu$. Each of these groups leaves invariant a point and is isomorphic with a group of order $(p^k + 1)p^k(p^k - 1)$, permuting the lines through that point.

* Cf. L. E. Dickson, *Linear Groups*, pp. 75–78.

2. Groups of order $(p^k+1)p^{3k}(p^k-1)^2/\mu$. Each of these groups leaves invariant a line and is isomorphic with a group of order $(p^k+1)p^k(p^k-1)$, permuting the points on that line.

3. Groups of order $6(p^k-1)^2/\mu$. Each of these groups leaves invariant a triangle with coördinates in the $GF(p^k)$, and makes all six permutations on its vertices.

4. Groups of order $3(p^{2k}+p^k+1)/\mu$. Each of these groups leaves invariant a triangle with coördinates in the $GF(p^{3k})$, but not in the $GF(p^k)$, and permutes its vertices cyclically.

5. Groups of order $(p^k+1)p^k(p^k-1)$. Each such group leaves invariant a conic.

6. Groups of the same structure as that of the $LF(3, p^k)$ itself, i. e., the $LF(3, p^m)$, where m is a factor of k.

7. Groups containing the $LF(3, p^m)$ as self-conjugate subgroups of index 3 if p^m-1 is divisible by 3, and k/m is divisible by 3.

8. The hyperorthogonal groups, $HO(3, p^{2m})$, where $2m$ is a factor of k.

9. Groups containing the $HO(3, p^{2m})$ as self-conjugate subgroups of index 3 if p^m+1 is divisible by 3, and $k/2m$ is divisible by 3.

10. The Hessian groups of order 216 (if p^k-1 is divisible by 9), 72 and 36 (if p^k-1 is divisible by 3).

11. Groups of order 168, which exist if $\sqrt{-7}$ exists in the $GF(p^k)$, i. e., if k is even, or (by the law of quadratic reciprocity) if k is odd and $p=7$, $7f+1$, $7f+2$, or $7f+4$. For $p=7$ any one of these groups leaves invariant a conic.

12. Groups of order 360, which exist if both $\sqrt{5}$ and a cube root of unity exist in the $GF(p^k)$; i. e., if k is even, or if k is odd, provided $p=15f+1$ or $15f+4$. For $p=3$ any one of these groups leaves invariant a conic.

13. Groups of order 720 containing the groups of order 360 as self-conjugate subgroups. These exist only for $p=5$ and k even.

14. Groups of order 2520, each isomorphic with the alternating group on seven letters. These exist only for $p=5$ and k even.

Since the largest subgroup is of order $(p^k+1)p^{3k}(p^k-1)^2/\mu$, we have

Theorem 29. *The smallest number of letters on which the group, $LF(3, p^k)$, may be represented as a permutation group is $p^{2k}+p^k+1$.*

§ 16. Subgroups of the Simple Groups $HO(3, p^{2k})$.

The subgroups of the $HO(3, p^{2k})$ are those groups determined above which have an invariant conjugate with

$$x_1^{p^k+1}+x_2^{p^k+1}+x_3^{p^k+1}=0,$$

together with the subgroups which leave invariant a point, line, or triangle. Each set of subgroups forms a single conjugate set provided the $HO(3, p^{2k})$ is invariant under no larger group under the whole collineation group of the plane, i. e., if $p^k + 1$ is not divisible by 3. If $p^k + 1$ is divisible by 3, each set of subgroups will form a single conjugate set if any one of them is invariant under a group of three times its order which is a subgroup of the group under which the $HO(3, p^{2k})$ is invariant but not of the $HO(3, p^{2k})$ itself; otherwise there will be three conjugate sets.

The subgroups are as follows:

1. Groups of order $(p^k + 1)p^{3k}(p^k - 1)/\nu$. Any such group leaves invariant the center and axis of a group of elations and is isomorphic with a metacyclic group on the axis of order $(p^{2k} - 1)p^{2k}/\nu$.

2. Groups of order $(p^k + 1)^2 p^k(p^k - 1)/\nu$. Any such group leaves invariant the center and axis of an homology and is isomorphic with the group of the points on the axis, which is of order $(p^k + 1)p^k(p^k - 1)$.

3. Groups of order $6(p^k + 1)^2/\nu$. Any such group leaves invariant a triangle whose coördinates are in the $GF(p^{2k})$ and makes all six permutations on its vertices.

4. Groups of order $3(p^{2k} - p^k + 1)/\nu$. Any such group leaves invariant a triangle whose coördinates lie in the $GF(p^{6k})$, but not in the $GF(p^{2k})$, and permutes its vertices cyclically.

5. Groups of order $(p^k + 1)p^k(p^k - 1)$. Any such group leaves invariant a conic.

6. The hyperorthogonal groups, $HO(3, p^{2m})$, where m is a factor of k and k/m is odd.

7. Groups containing the $HO(3, p^{2m})$ as self-conjugate subgroups of index 3 if $p^m + 1$ is divisible by 3 and k/m is odd and is divisible by 3.

8. The Hessian groups of order 216 (if $p^k + 1$ is divisible by 9), 72 and 36 (if $p^k + 1$ is divisible by 3).

9. Groups of order 168, which exist if $\sqrt{-7}$ does not exist in $GF(p^k)$, i. e., if k is odd and $p = 7f + 3$, $7f + 5$, or $7f + 6$. For $p = 7$ these groups also appear, but each leaves invariant a conic.

10. Groups of order 360, which exist if $\sqrt{5}$ exists and a cube root of unity does not exist in the $GF(p^k)$, i. e., if k is odd and $p = 5$, $15f - 1$, or $15f - 4$. If k is even and $p = 3$ these groups also appear, but in that case each leaves invariant a conic.

11. Groups of order 720, which exist if $p = 5$ and k is odd.

12. Groups of order 2520, which exist if $p = 5$ and k is odd.

The largest subgroup of the $HO(3, p^{2k})$ is of order $(p^k + 1)p^{3k}(p^k - 1)/\nu$, except for $p^k = 5$, in which case the largest subgroup is of order 2520. Hence we have

Theorem 30. *The smallest number of letters on which the group* $HO(3, p^{2k})$ *may be represented as a permutation group is* $p^{3k}+1$, *except for* $p^k = 5$, *in which case the smallest number is* 50.

That the $HO(3, p^{2k})$ can be represented as a permutation group on $p^{3k}+1$ letters has been shown by Dickson (Mathematische Annalen, vol. 55, p. 532). Similar results are obtained by him in that paper for the general hyperorthogonal group in m variables.

NEW HAVEN, CONN.,
September, 1910.

GENERAL THEORY OF LINEAR DIFFERENCE EQUATIONS*

BY

GEORGE D. BIRKHOFF

Introduction.

The theory of linear difference equations with rational coefficients was in a very backward state until Poincaré † in 1882 developed the notion of asymptotic representation, and its application to this branch of mathematics. Further important progress in this direction has been made only very recently, notably by Galbrun, ‡ who by means of the Laplace transformation has investigated the properties of a set of fundamental solutions in the entire plane of the complex variable x.

The aim of the present paper is first to study the nature of these solutions to which I have been led by means of direct methods, and secondly to show that there exists a purely Riemannian theory of linear difference equations. In particular certain rational functions of $e^{2\pi\sqrt{-1}\,x}$ are shown to play a part like that of the monodromic group constants of an ordinary linear differential equation.

To the best of my knowledge, the importance of the functional standpoint in the field of difference equations was emphasized first by Van Vleck in an inspiring series of lectures given at the University of Wisconsin in the spring of 1909, in which he conjectured the existence of sets of solutions analytic on either the left or the right side of the complex plane.

On account of the extreme simplicity of the matrix notation, I have found it convenient to deal with a linear difference system of n equations of the first order

$$g_i(x+1) = \sum_{j=1}^{n} a_{ij}(x)\, g_j(x), \qquad (i = 1, 2, \cdots, n), \tag{1}$$

rather than with a single equation of the nth order. In these equations the functions $a_{ij}(x)$ are taken to be rational functions of x with a pole at $x = \infty$ of order μ at most, so that one has

$$a_{ij}(x) = a_{ij}x^{\mu} + a_{ij}^{(1)}x^{\mu-1} + \cdots, \qquad (|x| > R). \tag{2}$$

* Presented to the Society, September 6, 1910.

† American Journal of Mathematics, vol. 7 (1885), pp. 203–258.

‡ Comptes Rendus, vol. 148 (1909), pp. 905–907.

The sole other restriction to be imposed on these functions below will, in particular, be satisfied if the determinantal *characteristic equation*

$$|a_{ij} - \delta_{ij}\rho_j| = 0 \tag{3}$$

[where $\delta_{ij} = 0$, $i \neq j$; $\delta_{ii} = 1$] has n distinct roots which are not zero. If the n sets of functions

$$\begin{array}{c} g_{11}(x), \cdots, g_{n1}(x) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ g_{1n}(x), \cdots, g_{nn}(x) \end{array}$$

are n linearly independent solutions of (1), the array of elements $g_{ij}(x)$ forms a *matrix solution* $G(x)$. The array of elements $a_{ij}(x)$ forms a second matrix $A(x)$, and the n^2 equations which the n solutions satisfy can be combined into a single matrix equation

$$G(x+1) = A(x)G(x). \tag{4}$$

If x be replaced by $x-1$, this may be written

$$G(x-1) = A^{-1}(x-1)G(x). \tag{4}'$$

The most general matrix solution of (4) is furnished by

$$H(x) = G(x)P(x), \tag{5}$$

in which $G(x)$ is any particular matrix solution, and $P(x)$ is any arbitrary matrix of periodic functions of period 1 whose determinant does not vanish identically.

The equation (4) admits of *two symbolic solutions*

$$\begin{cases} G(x) = A(x-1)A(x-2)\cdots, \\ G(x) = A^{-1}(x)A^{-1}(x+1)\cdots, \end{cases}$$

which converge to limit matrices only in particular cases. This convergence is secured to a sufficient extent by a suitable modification, made in § 1, as follows: The system (1) is known to be satisfied formally by taking for $g_1(x)$, $g_2(x)$, $\cdots, g_n(x)$ certain formal series, one set of them for each of the n roots ρ of the characteristic equation (3). The matrix formed from these sets taken in a certain order is denoted by $S(x)$, and the modified matrix obtained by breaking off these elements at the kth term is called $T(x)$. The sequences of matrices

$$\begin{array}{l} A(x-1)A(x-2)\cdots A(x-m)T(x-m), \\ A^{-1}(x)A^{-1}(x+1)\cdots A^{-1}(x+m-1)T(x+m), \end{array} \quad (m=1, 2, \cdots),$$

form the desired modification of the two symbolic solutions. It is demonstrated that all the determinants formed from the first λ columns and any λ rows of the

first of these matrices, ($\lambda = 1, 2, \cdots, n$), or from the last λ columns and any λ rows of the second of these matrices, converge to limit functions independent of k as m becomes infinite. By the aid of these functions and by a process of summation based on a contour integral suggested by the one which CARMICHAEL* has used, two remarkable particular matrix solutions $G(x)$ and $H(x)$ are obtained in §§ 2, 3, 4. These are termed the *first and second principal matrix solutions* respectively. The elements of $G(x)$ are analytic, save for poles at points $\sigma + 1, \sigma + 2, \cdots$, where σ is a pole of one of the elements of $A(x)$, and have the asymptotic form of the corresponding elements of $S(x)$ in any left half plane. The elements of $H(x)$ are analytic save for poles at points $\sigma - 1, \sigma - 2, \cdots$, where σ is a pole of one of the elements of $A^{-1}(x-1)$, and have the asymptotic form of the corresponding elements of $S(x)$ in any right half plane. The principal matrix solutions $G(x)$ and $H(x)$ are uniquely determined by this asymptotic property. A relation (5) will also, of course, subsist between $G(x)$ and $H(x)$.

Conversely if $G(x)$ and $H(x)$ are asymptotically represented by the elements of a matrix $S(x)$ *even to terms of the first order* and are related by means of a matrix $P(x)$ of periodic functions, as in (5), these matrices $G(x)$ and $H(x)$ will form the first and second principal matrix solutions of a system (4) in which the elements of $A(x)$ are rational (see § 7).

It is proved in § 5 that the elements of $P(x)$ are rational functions of $e^{2\pi\sqrt{-1}x}$. The structure of these functions is determined with particular reference to the case when the functions $a_{ij}(x)$ are polynomials of degree μ, to which case the more general one is reduced.

These results lead in § 6 to the determination of the nature of the elements of $G(x)$ and $H(x)$ in the remaining right and left half plane respectively; the asymptotic form is found to alter along certain *critical rays*. A complete description of the principal matrix solutions is thus obtained.

In § 7 it is shown that the number of *characteristic constants* involved in the complete characterization of $G(x)$ and $H(x)$ and their connection by means of $P(x)$ is precisely equal to the number of arbitrary constants in the coefficients $a_{ij}(x)$ assumed to be polynomials, and that the characteristic constants are independent. Thus I have ventured to propose the following as a fundamental problem of linear difference equations: *to determine whether there exists a linear difference system with prescribed characteristic constants.*

It would be interesting to know the exact relation between the characteristic constants and the monodromic group constants of the differential equation yielded by the Laplace transformation.

The method contained in the present paper is of wide generality and applies if the coefficients $a_{ij}(x)$ have the character of rational functions only at $x = \infty$,

*These Transactions, vol. 12 (1911), pp. 99–134.

in all cases where there exist n formal solutions. CARMICHAEL (loc. cit.) has studied this case to a considerable extent; in particular he has proved the existence of analytic solutions and has partly discussed their asymptotic form in the direction of the axis of reals.

§ 1. The Determinant Limits.

It is apparent that the given linear system (1), when written in the matrix notation (4), (4)′, possesses the two symbolic solutions

$$A(x-1)A(x-2)\cdots, \qquad A^{-1}(x)A^{-1}(x+1)\cdots.$$

These will converge to limit matrices in certain special cases, for example if $a_{ij}(x)$ has the form

$$\delta_{ij}+\frac{a_{ij}^{(2)}}{x^2}+\frac{a_{ij}^{(3)}}{x^3}+\cdots;$$

in this case the above symbolic solutions will yield the complete solution of (1). In general however these expressions must be properly modified in order to be made to lead indirectly to a complete solution of (1).

Let us first make precise the limitation which is to be made on the coefficients $a_{ij}(x)$ in equations (1). If the roots of the characteristic equation (3) are distinct, and if none of them are zero, there will exist precisely n sets of series

$$(6)\qquad \begin{aligned} s_{1j}(x) &= x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\left\{ s_{1j}+\frac{s_{1j}^{(1)}}{x}+\cdots\right\},\\ s_{2j}(x) &= x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\left\{ s_{2j}+\frac{s_{2j}^{(1)}}{x}+\cdots\right\}, \quad (j=1,2,\cdots,n),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ s_{nj}(x) &= x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\left\{ s_{nj}+\frac{s_{nj}^{(1)}}{x}+\cdots\right\}, \end{aligned}$$

each of which constitutes a formal solution of (1). The proof of this is a matter of direct reckoning, and the fact will be taken for granted.* The constants $\rho_1, \rho_2, \cdots, \rho_n$ are the roots of the characteristic equation, and furthermore we have the determinant

$$(7)\qquad d=|s_{ij}|\neq 0.$$

* The formal solutions of a single linear difference equation of the nth order have been frequently used (for example see HORN, Crelle's Journal, vol. 138 (1910), pp. 159–191) and admit of easy extension to a linear system. The formal solutions may be obtained by direct substitution if the roots of the characteristic equation are distinct and different from zero; the series expansion

$$\left(1+\frac{1}{x}\right)^{\mu x}=e^{\mu x\log\left(1+\frac{1}{x}\right)}=e^{\mu}-\frac{\mu e^{\mu}}{2x}+\cdots$$

enables one to make a comparison of coefficients and thus to determine the constants of the formal solutions.

The restrictive hypothesis which we shall make is not that $\rho_1, \rho_2, \cdots, \rho_n$ *are distinct but only that there exist* n *formal solutions* (6) *with the specified property* (7). The quantities $\rho_1, \rho_2, \cdots, \rho_n$ must of course still be the roots of the characteristic equation, and none of them can be zero. We shall choose our subscripts so that

$$|\rho_1| \geqq |\rho_2| \geqq \cdots \geqq |\rho_n|. \tag{8}$$

Without loss of real generality we may add the further condition that the linear homogeneous difference equation of the nth order at most which $g_1(x)$ satisfies, and which is obtained from (1) by elimination, is actually of the nth order.

The elements $s_{ij}(x)$ form a matrix $S(x)$ which is a formal matrix solution of the equation (4)

$$S(x+1) = A(x)\, S(x).$$

Let us compute the elements $s'_{ij}(x)$ of the inverse matrix $S^{-1}(x)$. The determinant $|S(x)|$ may be written

$$x^{n\mu x}(\rho_1\rho_2 \cdots \rho_n e^{-n\mu})^x x^{r_1+r_2+\cdots+r_n}\left\{d + \frac{d^{(1)}}{x} + \cdots\right\}.$$

The element in the ith row and jth column of the inverse matrix is the quotient of the cofactor of the element in the jth row and the ith column of this determinant by the determinant itself, and is therefore given by a formal series

$$s'_{ij}(x) = x^{-\mu x}(\rho_i e^{-\mu})^{-x} x^{-r_i}\left\{\sigma_{ij} + \frac{\sigma_{ij}^{(1)}}{x} + \cdots\right\}. \tag{9}$$

Let $T(x)$ denote any matrix obtained by breaking off the elements of $S(x)$ at the kth term, or more generally by replacing $s_{ij}(x)$ by convergent series $t_{ij}(x)$ of the same form as $s_{ij}(x)$ and having the same first k terms. Write then

$$T(x+1) = B(x)\, T(x). \tag{10}$$

The matrix $B(x)$, as thus defined, is clearly a matrix of functions $b_{ij}(x)$ each of which has an expansion like $a_{ij}(x)$ at $x = \infty$ with precisely the same first k terms. For if we compare the formulas

$$B(x) = T(x+1)\, T^{-1}(x) \qquad \text{and} \qquad A(x) = S(x+1)\, S^{-1}(x),$$

we observe first that the element $t'_{ij}(x)$ of $T^{-1}(x)$ is given by (9) to the first k terms, and that in consequence the right hand members of both the equations have the same first k terms. In consequence we may put

$$A(x) = B(x)\,[1 + x^{-k}\, C(x)], \tag{11}$$

1 being the identity matrix, and the elements of $C(x)$ being analytic at $x = \infty$. It is to be noted that the determinant formed from the principal coefficients of

$A(x)$ and $B(x)$, namely $|a_{ij}|$, cannot be zero, for that implies that zero is a root of the characteristic equation.

In this paper it is necessary to distinguish sharply between two kinds of asymptotic representation of a function by a series of the form (6), say

$$s(x) = x^{\mu x}(\rho e^{-\mu})^x x^r \left(s + \frac{s^{(1)}}{x} + \cdots\right).$$

Let $g(x)$ be the given function; if for each k the difference

$$g(x)x^{-\mu x}(\rho e^{-\mu})^{-x}x^{-r} - \left(s + \frac{s^{(1)}}{x} + \cdots + \frac{s^{(k)}}{x^k}\right)$$

becomes uniformly small of order x^{-k} for $\lim x = \infty$ in a certain region, we say that $g(x)$ is asymptotically represented by $s(x)$ in that region, *with respect to* x; if on the other hand the difference is uniformly small of order v^{-k} (where $x = u + \sqrt{-1}\,v$, u, v being real quantities) for $\lim v = \infty$ in the region, we say that the representation is *with respect to* v. In this connection two important inequalities are

$$\sum_{\nu=1}^{\infty} \left| \frac{1}{(x-\nu)^k} \right| < \frac{\pi}{2\,|x|^{k-1}} \qquad (k \geqq 2), \tag{12}$$

for x in the left half of the complex plane, and

$$\sum_{\nu=1}^{\infty} \left| \frac{1}{(x-\nu)^k} \right| < \frac{2}{|v|^{k-1}} \left[\frac{1}{|v|} + \frac{\pi}{2} \right] \qquad (k \geqq 2), \tag{13}$$

for x in the right half of the complex plane. To prove the first of these we note that when x lies in the left half plane u is negative, so that

$$|x-\nu|^2 = (u-\nu)^2 + v^2 > u^2 + v^2 + \nu^2,$$

whence the sum of the given series is less than

$$\sum_{\nu=1}^{\infty} \frac{1}{(|x|^2+\nu^2)^{\frac{1}{2}k}} < \int_0^{\infty} \frac{1}{|x|^{k-2}} \frac{d\nu}{(|x|^2+\nu^2)} = \frac{\pi}{2\,|x|^{k-1}}.$$

Again, for x in the right half plane, our series may be written

$$\sum_{\nu=1}^{\infty} \frac{1}{[(u-\nu)^2+v^2]^{\frac{1}{2}k}} < \sum_{\nu=-\infty}^{+\infty} \frac{1}{[(u-\nu)^2+v^2]^{\frac{1}{2}k}} < \frac{1}{|v|^{k-2}} \sum_{\nu=-\infty}^{+\infty} \frac{1}{(u-\nu)^2+v^2}.$$

If each quantity $|u-\nu|$ be replaced by the last of the integers $0, 1, 2, \cdots$ which does not exceed it, the terms in the last series are not decreased, and hence one has

$$\frac{1}{|v|^{k-2}} \sum_{\nu=-\infty}^{+\infty} \frac{1}{(u-\nu)^2+v^2} \leqq \frac{2}{|v|^{k-2}} \sum_{\nu=0}^{\infty} \frac{1}{\nu^2+v^2} < \frac{2}{|v|^{k-2}} \left[\frac{1}{v^2} + \int_0^{\infty} \frac{d\nu}{\nu^2+v^2} \right].$$

But the last integral reduces to $\pi/2\,|v|$, and thus the second inequality holds also.

We may now proceed to the existence theorem:

THEOREM I. *Form the two sequences of matrices*

$$(14)\qquad \begin{aligned} P_m(x) &= A(x-1)A(x-2)\cdots A(x-m)T(x-m),\\ Q_m(x) &= A^{-1}(x)A^{-1}(x+1)\cdots A^{-1}(x+m-1)T(x+m), \end{aligned} \qquad (m=1,2,\cdots).$$

Every determinant formed from the first λ *columns* ($\lambda=1,2,\cdots,n$) *and the* i*th,* j*th,* $\cdots$*,* l*th rows of* $P_m(x)$ $[Q_m(x)]$ *converges, for* k *sufficiently large, to a definite limit function* $u_{ij\cdots l}(x)$ $[v_{ij\cdots l}(x)]$, *independent of* k, *as* m *becomes infinite. This function is analytic throughout the entire* x*-plane excepting at points any number of units to the right (left) of poles of the elements of* $A(x)$ $[A^{-1}(x-1)]$ *and at these points may have a pole. The asymptotic form of* $u_{ij\cdots l}(x)$ $[v_{ij\cdots l}(x)]$ *in the vicinity of* $x=\infty$ *is given by the corresponding determinant* $s_{ij\cdots l}(x)$ *of* $S(x)$, *with respect to* x *in any left (right) half of the plane, and with respect to* v *in any right (left) half of the plane.*

Proof. We shall first prove the theorem in so far as it relates to the case $\lambda=1$, and then show how the cases $\lambda=2,3,\cdots,n$ may be reduced to this case.

It is necessary first to show that the elements of the first column of $P_m(x)$ converge to definite limit functions $u_1(x)$, $u_2(x)$, $\cdots$, $u_n(x)$ as m becomes infinite.

The matrix $P_m(x)$ may be written

$$\begin{aligned} P_m(x) &= T(x)\bar{P}_m(x)\\ &= T(x)\{[T^{-1}(x)A(x-1)T(x-1)][T^{-1}(x-1)A(x-2)T(x-2)]\\ &\qquad\cdots[T^{-1}(x-m+1)A(x-m)T(x-m)]\}. \end{aligned}$$

It is sufficient to prove that the elements of the first column of $\bar{P}_m(x)$ will converge, inasmuch as the elements of the first column of $P_m(x)$ arise from combination of the rows of $T(x)$ with the first column of $\bar{P}_m(x)$. The matrix $\bar{P}_m(x)$ is a product of matrices, the type of each of which is

$$T^{-1}(x+1)A(x)T(x).$$

By (10) and (11) this reduces to

$$1+\frac{1}{x^k}T^{-1}(x)C(x)T(x).$$

Let us examine the second term. The element in the ith row and jth column of $T^{-1}(x)$ will be given by an expression like that for $s'_{ij}(x)$ in (9) with the same first k terms; the elements of $C(x)$ are analytic at $x=\infty$, the element

in the ith row and jth column of $T(x)$ will be given by an expression like that for $s_{ij}(x)$ and with the same first k terms. The element in the ith row and jth column of the product $T^{-1}(x)\,C(x)\,T(x)$ has therefore the form

$$\left(\frac{\rho_j}{\rho_i}\right)^x x^{r_j-r_i} \text{ times a function analytic at } x=\infty .$$

Choose first k so large that it exceeds every difference $r_j - r_i$ in absolute value by more than an assigned integer $d \geqq 2$, and we see that the second term may be written

$$\frac{1}{x^d}\Theta(x) \quad \text{with elements} \quad \theta_{ij}(x) = \left(\frac{\rho_j}{\rho_i}\right)^x \phi_{ij}(x). \tag{15}$$

The function $\phi_{ij}(x)$ is finite in the vicinity of $x=\infty$, at least if x be restricted not to pass the positive axis of reals.

By means of this notation, $\bar{P}_m(x)$ reduces to the form

$$\left(1+\frac{1}{(x-1)^d}\Theta(x-1)\right)\left(1+\frac{1}{(x-2)^d}\Theta(x-2)\right)\cdots\left(1+\frac{1}{(x-m)^d}\Theta(x-m)\right),$$

or more fully,

$$\begin{aligned} &1+\sum_{k_1}\frac{1}{(x-k_1)^d}\Theta(x-k_1) \\ &\qquad +\sum_{k_1,k_2}^{k_1<k_2}\frac{1}{(x-k_1)^d(x-k_2)^d}\Theta(x-k_1)\Theta(x-k_2)+\cdots . \end{aligned} \tag{16}$$

It is the elements of the first column of this matrix which must be shown to converge. According to the notation of (15), the ith element is

$$\begin{aligned} &\delta_{i1}+\sum_{k_1}\frac{1}{(x-k_1)^d}\left(\frac{\rho_1}{\rho_i}\right)^{x-k_1}\phi_{i1}(x-k_1)+\sum_{k_1,k_2}^{k_1<k_2}\frac{1}{(x-k_1)^d(x-k_2)^d} \\ &\qquad \times\sum_{\tau=1}^{n}\left(\frac{\rho_\tau}{\rho_i}\right)^{x-k_1}\left(\frac{\rho_1}{\rho_\tau}\right)^{x-k_2}\phi_{i\tau}(x-k_1)\phi_{\tau 1}(x-k_2)+\cdots . \end{aligned} \tag{17}$$

But $\phi_{ij}(x)$ is limited in the vicinity of $x=\infty$ and is less than some constant M for $|x|>R_1$. Therefore if all the points $x-1, x-2, \cdots$ are exterior to the circle $|x|=R_1$, which is true if x lies in D of fig. 1, the typical element in

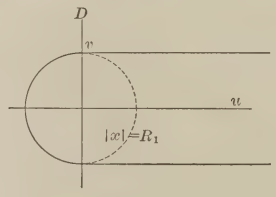

FIG. 1.

the $(l+1)$th term of (17),

$$\frac{1}{(x-k_1)^{\prime l}}\frac{1}{(x-k_2)^{\prime l}}\cdots\frac{1}{(x-k_l)^{\prime l}}\left(\frac{\rho_\tau}{\rho_i}\right)^{x-k_1}\left(\frac{\rho_\sigma}{\rho_\tau}\right)^{x-k_2}\cdots\left(\frac{\rho_1}{\rho_\nu}\right)^{x-k_l}$$

$$\times\,\phi_{i\tau}(x-k_1)\phi_{\tau\sigma}(x-k_2)\cdots\phi_{\nu 1}(x-k_l),$$

is less in absolute value than the quantity

$$\left(\frac{\rho_1}{\rho_i}\right)^{x}\frac{M^l}{(x-k_1)^{\prime l}\cdots(x-k_l)^{\prime l}}.$$

This inequality depends on the fact that we have

$$\left|\left(\frac{\rho_\tau}{\rho_i}\right)^{x-k_1}\left(\frac{\rho_\sigma}{\rho_\tau}\right)^{x-k_2}\cdots\left(\frac{\rho_1}{\rho_\nu}\right)^{x-k_l}\right|$$

$$=\left|\left(\frac{\rho_1}{\rho_i}\right)^{x}\left(\frac{\rho_i}{\rho_1}\right)^{k_1}\left(\frac{\rho_\tau}{\rho_1}\right)^{k_2-k_1}\cdots\left(\frac{\rho_\nu}{\rho_1}\right)^{k_l-k_{l-1}}\right|\leqq\left|\left(\frac{\rho_1}{\rho_i}\right)^{x}\right|$$

as a consequence of (8), and the fact that $k_1, k_2, \cdots, k_n$ are an increasing set of positive integers. Hence each term of (17) does not exceed the corresponding term of

$$\delta_{i1}+\left|\left(\frac{\rho_1}{\rho_i}\right)^{x}\right|\sum_{k_1}\frac{M}{|x-k_1|^{d}}+\left|\left(\frac{\rho_1}{\rho_i}\right)^{x}\right|\sum_{k_1,k_2}^{k_1<k_2}\frac{nM^2}{|x-k_1|^{d}|x-k_2|^{d}}+\cdots \tag{18}$$

in absolute value.

Allow m to approach infinity. The expression (17) becomes a multiple series. Since the sum of the terms after the first in (18) approaches the limit

$$\frac{1}{n}\left|\left(\frac{\rho_1}{\rho_i}\right)^{x}\right|\left\{\left[\left(1+\frac{nM}{|x-1|^{d}}\right)\left(1+\frac{nM}{|x-2|^{d}}\right)\cdots\right]-1\right\}, \tag{19}$$

it is obvious that the elements of the first column of $\bar{P}_m(x)$ and therefore of $P_m(x)$ converge absolutely and uniformly to an analytic function in the vicinity of points in D. Moreover as we may write

$$P_m(x)=A(x-1)A(x-2)\cdots A(x-r)P_{m-r}(x-r),$$

and as we may take r so large that $x-r-1, x-r-2, \cdots$ lie without the circle $|x|=R_1$, the elements of the first column of $P_m(x)$ will converge uniformly in the vicinity of any point of the plane excepting those only at which one of the elements of $A(x-1), A(x-2), \cdots$ has a pole. Hence the ith element of the first column of $P_m(x)$ converges to a function $u_i(x)$ which is analytic except at the points *congruent to* (any number of units to right or left of) the poles of the elements of $A(x)$ on the right. At these points $u_i(x)$ may have a pole.

It may easily be made evident that the definition of $u_1(x), u_2(x), \cdots, u_n(x)$

is independent of the choice of k and T. For if k' is any second choice of k and $T'(x)$ is the corresponding function $T(x)$, we may nevertheless write

$$P'_m(x) = T(x)\{[T^{-1}(x)A(x-1)T(x-1)]$$
$$\times [T^{-1}(x-1)A(x-2)T(x-2)]\cdots[T^{-1}(x-m+1)A(x-m)T'(x-m)].$$

If the elements of the first column be expanded in a sum like (17) and m be allowed to become infinite, the resulting multiple series for $u_i(x)$ will be term for term identical with that obtained before.

It remains to discuss the asymptotic nature of $u_1(x), \cdots, u_n(x)$ for large values of $|x|$. The limit $\bar{u}_i(x)$ of the ith element of the first column of $\bar{P}_m(x)$ differs from δ_{i1} by a quantity less than (19) in absolute value if x lies in D. This quantity is itself less than

$$\frac{1}{n}\left(\frac{\rho_1}{\rho_i}\right)^x\left(e^{\sum_{\nu=0}^{\infty}\frac{nM}{x-\nu_{i1}}}-1\right)$$

in absolute value. If now any line be drawn parallel to the axis of imaginaries and if x be restricted to lie in D to the left of the line, by inequality (12) the last factor will be of the form $M(x)/x^{d-1}$, where $M(x)$ is limited. Thus we may write

$$\bar{u}_i(x) = \delta_{i1} + \left(\frac{\rho_1}{\rho_i}\right)^x \frac{M_i(x)}{x^{d-1}}, \quad |M_i(x)| < M$$

for x in D and to the left of this line. Likewise if x is in D to the right of such a line, by inequality (13) we may write

$$\bar{u}_i(x) = \delta_{i1} + \left(\frac{\rho_1}{\rho_i}\right)^x \frac{M'_i(x)}{x^{d-1}}, \quad |M'_i(x)| < M.$$

But the relation $P_m(x) = T(x)\bar{P}_m(x)$ gives at once

$$u_i(x) = \sum_{\tau=1}^{n} t_{i\tau}(x)\bar{u}_\tau(x) .$$

If in this equation the first of the above expressions for $\bar{u}_1(x), \bar{u}_2(x), \cdots, \bar{u}_n(x)$ be substituted, we find at once

$$u_i(x) = t_{i1}(x) + \sum_{\tau=1}^{n} t_{i\tau}(x)\left(\frac{\rho_1}{\rho_\tau}\right)^x \frac{M_\tau(x)}{x^{d-1}}.$$

The functions $t_{ij}(x)$ are the same in form as $s_{ij}(x)$ up to the kth term. Substituting in their expansions we obtain

$$u_i(x) = x^{\mu_i x}\rho_i^x x^{r_1}\left\{\left(s_{i1} + \frac{s_{i1}^{(1)}}{x} + \cdots\right) + \sum_{\tau=1}^{n}\frac{x^{r_\tau}}{x^{r_1}}\frac{M_\tau(x)}{x^{d-1}}\left(s_{i\tau} + \frac{s_{i\tau}^{(1)}}{x} + \cdots\right)\right\}.$$

The second term in brackets can clearly be made infinitesimal in x of as high

order as may be desired by taking k, and therefore d, large enough. Consequently $u_i(x)$ is represented asymptotically by $s_{i1}(x)$ with respect to x to the left of any such line and likewise with respect to v to the right of such a line. The part of the theorem that relates to the case $\lambda = 1$ is proved.

It is clear that if the subscripts $1, 2, \cdots, n$ had not been so chosen that (8) was true, the only modification would be that the elements of any ith column of $P_m(x)$ for which the quantity ρ_i had a maximum absolute value would converge.

The case $\lambda = 2$ will next be discussed. This reduces at once to the case $\lambda = 1$ *inasmuch as the two-rowed determinants* $u^{(m)}_{\alpha\beta}(x)$ *formed from the elements of the first two columns of* $P^{(m)}(x)$ *which are in the* α*th and* β*th rows* $(\alpha < \beta)$ *are related to a certain difference system precisely as the elements of the first column are related to the original system* (1).

In fact, form the system with general solution

$$g_{ij:}(x) = \begin{vmatrix} g_i'(x) & g_i''(x) \\ g_j'(x) & g_j''(x) \end{vmatrix} \qquad (i, j = 1, 2, \cdots, n;\ i < j). \tag{20}$$

Here $g_1'(x), g_2'(x), \cdots, g_n'(x)$ and $g_1''(x), g_2''(x), \cdots, g_n''(x)$ are any pair of solutions of (1). One finds by the use of (1)

$$g_{ij:}(x+1) = \sum_{k,\,l} a_{ij:\,kl}(x)\, g_{kl:}(x), \tag{21}$$

where

$$a_{ij:\,kl}(x) = a_{ik}(x)\, a_{jl}(x) - a_{il}(x)\, a_{jk}(x).$$

Accordingly we have $\frac{1}{2}n(n-1)$ linear difference equations of precisely the type (1) to determine the functions $g_{ij:}(x)$. It is clear that the sets of formal expansions

$$s_{ij:\,kl}(x) = \begin{vmatrix} s_{ik}(x) & s_{il}(x) \\ s_{jk}(x) & s_{jl}(x) \end{vmatrix}$$

$$= x^{2\mu x}(\rho_k \rho_l)^x x^{r_k + r_l} \{ (s_{ik} s_{jl} - s_{il} s_{jk}) + \text{terms in } x^{-1}, x^{-2}, \cdots \},$$

k and l fixed, will satisfy (21). There are $\frac{1}{2}n(n-1)$ sets of these, and furthermore the determinant of the constant terms

$$|s_{ik} s_{jl} - s_{il} s_{ik}| \qquad (i < j;\ k < l)$$

is precisely equal to d^{n-1} and is therefore not zero by (7).* Hence the system (21) satisfies the restrictive hypothesis made at the beginning of § 1.

We are now in a position to apply the result already obtained. In matrix notation, equations (21) may be expressed as

$$G_2(x+1) = A_2(x) G_2(x).$$

* Pascal: *Die Determinanten*, p. 87.

Also let $T_2(x)$ be the matrix which the formal matrix solution $S_2(x)$ becomes when the formal series are replaced by convergent series which agree with them to the kth term. *The elements in that column of*

$$P_{2m}(x) = A_2(x-1)A_2(x-2)\cdots A_2(x-m)T_2(x-m)$$

for which the quantity $\rho_k\rho_l$ in the formal solution which appears in the same column of $S_2(x)$ has a maximum absolute value will converge; this results from an application of the preceding result for $\lambda = 1$ to the system of $\frac{1}{2}n(n-1)$ equations. Equations (8) show that we may take $k=1$, $l=2$ and it is also clear that we may take the elements of this column of $T_2(x)$ to be

$$t_{ij:12}(x) = \begin{vmatrix} t_{i1}(x) & t_{i2}(x) \\ t_{j1}(x) & t_{j2}(x) \end{vmatrix}.$$

The elements $p^{(m)}_{ij:12}(x)$ of this column of $P_{2m}(x)$ converge to functions analytic everywhere except for poles at points to the right of the poles of the elements of $A_2(x)$ and congruent to them. Moreover these functions are represented asymptotically by $s_{ij:12}(x)$ throughout the entire plane, with respect to x to the left of any line parallel to the axis of imaginaries, and with respect to v to the right of such a line.

But the sequence $p^{(m)}_{ij:12}(x)$, $m = 1, 2, \cdots$ is identical with the sequence of determinants $u^{(m)}_{ij}(x)$ formed from the elements of the first two columns which are in the ith and jth rows of $P_m(x)$. In fact the first member of both sequences is $t_{ij:12}(x)$, and in general $p^{(m+1)}_{ij:12}(x)$ is derived from $p^{(m)}_{ij:12}(x)$ just as $u^{(m+1)}_{ij}(x)$ is derived from $u^{(m)}_{ij}(x)$, namely we have

$$p^{(m+1)}_{ij:12}(x) = \sum_{k,\,l} a_{ij:kl}(x-1)\,p^{(m)}_{kl:12}(x-1),$$

$$u^{(m+1)}_{ij}(x) = \begin{vmatrix} p^{(m+1)}_{i1}(x), & p^{(m+1)}_{i2}(x) \\ p^{(m+1)}_{j1}(x), & p^{(m+1)}_{j2}(x) \end{vmatrix} = \begin{vmatrix} \sum_{\tau=1}^{n} a_{i\tau}(x-1)p^{(m)}_{\tau 1}(x-1), & \sum_{\tau=1}^{n} a_{i\tau}(x-1)p^{(m)}_{\tau 2}(x-1) \\ \sum_{\tau=1}^{n} a_{j\tau}(x-1)p^{(m)}_{\tau 1}(x-1), & \sum_{\tau=1}^{n} a_{j\tau}(x-1)p^{(m)}_{\tau 2}(x-1) \end{vmatrix}$$

$$= \sum_{k,\,l} a_{ij:kl}(x-1)\,u^{(m)}_{kl}(x-1).$$

Thus the theorem is true for $\lambda = 2$ also. Strictly speaking, we have only obtained functions $u_{ij}(x)$ for $i < j$; but we have obviously such functions for all i and j; these may be obtained by means of the equation $u_{ij}(x) = -u_{ji}(x)$. In like manner the theorem can be demonstrated for the remaining values of λ.

The existence of the functions $v_{ij\ldots l}(x)$ and their properties may be established in the same way. The rôle of right and left in the x-plane will be interchanged and the equation (4)′ replaces (4).

The functions $u_{ij\ldots l}(x)$ and $v_{ij\ldots l}(x)$ will be called the *determinant limits* on account of their origin.

In the case that the restrictive hypothesis is satisfied and all the roots ρ of the characteristic equation are equal, Theorem I supplies a complete existence theorem, every column of elements of $P_m(x)$ and $Q_m(x)$ converging to an analytic solution of (1). These conditions are met if one has

$$a_{ij}(x) = \delta_{ij} + \frac{a_{ij}^{(2)}}{x^2} + \frac{a_{ij}^{(3)}}{x^3} + \cdots.$$

In general however Theorem I supplies only one solution directly, and that is furnished by

$$g_1(x) = u_1(x),\ g_2(x) = u_2(x),\ \cdots,\ g_n(x) = u_n(x).$$

This set of functions must be a solution since the definition of $u_i(x)$ gives us

$$\begin{aligned} u_i(x+1) &= \lim_{m=\infty} \sum_{\tau,\ldots,\eta=1}^{n} a_{i\tau}(x)\cdots a_{\nu\eta}(x-m)\,t_{\eta 1}(x-m) \\ &= \sum_{\tau=1}^{n} a_{i\tau}(x)\,u_\tau(x). \end{aligned}$$

§ 2. The Solutions Associated with the Determinant Limits.

The functions $u_i(x)$, $u_{ij}(x)$, $\cdots$ which have been obtained behave as though they were constituent determinants of a matrix of solutions of (2).

Theorem II. *There exist n solutions $g_{1j}(x), g_{2j}(x), \cdots, g_{nj}(x)\,[h_{1j}(x), h_{2j}(x), \cdots, h_{nj}(x)]\,(j = 1, 2, \cdots, n)$ of (1) over any discrete set of values congruent to a given value x_0 of x, such that every λ-rowed determinant formed from the first [last] λ-columns of the matrix $G(x)[H(x)]$ and the ith, jth, $\cdots$, lth rows will equal $u_{ij\ldots l}(x)\,[v_{ij\ldots l}(x)]$ at these points. The first solution $g_{11}(x), g_{21}(x), \cdots, g_{n1}(x)\,[h_{1n}(x), h_{2n}(x), \cdots, h_{nn}(x)]$ is uniquely determined, the second solution $g_{12}(x), g_{22}(x), \cdots, g_{n2}(x)\,[h_{1,n-1}(x), h_{2,n-1}(x), \cdots, h_{n,n-1}(x)]$ is determined up to an additive term obtained by multiplying all the corresponding elements of the first solution by an arbitrary function of period 1, and so on, any one of these solutions being determined up to an additive term linear in the corresponding elements of the preceding solutions with arbitrary multipliers of period 1.*

Proof. The theorem is true in so far as it relates to $\lambda = 1$ since $u_1(x)$, $u_2(x)$, $\cdots$, $u_n(x)$ has been seen to give a solution of (1). We may write then

$$g_{11}(x) = u_1(x),\ g_{21}(x) = u_2(x),\ \cdots,\ g_{n1}(x) = u_n(x).$$

Next we shall demonstrate that there exists some solution $g_{12}(x), g_{22}(x), \cdots, g_{n2}(x)$ such that for all i and j

$$\begin{vmatrix} g_{i1}(x) & g_{i2}(x) \\ g_{j1}(x) & g_{j2}(x) \end{vmatrix} = u_{ij}(x). \tag{22}$$

For this purpose we shall recur to the sequence of elements of $P_m(x)$. By means of the identity

$$\begin{vmatrix} p_{i1}^{(m)}(x) & p_{i2}^{(m)}(x) & p_{i1}^{(m)}(x) \\ p_{j1}^{(m)}(x) & p_{j2}^{(m)}(x) & p_{j1}^{(m)}(x) \\ p_{k1}^{(m)}(x) & p_{k2}^{(m)}(x) & p_{k1}^{(m)}(x) \end{vmatrix} = 0,$$

we infer, by developing according to the elements of the third column and letting m become infinite, that

$$u_{ij}(x)u_k(x) - u_{ik}(x)u_j(x) + u_{jk}(x)u_i(x) = 0 \qquad (i, j, k = 1, 2, \cdots, n).$$

The n quantities $g_{12}(x_0)$, $g_{22}(x_0)$, $\cdots$, $g_{n2}(x_0)$ uniquely determine a solution of (1) on the set of points congruent to x_0. Choose these n quantities so that (22) is satisfied for some fixed i such that $g_{i1}(x_0) \neq 0$, and for every j. All the rest of the equations (22) will then be satisfied, as is evident from the preceding equation. This gives us

$$\begin{vmatrix} g_{i1}(x) & g_{i2}(x) \\ g_{j1}(x) & g_{j2}(x) \end{vmatrix} g_{k1}(x) - \begin{vmatrix} g_{i1}(x) & g_{i2}(x) \\ g_{k1}(x) & g_{k2}(x) \end{vmatrix} g_{j1}(x) + u_{jk}(x)g_{i1}(x) = 0$$

for $x = x_0$, which reduces to

$$u_{jk}(x) = \begin{vmatrix} g_{j1}(x) & g_{j2}(x) \\ g_{k1}(x) & g_{k2}(x) \end{vmatrix}$$

for $x = x_0$. Hence $g_{ij}(x)$ and $u_{ij}(x)$ are equal at $x = x_0$ for all i and j. But they are both constituent elements of a solution of (21), and therefore will agree at all congruent points, since only one solution of that equation exists on the discrete set of points congruent to x_0 whose elements have given values at this point. It is necessary to make an exception of points congruent to the poles of $A(x)$ or $A^{-1}(x-1)$, and such points are excluded.

The method by which this second solution was obtained clearly allows us to add to each element $g_{i2}(x)$ a term of the form $p(x)g_{i1}(x)$, where $p(x)$ is of period 1.

One may now in an entirely analogous way define a solution $g_{13}(x)$, $g_{23}(x)$, $\cdots$, $g_{n3}(x)$ on the set of points congruent to x_0. The identities to be employed in this case are

$$u_{ijk}(x)u_l(x) - u_{ijl}(x)u_k(x) + u_{ikl}(x)u_j(x) - u_{jkl}(x)u_i(x) = 0.$$

The function $g_{i3}(x)$ is only determined up to an additive term

$$p(x)g_{i1}(x) + q(x)g_{i2}(x).$$

By proceeding thus the theorem may be proved in a succession of steps in so far as it relates to the elements $g_{ij}(x)$.

The part of the theorem that relates to the elements $h_{ij}(x)$ is proved in an entirely similar way.

The solutions $g_{1j}(x), \cdots, g_{nj}(x)$ $[h_{1j}(x), \cdots, h_{nj}(x)]$ may be said to be the solutions *associated* with the determinant limits $u_i(x)$, $u_{ij}(x) \cdots [v_i(x), v_{ij}(x) \cdots]$.

It is important to obtain explicit formulas for these functions $g_{ij}(x)$, *and we shall show that it is possible to obtain the solutions* $g_{1j}(x), g_{2j}(x), \cdots, g_{nj}(x)$ *for* $j = 1, 2, \cdots, n$ *in terms of* $u_i(x), u_{ij}(x), \cdots, u_{12\ldots n}$, *and the operation* $\sum$ *defined by the condition*

$$\sum_{t=x+1} \phi(t) - \sum_{t=x} \phi(t) = \phi(x).$$

The functions $g_{11}(x), \cdots, g_{n1}(x)$ are equal respectively to $u_1(x), \cdots, u_n(x)$. It will accordingly be sufficient to obtain formulas for $g_{1k}(x), \cdots, g_{nk}(x)$ in terms of the solutions $g_{1j}(x), \cdots, g_{nj}(x) (j = 1, 2, \cdots, k-1)$ and the determinant limits.

Form the determinant

$$(23) \qquad \theta^{(k)}(x) = \begin{vmatrix} g_{11}(x) & g_{12}(x) & \cdots & g_{1k}(x) \\ g_{11}(x+1) & g_{12}(x+1) & \cdots & g_{1k}(x+1) \\ \cdots & \cdots & \cdots & \cdots \\ g_{11}(x+k-1) & g_{12}(x+k-1) & \cdots & g_{1k}(x+k-1) \end{vmatrix}.$$

If now in place of the elements $g_{1j}(x+m)$ $(m > 0)$ we insert their value as obtained from (1)

$$(24) \quad g_{1j}(x+m) = \sum_{\tau, \sigma, \ldots, \theta=1}^{n} a_{1\tau}(x+m-1)\, a_{\tau\sigma}(x+m-2) \cdots a_{\eta\theta}(x) g_{\theta j}(x),$$

in terms of the functions $g_{1j}(x), \cdots, g_{nj}(x)$, we see that the above determinant may be expressed as a sum of terms each of which contains as factor a determinant which reduces to one of the determinant limits of order k. In fact we have obviously

$$(25) \quad \theta^{(k)}(x) = \sum_{\tau, \sigma, \ldots=1}^{n} \left\{ a_{1\tau}(x) \left[\sum_{\tau=1}^{n} a_{1\tau}(x+1)\, a_{\tau\sigma}(x) \right] \cdots \right\} u_{\tau\sigma\ldots}(x).$$

Hence this determinant is to be regarded as a known function.

But the identity (23) may also be looked upon as a non-homogeneous linear difference equation of order $k-1$ in $g_{1k}(x)$, and may be written

$$(26) \quad \theta^{(k-1)}(x) g_{1k}(x+k-1) + \psi(x) g_{1k}(x+k-2) + \cdots \pm \theta^{(k-1)}(x+1) g_{1k}(x) = \theta^{(k)}(x).$$

The associated reduced homogeneous equation obtained by equating the left

hand member to zero has the solutions [see (23)] $g_{11}(x), \cdots, g_{1,k-1}(x)$, which are linearly independent unless $\theta^{(k-1)}(x)$ is zero. By a method analogous to that of variation of parameters for ordinary linear differential equations, let us put

$$g_{1k}(x) = \sum_{j=1}^{k-1} g_{1j}(x) l_j(x) \tag{27}$$

and determine $l_1(x), l_2(x), \cdots, l_{k-1}(x)$ by the equations

$$\begin{aligned} \sum_{j=1}^{k-1} g_{1j}(x+1)\Delta l_j(x) &= 0, \\ \sum_{j=1}^{k-1} g_{1j}(x+2)\Delta l_j(x) &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \sum_{j=1}^{k-1} g_{1j}(x+k-1)\Delta l_j(x) &= \frac{\theta^{(k)}(x)}{\theta^{(k-1)}(x)}, \end{aligned} \tag{28}$$

where as usual $\Delta\phi$ denotes the difference $\phi(x+1) - \phi(x)$. There are here $k-1$ linear equations to determine the $k-1$ unknowns $\Delta l_1(x), \cdots, \Delta l_{k-1}(x)$, and their determinant is precisely $\theta^{(k-1)}(x+1)$. Unless this determinant is identically zero, we have accordingly

$$\Delta l_j(x) = \frac{\theta^{(k)}(x) m_{j,k-1}(x)}{\theta^{(k-1)}(x)\theta^{(k-1)}(x+1)},$$

in which $m_{j,k-1}(x)$ is the cofactor of the element in the jth column and last row of the determinant $\theta^{(k-1)}(x+1)$:

$$\pm \begin{vmatrix} g_{11}(x+1) & \cdots & g_{1,j-1}(x+1) & g_{1,j+1}(x+1) & \cdots & g_{1,k-1}(x+1) \\ g_{11}(x+2) & \cdots & g_{1,j-1}(x+2) & g_{1,j+1}(x+2) & \cdots & g_{1,k-1}(x+2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{11}(x+k-1) & \cdots & g_{1,j-1}(x+k-1) & g_{1,j+1}(x+k-1) & \cdots & g_{1,k-1}(x+k-1) \end{vmatrix}. \tag{29}$$

The resulting expression for $g_{1k}(x)$ is

$$g_{1k}(x) = \sum_{j=1}^{k-1} g_{1j}(x) \sum_{t=x} \frac{\theta^{(k)}(t) m_{j,k-1}(t)}{\theta^{(k-1)}(t)\theta^{(k-1)}(t+1)}. \tag{30}$$

It remains to show that $g_{1k}(x)$ as thus defined is a solution of (26). We have

$$\begin{aligned} g_{1k}(x) &= \sum_{j=1}^{k-1} g_{1j}(x) l_j(x), \\ g_{1k}(x+1) &= \sum_{j=1}^{k-1} [(g_{1j}(x+1) l_j(x) + g_{1j}(x+1)\Delta l_j(x)] \\ &= \sum_{j=1}^{k-1} g_{1j}(x+1) l_j(x), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \end{aligned} \tag{31}$$

$$g_{1k}(x+k-1)=\sum_{j=1}^{k-1}[(g_{1j}(x+k-1)l_j(x)+g_{1j}(x+k-1)\Delta l_j(x)]$$
$$=\sum_{j=1}^{k-1}g_{ij}(x+k-1)l_j(x)+\frac{\theta^{(k)}(x)}{\theta^{(k-1)}(x)}.$$

Hence, if we multiply the last of these equations by $\theta^{(k-1)}(x)$, the one which precedes it by $\psi(x)$ and so on, and add corresponding members, the left hand member of the resulting equation will be precisely the left hand member of (26). Also all the terms except the last of the right hand member reduce to zero by reason of the fact that $g_{11}(x), \cdots, g_{1,k-1}(x)$ are solutions of the reduced equation, and there remains only a term $\theta^{(k)}(x)$. Therefore $g_{1k}(x)$ is the desired solution: furthermore the formula (30) does not determine $g_{1k}(x)$ uniquely but only up to a sum of terms

$$p_1(x)g_{11}(x)+\cdots+p_{k-1}(x)g_{1,k-1}(x),$$

where $p_1(x), \cdots, p_{k-1}(x)$ are arbitrary periodic multipliers since the operation $\sum_{t=x}$ is determined only up to an additive periodic term. But the result stated in Theorem II shows that $g_{1k}(x)$ has precisely this degree of indeterminateness, and hence *the most general determination of* $g_{1k}(x)$ *is furnished by* (30) *unless* $\theta^{(k-1)}(x)$ *is identically zero,* and this is an obvious case of failure.

When a choice of $g_{1k}(x)$ has been made, the remaining functions $g_{2k}(x), \cdots, g_{nk}(x)$ which make up the kth solution are determined. In fact the equations (24) for $m=1, 2, \cdots, n$ form n linear equations in $g_1(x), \cdots, g_n(x)$ of determinant not zero, since in the contrary case we should obtain a linear homogeneous relation between $g_1(x+1), \cdots, g_1(x+n)$, which contradicts the condition placed on $g_1(x)$ at the outset that it satisfies an equation of the nth order. By solving these equations we obtain

$$(32)\quad g_i(x)=\sigma_{i1}(x)g_1(x+1)+\sigma_{i2}(x)g_1(x+2)+\cdots+\sigma_{in}(x)g_1(x+n)$$
$$(i=1, 2, \cdots, n),$$

where the functions $\sigma_{ij}(x)$ are rational in x. These equations determine $g_2(x), \cdots, g_n(x)$ in terms of $g_1(x)$.

That none of functions $\theta^{(1)}(x), \cdots, \theta^{(n)}(x)$ can be identically zero may be proved as follows: In the first place $\theta^{(k)}(x)$ has a definite asymptotic form in the entire x-plane, namely that obtained by replacing $u_{\tau\sigma\cdots}(x)$ by $s_{\tau\sigma\cdots}(x)$ in the right hand member of (25). This is a consequence of Theorem I. But this form is

$$\begin{vmatrix} s_{11}(x) & \cdots & s_{1k}(x) \\ s_{11}(x+1) & \cdots & s_{1k}(x+1) \\ \cdots & \cdots & \cdots \\ s_{11}(x+k-1) & \cdots & s_{1k}(x+k-1) \end{vmatrix},$$

as may be shown readily by means of (23). Recalling now the explicit form of the series $s_{ij}(x)$, we obtain

$$(33)\qquad \theta^{(k)}(x) \sim x^{k\mu x}(\rho_1 e^{-\mu})^x \cdots (\rho_k e^{-\mu})^x x^{r_1+\cdots+r_k}\left(d_k + \frac{d_k^{(1)}}{x} + \cdots\right).$$

Since $\theta^{(k)}(x)$ is represented asymptotically by this expression with respect to x to the left of any line parallel to the axis of imaginaries, and with respect to v to the right of such a line (Theorem I), it is obvious that unless $d_k, d_k^{(1)}, \cdots$ are all zero, $\theta^{(k)}(x)$ does not vanish at a great distance from the positive axis of reals. But these constants can not all vanish. Otherwise the above determinant vanishes (formally) and $s_{i1}(x), \cdots, s_{ik}(x)$ are linearly dependent, and accordingly, by (32), $s_{i1}(x), \cdots, s_{ik}(x)$ $(i = 1, 2, \cdots, n)$, are linearly dependent, in contradiction to (7). Thus (30) is a valid formula.

§ 3. The Intermediate Associated Solutions.

The application of the formulas (30) *and* (32) *leads us immediately to a set of intermediate associated solutions*

$$g'_{1k}(x), \cdots, g'_{nk}(x) \qquad (k=1, 2, \cdots, n),$$

analytic above and below lines $A\infty$ *and* $B\infty$ *parallel to the axis of reals, and which have the property that*

$$g'_{1k}(x) \sim s_{1k}(x), \cdots, g'_{nk}(x) \sim s_{nk}(x) \qquad (k=1, 2, \cdots, n),$$

with respect to x *in any left half plane and with respect to* v *in any right half plane.*

We take

$$g'_{11}(x) = u_1(x), \cdots, g'_{n1}(x) = u_n(x),$$

and this first solution has the properties stated, by Theorem 1. We now define the solutions $g'_{12}(x), \cdots, g'_{n2}(x)$; $g'_{13}(x), \cdots, g'_{n3}(x)$; $\cdots$ in succession by choosing the summation operation Σ on $\phi(x)$ to be defined as

$$-\phi(x) - \phi(x+1) - \cdots$$

and employing formulas (30) and (32). It will suffice to show that a solution $g'_{1j}(x), \cdots, g'_{nj}(x)$ for $j = k$ will exist and have these properties under the hypothesis that the same is true for $j = 1, 2, \cdots, k-1$. The formula (30), which serves to define $g'_{1k}(x)$ if $|\rho_{k-1}| > |\rho_k|$, is then

$$(34)\qquad g'_{1k}(x) = -\sum_{j=1}^{k-1} g'_{1j}(x) \sum_{\nu=0}^{\infty} \frac{\theta^{(k)}(x+\nu) m'_{j,k-1}(x+\nu)}{\theta^{(k-1)}(x+\nu)\theta^{(k-1)}(x+\nu+1)}.$$

We have written $m'_{j,k-1}(x)$ to stand for what $m_{j,k-1}(x)$ becomes when g'_{ij} replaces g_{ij}. We shall first prove that the $k-1$ series which appear here are

convergent. To this end we need to compute the asymptotic form of the typical term of the above series

$$\frac{\theta^{(k)}(x)\,m'_{j,\,k-1}(x)}{\theta^{(k-1)}(x)\,\theta^{(k-1)}(x+1)},$$

and this is done by merely substituting the known asymptotic forms for the θ's as given in (33) and at the same time the known form of the elements $m'_{j,\,k-1}$. This latter [see (29)] is clearly given by

$$\begin{aligned} m'_{j,\,k-1}(x) \sim\ & x^{(k-2)\mu x}(\rho_1 e^{-\mu})^x \cdots (\rho_{j-1}e^{-\mu})^x (\rho_{j+1}e^{-\mu})^x \\ & \cdots (\rho_{k-1}e^{-\mu})^x x^{r_1+\cdots+r_{j-1}+r_{j+1}+\cdots+r_{k-1}} \left\{ m_j + \frac{m_j^{(1)}}{x} + \cdots \right\} \end{aligned} \tag{35}$$

as a result of our hypothesis. Combining (35) and (33) we find for the above term the asymptotic formula

$$\frac{\theta^{(k)}(x)\,m'_{j,\,k-1}(x)}{\theta^{(k-1)}(x)\,\theta^{(k-1)}(x+1)} \sim \left(\frac{\rho_k}{\rho_j}\right)^x x^{r_k-r_j} x^{s_j} \left\{ \eta_j + \frac{\eta_j^{(1)}}{x} + \cdots \right\}. \tag{36}$$

The integer s_j is zero unless $d_{k-1} = 0$.

Since we have $|\rho_j| > |\rho_k|$ for $j < k$ except when $|\rho_{k-1}| = |\rho_k|$ [see (8)], the terms in (34) clearly diminish in approximately geometrical ratio for v large enough, and in consequence the series converge uniformly in the vicinity of such points. Thus $g'_{1k}(x)$ has been actually defined and is analytic above and below suitable lines $A\infty$ and $B\infty$.

By the aid of relations (36) we may now determine the asymptotic form of each of the series in (34). In the first place, if each term of the jth series is replaced by its asymptotic formula, the series becomes

$$\sum_{\nu=0}^{\infty} \left(\frac{\rho_k}{\rho_j}\right)^{x+\nu} (x+\nu)^{r_k-r_j}(x+\nu)^{s_j} \left(\eta_j + \frac{\eta_j^{(1)}}{x+\nu} + \cdots \right).$$

The error committed in the final bracketed series by breaking off at the lth term is of order $(x+\nu)^{-l}$ if x lies to the left of the axis of imaginaries, and of order v^l if x lies to the right of this line, since the imaginary coefficient v of $x+\nu$ is the same as the imaginary coefficient of x. These bracketed series are, of course, finite above and below suitable lines $A\infty$ and $B\infty$.

Now separate the series into two parts: first the sum of the first m terms in which $\nu < \frac{1}{2}|x|$, secondly, the remaining terms.

In this first set of terms we have

$$(x+\nu)^\rho = x^\rho \left(1 + \frac{\rho\nu}{x} + \cdots \right),$$

where the error committed by breaking off the series at the kth term is of the order $(\nu/x)^k$. Hence the first m terms of the above series may be written

$$\left(\frac{\rho_k}{\rho_j}\right)^x x^{r_k-r_j}x^{s_j}\sum_{\nu=0}^{m}\left(\frac{\rho_k}{\rho_j}\right)^\nu\left[\psi_j+\frac{\psi_j^{(1)}}{x}+\cdots+M_j(x)\left(\frac{\nu}{x}\right)^k\right],$$

where $M_j(x)$ is limited. Now the terms involving $M_j(x)$ are of the order x^{-k}, since

$$\left|\sum_{\nu=0}^{m}\left(\frac{\rho_k}{\rho_j}\right)^\nu M_j(x)\left(\frac{\nu}{x}\right)^k\right|\leqq\left|\frac{M}{x^k}\sum_{\nu=0}^{\infty}\left(\frac{\rho_k}{\rho_j}\right)^\nu\nu^k\right|,\ M \text{ a fixed constant.}$$

As m becomes large we have also

$$\sum_{\nu=0}^{m}\left(\frac{\rho_k}{\rho_j}\right)^\nu\psi\sim\frac{\rho_j\psi}{\rho_j-\rho_k}$$

o all powers of x since the difference in absolute value is

$$\left|\frac{\left(\frac{\rho_k}{\rho_j}\right)^{m+1}\psi}{1-\frac{\rho_k}{\rho_j}}\right|<\left|\frac{\left(\frac{\rho_k}{\rho_j}\right)^{\left|\frac{x}{2}\right|}\psi}{1-\frac{\rho_k}{\rho_j}}\right|.$$

Hence the first m terms have the asymptotic form

$$\left(\frac{\rho_k}{\rho_j}\right)^x x^{r_k-r_j}x^{s_j}\left\{\psi_j+\frac{\psi_j^{(1)}}{x}+\cdots\right\} \tag{37}$$

with respect to x for x to the left of the axis of imaginaries, or similarly to the left of any parallel line. The terms after the mth diminish in approximately geometrical ratio for $|v|$ sufficiently large, and all contain a very small multiplier $(\rho_k/\rho_j)^m$. Hence the second group of terms does not affect the above asymptotic form.

Substituting the above expression in (34) and at the same time the known asymptotic formulas $s_{1j}(x)$ for $g'_{1j}(x)$, we find

$$g'_{1k}(x)\sim\sum_{j=1}^{k-1}x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\left\{s_{1j}+\frac{s_{1j}^{(1)}}{x}+\cdots\right\}$$
$$\times\left(\frac{\rho_k}{\rho_j}\right)^x x^{r_k-r_j}x^{s_j}\left\{\psi_j+\frac{\psi_j^{(1)}}{x}+\cdots\right\},$$

or multiplying out,

$$g'_{1k}(x)\sim(\rho_k e^{-\mu})^x x^{r_k}x^s\left\{\xi_{1k}+\frac{\xi_{1k}^{(1)}}{x}+\cdots\right\} \tag{38}$$

with respect to x in any left half plane.

By means of (32) we now obtain the asymptotic formulas for $g'_{2k}(x),\cdots,$ $g'_{nk}(x)$. Thus we obtain n formal expansions which must of course be the constituent elements of some formal solution. The possibility that $\xi_{1k}, \xi_{1k}^{(1)},\cdots$ are all zero must be excluded since from that fact we should infer that all the coefficients in the asymptotic expansion of $g'_{ik}(x)\,(i=1,2,\cdots,n)$, and consequently

of $\theta^{(k)}(x)$, were zero. The formal solution thus obtained must coincide with $s_{1k}(x), \cdots, s_{nk}(x)$ at least if $\rho_1, \cdots, \rho_n$ are distinct. For it is apparent that the only possible difference lies in a constant factor. By definition of $\theta^{(k)}(x)$ in terms of $g_{11}(x) = g'_{11}(x), \cdots, g_{1k} = g'_{1k}(x)$, and in virtue of the fact that the asymptotic form of $\theta^{(k)}(x)$ has been shown to be that obtained by replacing these functions by $s_{11}(x), \cdots, s_{1k}(x)$, it follows that this factor must be 1. Hence we have in this case $\xi_{ik} = s_{ik}$, $\xi^{(1)}_{ik} = s^{(1)}_{ik}, \cdots$. Under our restriction, these equations are algebraic identities persisting in the exceptional cases when $\rho_1, \cdots, \rho_n$ are not distinct also.

Thus we have demonstrated the existence of a kth solution having the desired properties provided that $|\rho_{k-1}|$ and $|\rho_k|$ are not equal.

We shall now indicate briefly the modifications that are necessary in the case $|\rho_{k-1}| = |\rho_k|$. Suppose that ρ_k is one of a group of roots

$$\rho_\alpha, \rho_{\alpha+1}, \cdots, \rho_\beta$$

of equal absolute values. If $\rho_k = \rho_\alpha$, we have seen how to construct the kth solution; if $\rho_k = \rho_{\alpha+1}$, one may invert the order of ρ_α and $\rho_{\alpha+1}$ in (8) and thus construct a $(k+1)$th solution possessing the requisite asymptotic properties, and so continue to the βth solution. It is however necessary to know that the solutions thus constructed are *associated* with $u_i(x), u_{ij}(x), \cdots$ as stated. According to the method used in § 2 there certainly exist αth, $(\alpha+1)$th, $\cdots$, βth solutions over congruent sets of points which satisfy *all* the limit determinant relations, obtained by interchanging the order of $\rho_\alpha, \cdots, \rho_\beta$ in (8); for the algebraic condition that this hold will be satisfied at all the approximations since these are actual determinants, and hence will be satisfied in the limit also. On this account at least certain determinations of the sum formula employed will give a kth associated solution. Since however this kth associated solution is determined up to a linear combination with periodic coefficients of the solutions before the αth, and the formula which gives it indicates a solution determined up to a linear combination with periodic coefficients of solutions before the αth, the general solution given by the sum formula is a kth associated solution. Thus the solutions, obtained in this exceptional case as indicated, are associated with the determinant limits.

It would be of considerable interest to determine the nature of these intermediate solutions near to the axis of reals.

Considerations of symmetry show that there exists a set of solutions $h'_{1j}(x), \cdots, h'_{nj}(x)$ $(j = 1, 2, \cdots, n)$ having the properties stated.

§ 4. The First and Second Principal Matrix Solutions.

We shall proceed now to prove the existence of certain two remarkable matrix solutions $G(x)$ and $H(x)$ of (4), and this will be done by means of the

formulas of § 2 and the intermediate solutions of § 3. It is clear that in order to define any matrix $G(x)$ it is merely necessary to choose properly the operations $\sum_{t=x}$ which occur in (30). We shall use a contour integral to effect these summations, similar to that which Carmichael has used, writing

$$\sum_{t=x}\phi(t) = -\int_L e^{2\pi\lambda\sqrt{-1}(x-t)}\frac{\phi(t)\,dt}{e^{2\pi\sqrt{-1}(x-t)}-1},$$

where λ is an integer and L is a path $\infty AB\infty$ like that in fig. 2. The contour AB is a simple curve which passes between x and $x-1$. The summation function $\phi(t)$ is supposed to be analytic except at points at a limited distance from the positive axis of reals and within the region $\infty AB\infty$. Furthermore $\phi(t)$ is supposed to tend toward zero along lines $A\infty$, $B\infty$ sufficiently distant from the axis of reals, and in such manner as to make the above integral absolutely and uniformly convergent in the vicinity of any x not congruent to a point of L. In this case the function represented by the integral will be analytic at a sufficient distance from the positive axis of reals, and will be independent of the particular lines $A\infty$, $B\infty$ or particular curve AB chosen. If we form the difference $\sum_{t=x+1}\phi(t) - \sum_{t=x}\phi(t)$ it will be given by an integral over $ABB'A'A$, the integrand being unchanged. But within this closed curve the integrand is analytic save for a pole at $t=x$; and by the calculus of residues this difference reduces to $\phi(x)$. Hence this is a possible determination of $\sum_{t=x}\phi(t)$.

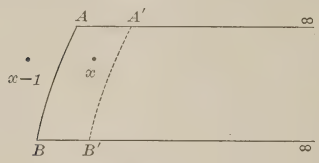

FIG. 2.

Having made this preliminary study of the operation $\sum_{t=x}$, we now proceed to the proof of

THEOREM III. *There exists a matrix solution $G(x)$ $[H(x)]$ of (4) whose elements have the following properties:*

(*a*) *the functions $g_{ij}(x)$ $[h_{ij}(x)]$ satisfy the conditions*

$$g_{i1}(x) = u_i(x),\ \begin{vmatrix} g_{i1}(x) & g_{i2}(x) \\ g_{j1}(x) & g_{j2}(x) \end{vmatrix} = u_{ij}(x),\ \cdots,$$

$$\left[h_{in}(x) = v_i(x),\ \begin{vmatrix} h_{i,n-1}(x) & h_{in}(x) \\ h_{j,n-1}(x) & h_{jn}(x) \end{vmatrix} = v_{ij}(x),\ \cdots \right];$$

(*b*) *the functions $g_{ij}(x)$ $[h_{ij}(x)]$ are analytic throughout the finite plane except possibly for poles to the right (left) of and congruent to the poles of the elements of $A(x)$ $[A^{-1}(x-1)]$;*

(*c*) *in any left (right) half plane $g_{ij}(x)$ $[h_{ij}(x)]$ is asymptotically represented by $s_{ij}(x)$ with respect to x.*

Proof. The proof of this theorem will be made by actually obtaining the solutions $g_{11}(x), \cdots, g_{n1}(x); \cdots; g_{1n}(x), \cdots, g_{nn}(x)$ in succession. The conditions of the theorem are evidently satisfied by taking the first solution to be

$$g_{11}(x) = u_1(x), \cdots, g_{n1}(x) = u_n(x).$$

It suffices therefore to exhibit a solution $g_{1k}(x), \cdots, g_{nk}(x)$ which satisfies the conditions of the theorem *under the hypothesis that the solutions* $g_{1j}(x), \cdots, g_{nj}(x)$ $(j < k)$ *have been constructed satisfying all of these conditions.* This is the way in which the theorem will be demonstrated.

The formula (30) gives the most general determination of $g_{1k}(x)$ which is possible if $g_{1k}(x)$ is to be the first member of a kth solution satisfying restriction (a). We shall consider first the case $|\rho_{k-1}| > |\rho_k|$. We choose the operation $\sum_{t=x}$ as indicated above, and obtain

$$(39)\quad g_{1k}(x) = -\sum_{j=1}^{k-1} g_{1j}(x) \int_L e^{2\pi\lambda_{jk}\sqrt{-1}(x-t)} \frac{\theta^{(k)}(t)\, m_{j,\,k-1}(t)}{\theta^{(k-1)}(t)\,\theta^{(k-1)}(t+1)} \frac{dt}{e^{2\pi\sqrt{-1}(x-t)} - 1};$$

the integer λ_{jk} *will be chosen to be the least integer as great as the real part of*

$$(40)\qquad \frac{1}{2\pi\sqrt{-1}} (\log \rho_k - \log \rho_j).$$

The functions $g_{2k}(x), \cdots, g_{nk}(x)$ are then obtained from (32).

The first step is to establish the fact that the summation functions in (39) satisfy the restriction on the summation function $\phi(t)$. Now $\theta^{(k)}(x)$, $\theta^{(k-1)}(x)$ are analytic save at points to the right of and congruent to the poles of elements $a_{ij}(x)$ by (25). Likewise, by the initial hypothesis and by definition of $m_{j,\,k-1}(x)$ as the cofactor of the element in the jth column and last row of $\theta^{(k-1)}(x+1)$ [see the form (29)], the singularities of $m_{j,\,k-1}(x)$ $(j = 1, 2, \cdots, k-1)$ are restricted in the same way. Hence the summation function

$$\phi(t) = \frac{\theta^{(k)}(t)\, m_{j,\,k-1}(t)}{\theta^{(k-1)}(t)\,\theta^{(k-1)}(t+1)}$$

will be analytic at all points sufficiently far from the positive axis of reals, provided that $\theta^{(k-1)}(t)$ does not vanish except at points within a certain distance of that axis. This fact is an immediate consequence of the relation (33) in which $d_k, d_k^{(1)}, \cdots$ have been seen not all to be zero. It remains to show that the function $\phi(t)$ tends to zero in such a way that the integrals appearing in (39) converge as stated for all positions of the lines $A\infty$ and $B\infty$ at a sufficient distance from the axis of the reals. In order to demonstrate this essential fact we make use of the intermediate solutions obtained in § 3. By Theorem II the general determination of the solutions $g_{1j}(x), \cdots, g_{nj}(x)$ associated with

the determinant limits is given by any one of them plus arbitrary periodic multiples of the preceding ones. Hence we have

$$g_{ij}(x) = g'_{i1}(x)\,p_{1j}(x) + \cdots + g'_{i,j-1}(x)\,p_{j-1,j}(x) + g'_{ij}(x) \qquad (41)$$
$$(i = 1, 2, \cdots, n\,;\ j = 1, 2, \cdots, k-1),$$

where the functions $p_{ij}(x)$ are periodic functions of period 1, analytic for x sufficiently distant from the axis of reals since the determinant $|g'_{ij}(x)| \sim |s_{ij}(x)|$ does not then vanish. Substitute these expressions (41), for $i = 1$, in $m_{j,k-1}(t)$. A reference to (29) makes it clear that the $m_{j,k-1}(t)$ may be written as a sum of products of periodic multipliers analytic for x sufficiently distant from the axis of reals and of factors $m'_{j,k-1}(t)$, where $m'_{j,k-1}$ signifies as before that which $m_{j,k-1}$ becomes when g' is put in place of g. Hence the integrals will converge in the manner desired if the expressions

$$\frac{\theta^{(k)}(t)\,m'_{j,k-1}(t)}{\theta^{(k-1)}(t)\,\theta^{(k-1)}(t+1)}$$

tend to zero in exponential fashion along $A\infty$ and $B\infty$. This is true if $A\infty$ and $B\infty$ are sufficiently distant from the axis of reals, in view of the asymptotic relationship (36) which holds with respect to v in the right half plane.

The function $g_{1k}(x)$ is therefore analytic at a sufficient distance from the positive axis of reals, and by (32) the same may be said of the remaining elements $g_{2k}(x), \cdots, g_{nk}(x)$ which constitute the kth solution. Working to the right across the plane by the aid of the original equations one proves readily that the only possible singularities of the functions of this solution will be poles congruent to the poles of the elements of $A(x)$ and to the right of them. Hence (b) is true for the kth solution. Also (a) holds in so far as it relates to this solution since every solution $g_{1k}(x), \cdots, g_{nk}(x)$ determined by (30) had this property.

Statement (c) has to do with the asymptotic nature of $g_{1k}(x), \cdots, g_{nk}(x)$. In order to prove (c) let us break up each integral in (39) into two others over contours L_1 and L_2 as follows: the contour L_1 is a fixed contour $\infty A_1 B_1 \infty$ (fig. 3). If x lies above $A_1\infty$, L_2 consists of a loop-circuit to ∞ which includes the points $x, x+1, \cdots$ within it but not $x-1, x-2, \cdots$. If x lies between $A_1\infty$ and $B_1\infty$, L_2 consists of a loop which includes $x, x+1, \cdots, x+l$ within it, where $x+l$ is the last of this series of points to the left of A_1B_1. If x lies below $B_1\infty$, L_2 is a loop-circuit of the same nature as when x lies above $A_1\infty$. Any combination of paths L_1 and L_2 is clearly equivalent to the single path $\infty AB\infty$ of fig. 2. If x lies on $A\infty$ or $B\infty$ or on a line congruent to a point A_1B_1 on the left, the two integrals are undefined.

Let us consider first the part of the integrals contributed by L_1. The integrands are periodic functions of x of period 1 and therefore the integrals will

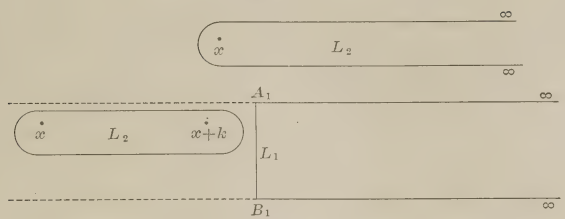

FIG. 3.

also be periodic since the contour L_1 is fixed. This part of the jth one of these integrals may be written as either of the products

$$e^{2\pi\lambda_{jk}\sqrt{-1}x}\int_{L_1} e^{-2\pi\lambda_{jk}\sqrt{-1}t}\,\frac{\theta^{(k)}(t)\,m_{j,\,k-1}(t)}{\theta^{(k-1)}(t)\,\theta^{(k-1)}(t+1)}\,\frac{dt}{e^{2\pi\sqrt{-1}(x-t)}-1},$$

$$e^{2\pi(\lambda_{jk}-1)\sqrt{-1}x}\int_{L_1} e^{-2\pi(\lambda_{jk}-1)\sqrt{-1}t}\,\frac{\theta^{(k)}(t)\,m_{j,\,k-1}(t)}{\theta^{(k-1)}(t)\,\theta^{(k-1)}(t+1)}\,\frac{dt}{1-e^{-2\pi\sqrt{-1}(x-t)}}.$$

It is apparent that in the first of these the second factor remains limited as v becomes positively infinite, inasmuch as $e^{2\pi\sqrt{-1}x}$ tends toward zero; likewise that in the second of these the second factor remains limited as v becomes negatively infinite. Hence the part of g_{1k} thus obtained is

$$(42)\qquad C_{L_1}=\begin{cases} g_{11}(x)\,e^{2\pi\lambda_{1k}\sqrt{-1}x}\,q_{1k}(x)+\cdots+g_{1,\,k-1}(x)\,e^{2\pi\lambda_{k-1,\,k}\sqrt{-1}x}\,q_{k-1,\,k}(x), \\ g_{11}(x)\,q_{1k}(x)+\cdots+g_{1,\,k-1}(x)\,q_{k-1,\,k}(x), \\ g_{11}(x)\,e^{2\pi(\lambda_{1k}-1)\sqrt{-1}x}\,q_{1k}(x)+\cdots+g_{1,\,k-1}(x)\,e^{2\pi(\lambda_{k-1,\,k}-1)\sqrt{-1}x}\,q_{k-1,\,k}(x), \end{cases}$$

according as x lies above $A_1\infty$, between $A_1\infty$ and $B_1\infty$, or below $B_1\infty$. The functions $q_{1k}(x),\ \cdots,\ q_{k-1,\,k}(x)$ are periodic and limited. This is obviously true unless x is in the vicinity of a point congruent to a point of the contour of integration. However it is true in this case also since each of the three expressions above is unchanged if the contour be varied continuously; of course the function $q_{ik}(x)$ is not one and the same analytic function in the three cases.

Next let us consider the contribution C_{L_2} yielded by the integrals taken over L_2. Let us suppose first that x lies above $A_1\infty$ or below $B_1\infty$. In this event the integrals in C_{L_2} may be evaluated as a sum of residues, each of which may be expressed in two ways, as indicated in the following formulas

$$(43)\qquad C_{L_2}=-\sum_{j=1}^{k-1} g_{1j}(x)\sum_{\nu=0}^{\infty}\left[\frac{\theta^{(k)}(x+\nu)\,m_{j,\,k-1}(x+\nu)}{\theta^{(k-1)}(x+\nu)\,\theta^{(k-1)}(x+\nu+1)}\right]$$

or

$$C_{L_2}=-\sum_{j=1}^{k-1} g_{1j}(x)\sum_{\nu=0}^{\infty}\left[\frac{\theta^{(k)}(x+\nu)\,m'_{j,\,k-1}(x+\nu)}{\theta^{(k-1)}(x+\nu)\,\theta^{(k-1)}(x+\nu+1)}\right].$$

The first form is the obvious one derived from equation (39). If we observe however that we have, by definition of $m_{j,k-1}(x)$,

$$\sum_{j=1}^{k-1} g_{1j}(x)\, m_{j,k-1}(x+\nu) = \begin{vmatrix} g_{11}(x+\nu+1) & \cdots & g_{1,k-1}(x+\nu+1) \\ g_{11}(x+\nu+2) & \cdots & g_{1,k-1}(x+\nu+2) \\ \cdot & \cdots & \cdot \\ g_{11}(x) & \cdots & g_{1,k-1}(x) \end{vmatrix},$$

and then write in place of g_{1j} their equals as given in (41), the resulting determinant may be simplified because the periodic coefficients are the same at $x+\nu+1, x+\nu+2, \cdots$ as at x. By a combination of successive columns the determinant reduces to a new determinant identical with the above except that g' replaces g, i. e., reduces to

$$\sum_{j=1}^{k-1} g'_{1j}(x)\, m'_{j,k-1}(x+\nu).$$

Thus the second of the above forms is obtained except that the two signs of summation will appear in inverted order. It is permissible to change this order if all the infinite series converge. This is certainly the case as the second form is precisely that which defined $g'_{1k}(x)$ in § 3 [see (34)]. Thus we have for x above $A_1\,\infty$ or below $B_1\,\infty$

$$C_{L_2} = g'_{1k}(x). \tag{44}$$

When x lies between $A_1\,\infty$ and $B_1\,\infty$ to the left of A_1B_1 we use a form like (43). We will agree to denote C_{L_2} by $g'_{1k}(x)$ for the sake of convenience, because then (44) holds in all cases and $g'_{1k}(x)$ *as thus defined will have the asymptotic form* s_{1k} *with respect to* x *in any left half plane, and with respect to* v *in any right half plane.* In § 3 these facts have already been proved for $g'_{1k}(x)$ as there defined, that is for x not between $A_1\,\infty$ and $B_1\,\infty$. We need therefore to prove that $g'_{1k}(x) \sim s_{1k}(x)$ with respect to x between $A_1\,\infty$ and $B_1\,\infty$ only. The discussion is entirely similar to a discussion that has been given in § 3. In the first place the integrals in the expression $g'_{1k}(x)$ for C_{L_2} may be evaluated as a sum of residues

$$g'_{1k}(x) = -\sum_{j=1}^{k-1} g_{1j}(x) \sum_{\nu=0}^{l} \frac{\theta^{(k)}(x+\nu)\, m_{j,k-1}(x+\nu)}{\theta^{(k-1)}(x+\nu)\,\theta^{(k-1)}(x+\nu+1)}, \tag{45}$$

where $x+l$ is the last point of the set $x, x+1, \cdots$ to the left of A_1B_1. If this equation be compared with (34) one sees that the right hand members differ in that a finite sum replaces an infinite series. Proceeding precisely as we did with (34) we arrive at the asymptotic relation $g'_{1k}(x) \sim s_{1k}(x)$ with respect to x by means of the hypothesis made that $g_{1j}(x), \cdots, g_{nj}(x)$ are asymptotically represented by $s_{1j}(x), \cdots, s_{nj}(x)$ with respect to x in any left half plane and

for $j = 1, 2, \cdots, k-1$. The sole modification consists in this that instead of having an *infinite* set of terms in which those after the mth (m the greatest integer less than $\frac{1}{2}|x|$) do not affect the asymptotic form, one has only a *finite* set of terms in which however those after the mth do not affect the asymptotic form.

The proof of (c) of the theorem in so far as it relates to $g_{1k}(x), \cdots, g_{nk}(x)$ may now be given. In the first place, for x above $A_1\infty$, (42) and (44) give

$$g_{1k}(x) \sim s_{11}(x)e^{2\pi\lambda_{1k}\sqrt{-1}x}q_{1k}(x) + \cdots + s_{1,k-1}(x)e^{2\pi\lambda_{k-1,k}\sqrt{-1}x}q_{k-1,k} + s_{1k}(x).$$

The dominating term will depend upon the dominating term of

$$e^{(2\pi\lambda_{1k}\sqrt{-1}+\log\rho_1)x}, \quad \cdots, \quad e^{(2\pi\lambda_{k-1,k}\sqrt{-1}+\log\rho_{k-1})x}, \quad e^{(\log\rho_k)x}$$

which begin the formal expansions. On dividing by the last of these, the exponents become

$$2\pi\sqrt{-1}\left(\lambda_{1k} + \frac{\log\rho_1 - \log\rho_k}{2\pi\sqrt{-1}}\right)x, \cdots, 2\pi\sqrt{-1}\left(\lambda_{k-1,k} + \frac{\log\rho_{k-1} - \log\rho_k}{2\pi\sqrt{-1}}\right)x, 0.$$

The bracketed expressions here all have a positive real part less than 1: this follows from the definition of $\lambda_{1k}, \cdots, \lambda_{k-1,k}$ and the fact that $|\rho_{k-1}| > |\rho_k|$. Hence the coefficients of x are represented by points in the upper half plane below the line $v = 2\pi$. Therefore if x lies in the second quadrant, all of these exponents have a negative real part, and the last term dominates, that is

$$g_{1k}(x) \sim s_{1k}(x)$$

for x above $A_1\infty$ and to the left of the axis of imaginaries, or likewise of any parallel line.

For x between $A_1\infty$ and $B_1\infty$ we have similarly

$$g_{1k}(x) \sim s_{11}(x)q_{1k}(x) + \cdots + s_{1,k-1}(x)q_{k-1,k}(x) + s_{1k}(x)$$

and the last term clearly dominates again by (8).

For x below $B_1\infty$, $g_{1k}(x)$ is represented by

$$s_{11}(x)e^{2\pi(\lambda_{1k}-1)\sqrt{-1}x}q_{1k}(x) + \cdots + s_{1,k-1}(x)e^{2\pi(\lambda_{k-1,k}-1)\sqrt{-1}x}q_{k-1,k}(x) + s_{1k}(x).$$

The dominating term depends on the exponents

$$2\pi\sqrt{-1}\left(\lambda_{1k} - 1 + \frac{\log\rho_1 - \log\rho_k}{2\pi\sqrt{-1}}\right)x, \cdots, 2\pi\sqrt{-1}\left(\lambda_{k-1,k} - 1 + \frac{\log\rho_{k-1} - \log\rho_k}{2\pi\sqrt{-1}}\right)x, 0.$$

The coefficients of x will in this case clearly lie below the real axis and above the line $v = 2\pi$, and if x lies in the third quadrant, the last term $s_{1k}(x)$ again dominates.

Hence in any left half plane $g_{1k}(x)$ is asymptotically represented by $s_{1k}(x)$

with respect to x, and of course $g_{2k}(x), \cdots, g_{nk}(x)$ are at the same time represented by $s_{2k}(x), \cdots, s_{nk}(x)$ respectively, as follows from (32). Thus a solution $g_{1k}(x), \cdots, g_{nk}(x)$ exists possessing properties $(a), (b), (c)$ unless $|\rho_{k-1}| = |\rho_k|$, that is unless ρ_k is one of a group $\rho_\alpha, \rho_{\alpha+1}, \cdots, \rho_p$ of roots ρ of equal absolute value. The same argument which enabled us to treat this exceptional case in § 3 may now be applied here; that is, we may take ρ_k as the first of this group without violating (8) and construct a kth solution which satisfies $(a), (b), (c)$ corresponding to each of these values of ρ.

The second set of solutions $h_{ij}(x), \cdots, h_{nj}(x), (j = 1, 2, \cdots, n)$ may be obtained by considerations of symmetry.

There is one and only one solution $g_{1k}(x), \cdots, g_{nk}(x)$ $[h_{1k}(x), \cdots, h_{nk}(x)]$ $(k = 1, 2, \cdots, n)$ analytic except for poles and asymptotically represented by $s_{1k}(x), \cdots, s_{nk}(x)$ even to terms of the first order in any left (right) half plane. The matrix $G(x)\,[H(x)]$ with elements $g_{ij}(x)\,[h_{ij}(x)]$ is termed the *first (second) principal matrix solution* of (4).

In order to justify this definition it is necessary to show that there exists but one such solution $g_{1k}(x), \cdots, g_{nk}(x)\,[h_{1k}(x), \cdots, h_{nk}(x)]$. Suppose if possible that there exists a second solution $\bar{g}_{1k}(x), \cdots, \bar{g}_{nk}(x)$ with the properties stated. Now we can express $\bar{g}_{ik}(x)$ as follows:

$$\bar{g}_{ik}(x) = g_{i1}(x)p_1(x) + \cdots + g_{in}(x)p_n(x) \quad (i = 1, 2, \cdots, n),$$

where $p_1(x), \cdots, p_n(x)$ are periodic of period 1, inasmuch as $g_{1j}(x), \cdots, g_{nj}(x)$ $(j = 1, 2, \cdots, n)$ form a fundamental system of solutions. If we solve these equations for $p_i(x)$, we shall have

$$p_i(x) = \frac{\begin{vmatrix} g_{11}(x) & \cdots & \bar{g}_{1k}(x) & \cdots & g_{1n}(x) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ g_{n1}(x) & \cdots & \bar{g}_{nk}(x) & \cdots & g_{nn}(x) \end{vmatrix}}{\begin{vmatrix} g_{11}(x) & \cdots & g_{1i}(x) & \cdots & g_{1n}(x) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ g_{n1}(x) & \cdots & g_{ni}(x) & \cdots & g_{nn}(x) \end{vmatrix}}.$$

Consider now some definite period-strip formed by two lines parallel to the axis of imaginaries and one unit apart, and let x become infinite in this strip. On account of the known asymptotic form of the elements g and $\bar{g}$ in the strip to terms of the first order we find at once, if $i \neq k$,

$$p_i(x) \sim \left(\frac{\rho_k}{\rho_i}\right)^x x^{r_k - r_i}[0]$$

for x at either end of the strip. If we write $z = e^{2\pi\sqrt{-1}\,x}$, this may be written

$$p_i(x) \sim z^{\frac{1}{2\pi\sqrt{-1}}(\log \rho_k - \log \rho_i)}(\log z)^{r_k - r_i}[0].$$

The upper end of the strip corresponds to $z = 0$, and the lower end to $z = \infty$. The above relation shows that $p_i(x)$ has a zero of at least the order λ_{ik} at $z = 0$. Likewise, this relation shows that $p_i(x)$ has a pole of order $\lambda_{ik} - 1$ at most at $z = \infty$. Consequently $p(x)z^{-\lambda_{ik}}$ is analytic at $z = 0$ and vanishes at $z = \infty$. Moreover, by taking x far enough to the left, the original determinant expression for $p_i(x)$ is analytic since every element of the two determinants is analytic, and the denominator is asymptotic to the formal determinant $|s_{ij}(x)|$ which is not zero if x lies far to the left. Hence $p_i(x)z^{-\lambda_{ik}}$, for $i \neq k$, is analytic for all finite values of z and vanishes at $z = \infty$; it must be identically zero. We conclude that

$$g_{ik}(x) = g_{ik}(x)p_k(x).$$

Now let i be chosen so that s_{ik}, the constant of the principal term of $s_{ik}(x)$, is not zero, as is possible by (7). Consider any point x_0 and the points $x_0 - 1$, $x_0 - 2$, $\cdots$ to its left. As x takes on these values, $\bar{g}_{1k}(x)/g_{1k}(x)$ tends toward 1, and thus we see that $p_k(x)$ is 1 at the point x_0. That is, $\bar{g}_{1k}(x)$ and $g_{1k}(x)$ are one and the same function, in contradiction to our hypothesis.

If the functions $a_{ij}(x)$ had not been restricted to be rational, the only modification in §§ 1–4 would be that in the statement and proof of (*b*), Theorem III, other singularities than poles would enter.

§ 5. The Fundamental Periodic Functions.

Either of the two principal matrix solutions $G(x)$ and $H(x)$ of (4) may be expressed in terms of the other, e. g.,

$$G(x) = H(x)P(x), \tag{46}$$

in which the elements $p_{ij}(x)$ of $P(x)$ are periodic functions of period 1 which may be termed the *fundamental periodic functions*. We shall now determine the nature of these functions. The fact that will enable us to do this is that $G(x)$ and $H(x)$ are composed of elements of known asymptotic form in any period strip which may be taken to be one and the same in both cases.

On account of the gain in simplicity we shall make a preliminary transformation. Let

$$d(x) = (x - \alpha)(x - \beta)\cdots(x - \lambda)$$

be the least common denominator of the set of rational functions $a_{ij}(x)$, and write

$$\bar{g}_i(x) = \Gamma(x - \alpha)\Gamma(x - \beta)\cdots\Gamma(x - \lambda)g_i(x) \qquad (i = 1, 2, \cdots, n),$$

where $\Gamma(x)$ is the gamma function. The transformed system will be altered in that the common denominator has been removed. The formal solutions of the new system will be those of the old multiplied by a certain factor, and hence will satisfy the restrictions imposed in § 1.

The new coefficients are polynomials in x. It is then permissible to take all the functions $a_{ij}(x)$ as polynomials, say of degree μ or less, and this we shall do. The first principal matrix solution $G(x)$ is constituted of elements analytic throughout the finite plane, in this case, inasmuch as the elements $a_{ij}(x)$ have no poles in the finite plane.

From the equation (46) we have

$$P(x) = H^{-1}(x)\, G(x); \tag{47}$$

but if we take our period strip far enough to the right, the elements of $H(x)$ are analytic functions of x and the determinant of $H(x)$ will not vanish in the strip inasmuch as its asymptotic form is given by the determinant $|s_{ij}(x)|$. Consequently the elements $p_{ij}(x)$ are functions of $z = e^{2\pi\sqrt{-1}x}$ which are analytic except perhaps at $z = 0$ and $z = \infty$, corresponding to the upper and lower end of the period strip. In the case when the functions $a_{ij}(x)$ in (1) are merely rational it is clear that the periodic functions $p_{ij}(x)$ are analytic or have poles except at $z = 0$ or $z = \infty$.

It is necessary now to remove a certain ambiguity. The expression $s_{ij}(x)$ represents the elements $g_{ij}(x)$ and $h_{ij}(x)$ asymptotically in any right and left hand side of the plane respectively, and is a many-valued expression which is changed to

$$e^{2\pi\mu\sqrt{-1}x}\, e^{2\pi r_j\sqrt{-1}}\, s_{ij}(x) \tag{48}$$

after a complete positive circuit of $x = 0$ [see (6)]. Hence if the same determination of $s_{ij}(x)$ is chosen for $g_{ij}(x)$ and $h_{ij}(x)$ along the direction of the positive axis of imaginaries, the elements $g_{ij}(x)$ and $h_{ij}(x)$ will have a different asymptotic form along the direction of the negative axis of imaginaries, since in one case we pass from the positive axis of imaginaries to the negative by a positive partial circuit, and in the other by a negative partial circuit. (It must be remembered that $g_{ij}(x)$ and $h_{ij}(x)$ preserve their asymptotic form in any *left* and *right* half plane respectively.) Let the determination for $h_{ij}(x)$ be termed $s_{ij}(x)$. Then that of $g_{ij}(x)$ is obviously obtained by a complete positive circuit and is given by (48) in the direction of negative imaginaries.

In the upper half of the period strip $g_{ij}(x)$ and $h_{ij}(x)$ have a common asymptotic representation $s_{ij}(x)$, and this proves that

$$P(x) \sim S^{-1}(x)\, S(x).$$

But the elements of $s'_{ij}(x)$ and $s_{ij}(x)$ of $S^{-1}(x)$ and $S(x)$ are of the respective asymptotic forms

$$x^{\mu x}\rho_i^{-x}x^{-r_i}\left\{\sigma_{ij} + \frac{\sigma_{ij}^{(1)}}{x} + \cdots\right\}, \qquad x^{\mu x}\rho_j^{x}x^{r_j}\left\{s_{ij} + \frac{s_{ij}^{(1)}}{x} + \cdots\right\}$$

[see (9) and (6)], so that in consequence the relation holds

$$p_{ij}(x) \sim \rho_i^{-x} \rho_j^{x} x^{-r_i+r_j} \delta_{ij}.$$

Therefore, if we think of $p_{ij}(x)$ as a function of z, we have

$$\lim_{z=0} p_{ij}(x) z^{-\frac{\log \rho_j - \log \rho_i}{2\pi\sqrt{-1}}} \left(\frac{\log z}{2\pi\sqrt{-1}}\right)^{r_i-r_j} = \delta_{ij}.$$

The single-valued analytic function $p_{ij}(x)$ of z is accordingly infinite only to a finite order at $z = 0$ and either has a pole there, or is analytic. For $i = j$ it is clear that $p_{ij}(x)$ is an entire function of z taking on the value 1 at $z = 0$, that is

$$p_{ii}(x) = 1 + c_{ii}^{(1)} e^{2\pi\sqrt{-1}x} + \cdots. \tag{49}$$

For $i \neq j$ we have obviously

$$\lim_{z=0} p_{ij}(x) z^{-\lambda_{ij}} = \text{constant}$$

since λ_{ij} was defined as the least integer as great as the real part of

$$\frac{\log \rho_j - \log \rho_i}{2\pi\sqrt{-1}}.$$

Hence $p_{ij}(x)$ has a zero of order λ_{ij}, at least, at $z = 0$, and may be written

$$p_{ij}(x) = e^{2\pi\lambda_{ij}\sqrt{-1}x}(c_{ij}^{(0)} + c_{ij}^{(1)} e^{2\pi\sqrt{-1}x} + \cdots), \tag{50}$$

the coefficient of the exponential factor being an entire function of z.

It remains to consider the functions $p_{ij}(x)$ in the lower half of the period strip. Here the functions $g_{ij}(x)$ and $h_{ij}(x)$ have the respective asymptotic forms

$$e^{2\pi\mu\sqrt{-1}x} e^{2\pi r_j\sqrt{-1}} s_{ij}(x) \quad \text{and} \quad s_{ij}(x),$$

and the matrices $G(x)$ and $H(x)$ are asymptotically represented by

$$S(x) \quad \text{and} \quad S_1(x)$$

respectively where

$$S_1(x) = S(x)(e^{2\pi\mu\sqrt{-1}x} e^{2\pi r_j\sqrt{-1}} \delta_{ij}).*$$

Hence we obtain

$$P(x) \sim S^{-1}(x) S_1(x) \quad \text{or} \quad [S^{-1}(x) S(x)](e^{2\pi\mu\sqrt{-1}x} e^{2\pi r_j\sqrt{-1}} \delta_{ij})$$
$$\text{or} \quad \rho_i^{-x} \rho_j^{x} x^{-r_i+r_j}(e^{2\pi\mu\sqrt{-1}x} e^{2\pi r_j\sqrt{-1}} \delta_{ij}).$$

Thus each function $p_{ij}(x)$ satisfies the condition

$$\lim_{z=\infty} p_{ij}(x) z^{-\mu - \frac{\log \rho_j - \log \rho_i}{2\pi\sqrt{-1}}} \left(\frac{\log z}{2\pi\sqrt{-1}}\right)^{r_i-r_j} e^{-2\pi r_j\sqrt{-1}} = \delta_{ij}.$$

* The notation (θ_{ij}) signifies a matrix with term θ_{ij} in the ith row and jth column.

We see again that $p_{ij}(x)$ becomes infinite only to a finite order at $z=\infty$. This demonstrates that $p_{ii}(x)$ is a polynomial in z and that $p_{ij}(x)$ $(i \neq j)$ is equal to $z^{\lambda_{ij}}$ multiplied by a polynomial in z.

In the case $i=j$ we have

$$\lim_{z=\infty} p_{ii}(x) z^{-\mu} = e^{2\pi r_i \sqrt{-1}},$$

so that the polynomial $p_{ii}(x)$ is of degree μ in z with leading coefficient $e^{2\pi r_i \sqrt{-1}}$. In the case $i \neq j$ we have

$$\lim_{z=\infty} p_{ij}(x) z^{-\mu-\lambda_{ij}+1} = \text{constant},$$

so that the polynomial coefficient of $z^{\lambda_{ij}}$ is of degree $\mu - 1$.

Thus we have proved the following theorem:

THEOREM IV. *Let $G(x)$ and $H(x)$ be the first and second principal matrix solutions of* (4) *in which the elements of $H(x)$ are polynomials of degree μ at most in x, and write*

$$G(x) = H(x) P(x).$$

The fundamental periodic functions $p_{ij}(x)$ which constitute $P(x)$ are in this case of the form

$$\begin{aligned} p_{ii}(x) &= 1 + c_{ii}^{(1)} e^{2\pi \sqrt{-1} x} + \cdots + e^{2\pi r_i \sqrt{-1}}\, e^{2\pi \mu \sqrt{-1} x}, \\ p_{ij}(x) &= e^{2\pi \lambda_{ij} \sqrt{-1} x} \left[c_{ij}^{(0)} + c_{ij}^{(1)} e^{2\pi \sqrt{-1} x} + \cdots + c_{ij}^{(\mu-1)} e^{2\pi(\mu-1)\sqrt{-1} x} \right] \quad (i \neq j), \end{aligned} \tag{51}$$

in which λ_{ij} denotes the least integer as great as the real part of

$$\frac{1}{2\pi \sqrt{-1}} (\log \rho_j - \log \rho_i).$$

The simplest illustration is given by the difference equation for the Gamma function, when one has

$$g(x) = e^{\pi \sqrt{-1} x} - \frac{2\pi \sqrt{-1}}{\Gamma(1-x)}, \quad h(x) = \Gamma(x), \quad p(x) = 1 - e^{2\pi \sqrt{-1} x}.$$

A similar analysis of the nature of the functions $p_{ij}(x)$ at $z=0$ and $z=\infty$ may be made in the more general case when the functions $a_{ij}(x)$ are not restricted to be rational functions, and shows that $p_{ij}(x)$ is analytic or has a pole at $z=0$ and $z=\infty$ with leading terms of the same form as those of (51) at these points. If the $a_{ij}(x)$ are rational, it is obvious that *the functions $p_{ij}(x)$ are rational in $e^{2\pi \sqrt{-1} x}$*; we shall not stop to investigate further the structure of these functions.

§ 6. The Principal Matrix Solutions in the Entire Plane.

The results of the preceding paragraph make it possible to discuss the asymptotic form of the elements of $G(x)$ [and likewise of $H(x)$] in the entire complex plane. In the first place it has been shown that

$$g_{ij}(x) \sim s_{ij}(x)$$

along any ray from $x = 0$ in the second or third quadrants. To investigate the asymptotic form along rays in the first and fourth quadrants we make use of the equation

$$(52) \quad G(x) = H(x)P(x) \quad \text{or} \quad g_{ij}(x) = \sum_{\tau=1}^{n} h_{i\tau}(x)p_{\tau j}(x) \quad (i, j = 1, 2, \cdots, n),$$

in which the elements $h_{ij}(x)$ are of known asymptotic form $s_{ij}(x)$ in any right half plane, and the elements $p_{ij}(x)$ are known functions of x.

Consider now a ray from $x = 0$ in the first quadrant and lying above the axis of reals. Along this ray, by equations (51), we shall have

$$p_{ij}(x) \sim e^{2\pi\lambda_{ij}\sqrt{-1}x}c_{ij}^{(0)}, \quad \text{where } \lambda_{ii} = 0, \quad c_{ii}^{(0)} = 1.$$

Thus the preceding equation for $g_{ij}(x)$ leads to

$$(53) \qquad g_{ij}(x) \sim s_{i1}(x)e^{2\pi\lambda_{1j}\sqrt{-1}}c_{1j}^{(0)} + \cdots + s_{in}(x)e^{2\pi\lambda_{nj}\sqrt{-1}x}c_{nj}^{(0)},$$

a formula which determines the asymptotic form in the first quadrant.

To determine the dominant term we need to compare the relative magnitude of the terms

$$\rho_1^r e^{2\pi\lambda_{1j}\sqrt{-1}x}, \ \cdots, \ \rho_j^r, \ \cdots, \ \rho_n^r e^{2\pi\lambda_{nj}\sqrt{-1}x}.$$

If these be divided by ρ_j^r, the logarithms become

$$2\pi\sqrt{-1}\left(\frac{\log\rho_1 - \log\rho_j}{2\pi\sqrt{-1}} + \lambda_{1j}\right)x, \ \cdots, \ 0x, \ \cdots,$$

$$2\pi\sqrt{-1}\left(\frac{\log\rho_n - \log\rho_j}{2\pi\sqrt{-1}} + \lambda_{nj}\right)x.$$

By the very definition of λ_{ij}, the coefficients $\epsilon_{1j}, \cdots, \epsilon_{jj}=0, \cdots, \epsilon_{nj}$ of x in these expressions will lie in the strip $0 \leqq v < 2\pi$. Let $P_1, \cdots, P_n$ be the points which represent these coefficients in the complex plane. On account of (8) these coefficients will have decreasing real parts and the points $P_1, \cdots, P_n$ are ordered from right to left. As $\arg x$ increases from 0 to $\frac{1}{2}\pi$ the above logarithms maintain their relative positions but are rotated about $x = 0$ through the same angle. The dominant term along the ray $\arg x = \tau$ is that one for which the corresponding point is farthest to the right (i. e., the logarithm has the greatest real part) when a rotation τ has been effected. None of the terms after the j-th

can dominate since the corresponding points are to the left of $P_j = 0$ for all values of $\tau < \pi/2$.

Draw the convex broken line $P_1 P_\sigma \cdots P_j$ below which none of the points $P_1, \cdots, P_j$ lie and whose vertices are at some of these points. If the plane be rotated through an angle τ, it is clear that these vertex-points, and only these, will correspond to dominant terms. Let the acute angles which the successive sides of $P_1 P_\sigma \cdots P_j$ make with the axis of reals be $\phi_1, \phi_\sigma, \cdots$. The *critical rays* along which the dominant term changes are then in order of increasing magnitude

$$\frac{\pi}{2} - \phi_1, \quad \frac{\pi}{2} - \phi_\sigma, \quad \cdots.$$

For small values of arg x the dominant term is clearly given by

$$s_{i1}(x)\, e^{2\pi\lambda_{1j}\sqrt{-1}\,x}\, c_{1j}^{(0)},$$

and as the first critical ray is passed this changes to

$$s_{i\sigma}(x)\, e^{2\pi\lambda_{\sigma j}\sqrt{-1}\,x}\, c_{\sigma j}^{(0)}.$$

Along the ray the form is given by a combination of two or more terms, a fact which we shall express by saying that the change is *uniform* along the ray. A similar change takes place along each of the successive critical rays until the last is passed when the asymptotic form becomes $s_{ij}(x)$.

Likewise in the fourth quadrant the asymptotic nature is deduced from a relation

$$g_{ij}(x) \sim s_{i1}(x)\, e^{2\pi(\mu+\lambda_{1j}-1)\sqrt{-1}\,x}\, c_{1j}^{(\mu-1)} + \cdots + s_{ij}(x)\, e^{2\pi\mu\sqrt{-1}\,x}\, e^{2\pi r_j\sqrt{-1}} + \cdots + s_{in}(x)\, e^{2\pi(\mu+\lambda_{nj}-1)\sqrt{-1}\,x}\, c_{nj}^{(\mu-1)}.$$

The expressions $s_{i1}(x), \cdots, s_{in}(x)$ which appear here are the determinations for the functions $h_{i1}(x), \cdots, h_{in}(x)$ in the right half plane. Consequently the determination of $s_{ij}(x)$ obtained by a positive rotation from the first to the fourth quadrant along the negative axis of reals is this determination multiplied by the factor $e^{2\pi\mu\sqrt{-1}\,x}\, e^{2\pi r_j\sqrt{-1}}$. Hence in terms of the second determination

$$g_{ij}(x) \sim s_{i1}(x)\, e^{2\pi(\lambda_{1j}-1)\sqrt{-1}\,x}\, e^{-2\pi r_j\sqrt{-1}}\, c_{1j}^{(\mu-1)} + \cdots + s_{ij}(x) + \cdots + s_{in}(x)\, e^{2\pi(\lambda_{nj}-1)\sqrt{-1}\,x}\, e^{-2\pi r_j\sqrt{-1}}\, c_{nj}^{(\mu-1)}.$$

After division the corresponding logarithms are

$$2\pi\sqrt{-1}\left[\frac{\log\rho_1 - \log\rho_j}{2\pi\sqrt{-1}} + \lambda_{1j} - 1\right]x, \ \cdots, \ 0x, \ \cdots, \ 2\pi\sqrt{-1}\left[\frac{\log\rho_n - \log\rho_j}{2\pi\sqrt{-1}} + \lambda_{nj} - 1\right]x.$$

The coefficients of x are represented by points $Q_1, \cdots, Q_j = 0, \cdots, Q_n$, 2π units below the corresponding points $P_1, \cdots, P_n$ (except for $P_j = Q_j = 0$). These points all lie in the strip $-2\pi \leqq v < 0$ and are given in order from right to left. As arg x decreases from 0 to $-\pi/2$, the figure formed by these points rotates through an angle $-\pi/2$, and the dominant terms correspond to the point farthest to the right. None of the terms after the jth can dominate.

In this case we are led to the following rule: draw a convex broken line $Q_1 Q_\theta \cdots Q_j$ above which none of the points $Q_1, Q_2, \cdots, Q_j$ lie and whose vertices are at these points. Let the acute angle which the successive sides $Q_1 Q_\theta \cdots Q_j$ make with the axis of x be $\psi_1, \psi_\theta, \cdots$. The critical rays along which the dominant term changes are then

$$-\frac{\pi}{2} + \psi_1, \quad -\frac{\pi}{2} + \psi_\theta, \quad \cdots$$

in order of decreasing magnitude. For small arg x the dominant term has the asymptotic form

$$s_{i1}(x)\, e^{2\pi(\lambda_{1j}-1)\sqrt{-1}\,x}\, e^{-2\pi r_j \sqrt{-1}}\, c_{1j}^{(\mu-1)},$$

and as the first critical ray is passed, this changes uniformly to

$$s_{i\theta}(x)\, e^{2\pi(\lambda_{\theta j}-1)\sqrt{-1}\,x}\, e^{-2\pi r_\theta \sqrt{-1}}\, c_{\theta_j}^{(\mu-1)},$$

and similar changes take place along the remaining critical rays until the last ray is passed, when the form is $s_{ij}(x)$.

THEOREM V. *Let the coefficients $a_{ij}(x)$ be polynomials of degree not greater than μ in x. Write*

$$2\pi\sqrt{-1}\left(\frac{\log \rho_i - \log \rho_j}{2\pi\sqrt{-1}} + \lambda_{ij}\right) = \epsilon_{ij}$$

where λ_{ij} is the least integer as great as the real part of the negative of the first term in parenthesis, and where $\rho_1, \rho_2, \cdots, \rho_n$ are the roots of the characteristic equation ordered according to decreasing absolute magnitude. Mark the two sets of points

$$\epsilon_{1j}, \ \epsilon_{2j}, \ \cdots, \ \epsilon_{j-1,j}, \ 0 \quad \textit{and} \quad \epsilon_{1j} - 1, \ \epsilon_{2j} - 1, \ \cdots, \ \epsilon_{j-1,j} - 1, \ 0$$

and call them respectively

$$P_1, \ P_2, \ \cdots, \ P_j \quad \textit{and} \quad Q_1, \ Q_2, \ \cdots, \ Q_j.$$

Construct the convex broken line $P_1 P_\sigma \cdots P_j$ above which all the remaining points P lie, and likewise construct the convex broken line $Q_1 Q_\theta \cdots Q_j$ below which all the remaining points Q lie. Let the acute angles which the successive sides of $P_1 P_\sigma \cdots P_j$ and $Q_1 Q_\theta \cdots Q_j$ make with the

axis of reals be $\phi_1, \phi_\sigma, \cdots$ *and* $\psi_1, \psi_\theta, \cdots$ *respectively. The critical rays between* $\arg x = 0$ *and* $\arg x = 2\pi$ *are then*

$$\frac{\pi}{2} - \phi_1, \ \frac{\pi}{2} - \phi_\sigma, \ \cdots, \ -\frac{\pi}{2} + \psi_\theta, \ -\frac{\pi}{2} + \psi_1,$$

in angular order. The asymptotic form of $g_{ij}(x)$ *changes uniformly from*

$$s_{i\alpha}(x)\, e^{2\pi\lambda_{\alpha j} x^{\nu}\sqrt{-1}}\, c_{\alpha j}^{(0)} \quad \text{to} \quad s_{i\beta}(x)\, e^{2\pi\lambda_{\beta j} x^{\nu}\sqrt{-1}}\, c_{\beta j}^{(0)}$$

along the critical ray $\arg x = \frac{1}{2}\pi - \phi_\alpha$, *and likewise from*

$$s_{i\gamma}(x)\, e^{2\pi(\lambda_{\gamma j}-1) x^{\nu}\sqrt{-1}}\, e^{-2\pi r_\delta x^{\nu}\sqrt{-1}}\, c_{\gamma j}^{(\mu-1)} \quad \text{to} \quad s_{i\delta}(x)\, e^{-2\pi(r_\delta - 1) x^{\nu}\sqrt{-1}}\, e^{-2\pi r_\delta x^{\nu}\sqrt{-1}}\, c_{\delta j}^{(\mu-1)}$$

along the critical ray $\arg x = -\frac{1}{2}\pi + \psi_\delta$. *Between the last critical ray of the first set and the first of the last set the asymptotic form is given by* $s_{ij}(x)$.

It is interesting to observe that with $g_{1j}(x), \cdots, g_{nj}(x)$ there are associated at most $j-1$ critical rays in the first and in the fourth quadrants so that, for example, when $j = 1$ there are no critical rays. But we have $g_1(x) = u_1(x)$, $\cdots$, $g_n(x) = u_n(x)$ and this result is in accord with Theorem 1.

There are two special cases of exceptional interest. The first is that in which $\rho_1, \rho_2, \cdots, \rho_n$ are of equal absolute value. The quantities ϵ_{ij} are in this case all pure imaginaries and the broken lines $P_1, P_\sigma, \cdots$ and $Q_1, Q_\theta, \cdots$ reduce to segments of the imaginary axis, and we have $\phi_1 = \pi/2$, $\psi_1 = \pi/2$, so that the critical rays fall along the positive axis of reals, i. e., *do not exist* since this line is excluded throughout. In this event the elements of $G(x)$ preserve their asymptotic form throughout the entire plane, a result which agrees with that obtained in § 1. The second case is that in which the arguments of $\rho_1, \cdots, \rho_n$ are equal. In this case the quantities ϵ_{ij} are all real. In the first quadrant the only critical ray is the positive axis of imaginaries. In the fourth quadrant one finds critical rays different from the negative axis of imaginaries. How shall we reconcile these unsymmetric and apparently contradictory results, since by Theorem III the positive axis of imaginaries is not a critical ray? If one of the constants $c_{ij}^{(0)}$ which determine the change in asymptotic form *vanishes*, the formulas are still valid but do not give the dominant term.* The true critical rays in the first quadrant are symmetrically placed with respect to those in the fourth quadrant.

In general the effect of a zero value of $c_{ij}^{(0)}$ (or of $c_{ij}^{(\mu-1)}$) is to shift the actual change in asymptotic form to a *secondary* critical ray making a smaller angle with the positive axis of reals. The complete determination of its position may be effected when the constants $c_{ij}^{(k)}$ are known.

The critical rays for these solutions have been determined by Galbrun by

*Just as, for example, if $u = x^2$ we have $u \sim e^x(0)$ along the positive axis of reals, but this does not give the dominant term.

use of the Laplace transformation. It may be noted that the above results hold when the functions $a_{ij}(x)$ are only restricted as in (2), since only the form of the functions $p_{ij}(x)$ at $z = 0$ and $z = \infty$ has been made use of. An interesting difference between the case when the functions $a_{ij}(x)$ are rational and the case when they are not lies in the fact that there are a limited number of possible positions for the secondary rays in the first case, and an unlimited number in the second.

§ 7. The Fundamental Problem.

The properties which we have thus far adduced completely characterise the solutions of a linear difference system (1) with rational coefficients, as is evidenced by the following theorem:

Theorem VI. *Let $G(x)$ and $H(x)$ be two matrices of single-valued functions, analytic save for poles in the finite plane and such that their elements have the property*

$$\lim_{x=\infty} g_{ij}(x)x^{-\mu x}\rho_j^{-x}x^{-\gamma_j} = s_{ij} \quad \text{and} \quad \lim_{x=\infty} h_{ij}(x)x^{-\mu x}\rho_j^{-x}x^{-\gamma_j} = s_{ij}, \quad (|s_{ij}| \neq 0)$$

for x in any left or right half plane respectively, and furthermore let

$$G(x) = H(x)\,P(x)$$

where the elements of $P(x)$ are periodic of period 1. Then $G(x)$ and $H(x)$ are the first and second principal matrix solutions of a system (4) in which the elements of $A(x)$ are rational in x, at least if $\rho_1, \rho_2, \cdots, \rho_n$ are distinct.

Proof. If we put

$$A(x) = G(x+1)G^{-1}(x) = H(x+1)H^{-1}(x),$$

as we may by the hypothesis of the theorem, it is clear that the elements $a_{ij}(x)$ of $A(x)$ are single-valued and analytic in the finite plane except for poles. Furthermore, by direct computation, the elements $g'_{ij}(x)$ and $h'_{ij}(x)$ of $G^{-1}(x)$ and $H^{-1}(x)$ are seen to be such that

$$\lim_{x=\infty} g'_{ij}(x)x^{\mu x}(\rho_i e^{\mu})^x x^{\gamma_i} = \sigma_{ij} \qquad \text{and} \qquad \lim_{x=\infty} h'_{ij}(x)x^{\delta\mu x}(\rho_i e^{-\mu})^{\delta x}x^{\gamma_i} = \sigma_{ij},$$

in which σ_{ij} denotes as before the element in the ith row and jth column of the matrix inverse to (s_{ij}). The variable x lies in any left and right half plane respectively. Therefore we have

$$a_{ij}(x) = \sum_{\tau=1}^{n} g_{i\tau}(x+1)g'_{\tau j}(x) = \sum_{\tau=1}^{n} h_{i\tau}(x+1)h'_{\tau j}(x),$$

$$= \sum_{\tau=1}^{n} (x+1)^{\mu(x+1)}(\rho_\tau e^{-\mu})^{x+1}(x+1)^{\gamma_\tau}(s_{i\tau}+\epsilon_{i\tau})x^{-\mu x}(\rho_\tau e^{-\mu})^{-x}x^{-\gamma_\tau}(\sigma_{\tau j}+\epsilon'_{1j}),$$

where $\epsilon_{i\tau}$ and $\epsilon'_{\tau j}$ tend to zero *for all large values of* x, that is, in overlapping right and left half planes. This may be written

$$a_{ij}(x) = x^{\mu} e^{-\mu} \left(1 + \frac{1}{x}\right)^{\mu(x+1)} \sum_{\tau=1}^{n} (s_{i\tau}\rho_{\tau}\sigma_{\tau j} + \epsilon''_{i\tau j}),$$

where $\epsilon''_{i\tau j}$ tends to zero as x becomes infinite.

Thus $a_{ij}(x)$ must have a pole of order μ at $x = \infty$ with leading coefficient

$$a'_{ij} = \sum_{\tau=1}^{n} s_{i\tau}\rho_{\tau}\sigma_{\tau j},$$

and is a rational function of x. In order to find the roots of the characteristic equation (3) in this case, we make an initial linear transformation with constant coefficients on $g_1(x), \cdots, g_n(x)$ in the linear difference system

$$g_i(x) = \sum_{i=1}^{n} a_{ij}(x) g_j(x)$$

which has been constructed. The constants s_{ij} undergo the same transformation and may be reduced to δ_{ij}; thus we shall have $a_{ij} = \delta_{ij}\rho_j$, so that the invariant characteristic equation is

$$(\rho - \rho_1)(\rho - \rho_2)\cdots(\rho - \rho_n) = 0$$

with distinct roots $\rho_1, \cdots, \rho_n$ not zero. Hence the above system satisfies the restrictions of § 1. According to the definition of principal matrix solutions, $G(x)$ and $H(x)$ must be the first and second principal matrix solutions, since the principal matrix solutions are the only ones which preserve a definite asymptotic form $s_{ij}(x)$ even to terms of the first order in any left or right half plane respectively.

It is evident that the $n^2 + 2n$ *exponential constants* ρ_i, r_i, s_{ij} together with the functions $p_{ij}(x)$ characterize the principal solutions more or less completely. Let us turn again to the case in which the coefficients $a_{ij}(x)$ are polynomials of degree μ in x. In this case the equations (51) show that the functions $p_{ij}(x)$ involves $n^2\mu - n$ further *polynomial constants* $c_{ij}^{(l)}$. But among the exponential constants there are n which are not essential since we can alter any set $s_{1k}, \cdots, s_{nk}$ by an arbitrary multiplier and not affect the associated linear difference system (1). Hence there have been specified in all

$$(n^2 + 2n) + (n^2\mu - n) - n \qquad \text{or} \qquad n^2(\mu + 1)$$

characteristic constants counting only the ratios $s_{1k}, s_{2k}, \cdots, s_{nk}$ as characteristic constants. This number is precisely the number of arbitrary coefficients in the n^2 polynomials $a_{ij}(x)$, each of degree μ or less in x. Now the system (1)

determines the constants, but not uniquely since $r_1, \cdots, r_n$ may be replaced by any determination $r_1 + 2m_1\sqrt{-1}, \cdots, r_n + 2m_n\sqrt{-1}$ ($m_1, \cdots, m_n$ integers) in the formal expansions. There is however exactly one set of characteristic constants for each such determination. If the characteristic constants are independent, one may conversely regard the coefficients in the polynomials $a_{ij}(x)$ as functions of the characteristic constants. The following definition and theorem go to show that conversely the characteristic constants do describe all the essential properties of the solutions:

Definition. Two linear systems (1) are said to belong to the same *class* if one may be transformed into the other by a linear transformation with rational coefficients.

THEOREM VII. *Two linear difference systems* (1) *in which* $\rho_1, \rho_2, \cdots, \rho_n$ *are distinct and in which the functions* $a_{ij}(x)$ *are polynomials in* x *of degree* μ *will belong to the same class if their characteristic constants are the same.*

Proof. Let

$$(54) \quad G_1(x+1) = A_1(x)G_1(x) \qquad \text{and} \qquad G_2(x+1) = A_2(x)G_2(x)$$

be two systems of this kind, expressed in matrix notation, which possess the same characteristic constants. For convenience we suppose both of the equations transformed into a normal form in which

$$(55) \qquad a_{ij}(x) = \rho_j(\delta_{ij}x^\mu + a_{ij}^{(1)}x^{\mu-1} + \cdots).$$

This may always be effected by one and the same linear transformations with constant coefficients,

$$G(x) = C\bar{G}(x),$$

namely by that transformation which reduces the constants s_{ij} to δ_{ij},

$$(\delta_{ij}) = (c_{ij})(s_{ij}).$$

This is always possible by (7), and if one substitutes in the new formal expansions, it appears at once that the transformed elements of $A_1(x)$ and $A_2(x)$ have the form (55). The exponential constants ρ_j, r_j and the polynomial constants are of course unaltered by this transformation, and we have explicitly in each case

$$(56) \qquad r_i = a_{ii}^{(1)}, \qquad (i = 1, 2, \cdots, n).$$

This simplification being effected, let us consider the relation between the first principal matrix solutions $G_1(x)$ and $G_2(x)$. Write

$$(57) \qquad G_1(x) = R(x)G_2(x).$$

We find then

$$(58) \quad R(x) = G_1(x)G_2^{-1}(x) = H_1(x)P(x)[H_2(x)P(x)]^{-1} = H_1(x)H_2^{-1}(x),$$

where $H_1(x)$ and $H_2(x)$ are the second principal matrix solutions and $P(x)$ is the common matrix of fundamental periodic functions. It is now easy to demonstrate that $R(x)$ is a matrix of rational functions of x. The elements $a_{ij}(x)$ of this matrix have only poles in the finite plane and the relation (58) shows that

$$\lim_{x=} R(x) = \delta_{ij} \tag{59}$$

in the complete vicinity of $x = \infty$ on account of the common asymptotic form of G_1 and G_2, H_1 and H_2. In consequence of (59) the elements $r_{ij}(x)$ must be rational functions of x, analytic at $x=\infty$ and taking on the value δ_{ij} at that point.

Substitute the value of $G_1(x)$ given in (57) in the first of the given difference systems. There results

$$G_2(x+1) = [\, R^{-1}(x+1)\, A_1(x)\, R(x)\,]\, G_2(x).$$

A comparison with the second of the given systems shows then that

$$A_1(x)\, R(x) = R(x+1)\, A_2(x). \tag{60}$$

This is the condition that the two systems belong to the same class and are transformable, the first into the second, by (57).

Even if this theorem is true there remains the possibility that not all of the characteristic constants are independent, although we are now certain that these constants effectively determine the transcendental properties of the solutions. We may however say something more of the dependence of solutions on the characteristic constants.

In the first place we observe that the exponential constants are expressed in terms of the coefficients in (56) when the reduction to normal form has been effected. Now the principal matrix solutions of (1) were constructed by means of certain limit determinants and a process of summation based on a contour integral. All the processes employed are uniformly convergent in the coefficients $a_{ij}^{(k)}$ and involve them analytically. *It follows that the two principal solutions* $G(x)$ *and* $H(x)$ *of* (1) *are analytic functions of the constants* $a_{ij}^{(k)}$ as well as of x, and hence the elements of

$$P(x) = G(x) H^{-1}(x)$$

have the same property. This necessitates, of course, that the coefficients $c_{ij}^{(l)}$ be analytic in the coefficients $a_{ij}^{(l)}$. Let us take $\rho_1, \cdots, \rho_n, r_1, \cdots, r_n$ as fixed, which amounts to choosing $a_{ii}^{(1)}$ in (56) once for all. The coefficients $c_{ij}^{(l)}$ are then single-valued analytic functions of the remaining coefficients $a_{ij}^{(l)}$, which appear in precisely equal numbers.

Now only two things are possible: either the characteristic constants are independent and one may conversely regard the quantities $a_{ij}^{(l)}$ as analytic functions

of the quantities $c_{ij}^{(l)}$, or else they are *dependent*. In this latter case we should be able to choose

$$a_{ij}^{(k)} = \phi_{ijk}(\lambda) \qquad \text{for all } i, j, k,$$

where $\phi_{ijk}(\lambda)$ is an analytic function of the parameter λ, in such a way that the characteristic constants are independent of λ. The functions $a_{ij}(x)$ are then dependent on λ, so that we may write $A(x) = A(x, \lambda)$. Furthermore we can take generally

$$A_1(x) = A(x, \lambda_0), \qquad A_2(x) = A(x, \lambda)$$

as two determinations of $A(x)$ which give equations (54) with the same characteristic constants, and we may take $\lambda = \lambda_0$ as a value of λ for which the functions $\phi_{ijk}(x)$ are analytic. The matrix $R(x)$ is defined by (57), where $G_1(x)$ and $G_2(x)$ denote again the first principal solutions in (54). The elements $r_{ij}(x)$ are rational functions of x by the preceding theorem.

We will now show that the functions $r_{ij}(x)$, which depend also on λ, must be polynomials in x. To this end we make the further assumption that no two of the μ^n roots of the determinant equation $|A(x, \lambda^0)| = 0$ differ by integers. This is manifestly the general case.

If one of the elements of R has a pole at a point, it will be possible to form a group of such points

$$\alpha, \alpha + 1, \cdots, \beta$$

containing the special point and such that $\alpha - 1$ and $\beta + 1$ are poles of no element of R. Now if we consider (60) at $x = \beta$, we see that the right-hand member is analytic at this point, and that the left-hand member will have at least one element with a pole at $x = \beta$ unless $|A_1(x)| = 0$ at $x = \beta$. Hence we must have $A_1(\beta) = 0$. Likewise if we consider the same equation at $x = \alpha - 1$, we see that the left-hand member is analytic at this point, and that the right-hand member will have at least one element with a pole unless $|A_2(x)| = 0$ at $x = \alpha - 1$. Hence we must have $A_2(\alpha - 1) = 0$. But for λ near to λ_0, the roots of

$$|A_1(x)| = |A(x, \lambda_0)| = 0 \qquad \text{and} \qquad |A_2(x)| = A(x, \lambda)| = 0$$

will nearly coincide and no two of them, as $\alpha - 1$ and β respectively, can be congruent and distinct on account of the hypothesis concerning the roots of $|A(x, \lambda_0)| = 0$. It follows that the elements of $R(x)$ can have no poles in the finite plane, and hence are all polynomials in x. According to (59) these polynomials reduce to δ_{ij} at $x = \infty$, and hence are identically equal to δ_{ij}. *The elements of* $A(x, \lambda)$ *will not depend on the parameter* λ, *contrary to hypothesis.*

Hence the characteristic constants are independent, and the coefficients in the polynomials $a_{ij}(x)$ may be conversely regarded as analytic functions of these characteristic constants. A fundamental problem of the linear difference system is to study this functional relation and in particular *to determine whether there exists a system* (1) *with coefficients polynomial in x of degree not greater than* μ *having any assigned set of characteristic constants.*

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ON PROPERTIES OF A DOMAIN FOR WHICH ANY DERIVED SET IS CLOSED*

BY

EARLE RAYMOND HEDRICK

1. Introduction. FRÉCHET, in his thesis †, defines a general class of assemblages (L) of objects of any sort, with which some definition of limiting element of a sequence is associated. ‡

The most interesting type of such assemblages is that which FRÉCHET calls *compact*. § Among such assemblages one very important class are those which have the property that the *first derived set* (E') *of every subset* (E) *of* (D) *is closed*. ‖

In this paper, some of the properties of any fundamental assemblage (D) which has both the properties described above will be discussed. We shall call (D) the fundamental domain. ¶

2. Elements Interior to Assemblages. Let there be given a fundamental domain (D) with the properties ** described in § 1. An element A is said to be *interior* †† to a set (E) of (D) if A is not a limiting element of any sequence of elements of (D) which do not belong to (E). One set (E) is interior to another set (F) if every element of (E) is interior to (F).

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* Presented to the Society (Columbia, Mo.) November 27, 1909.

† Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906).

‡ The only restrictions are (1) that a sequence of identical elements, $A, A, \cdots, A, \cdots$ be said to have A as a limiting element, (2) that if a sequence $A_1, A_2, \cdots, A_n, \cdots$ has a unique limit, any infinite subsequence selected from it, in the same order, has the same limit. (FRÉCHET, loc. cit., § 7.)

§ A fundamental domain is *compact* if every infinite set of its elements has at least one limiting element. (FRÉCHET, loc. cit., § 9.)

‖ Compare FRÉCHET, loc. cit., § 22. It is evident that domains exist which do not have this property. FRÉCHET cites the case of the domain of all continuous functions defined in a closed interval. See also SCHOENFLIES, *Punktmannigfaltigkeiten*, Bericht der Deutschen Mathematiker-Vereinigung, Ergänzungsband II (1908), page 86, lines 1–6, and page 279. On page 282 it is proved that this property is not a consequence of the compactness of (D).

¶ The notation (L) is used by FRÉCHET for any domain for which an acceptable definition of limit is given. In this paper we shall assume that (D) is *compact* and has the closure property mentioned above. The investigation of the properties of sets having these two characteristics is suggested by SCHOENFLIES, loc. cit., p. 279.

** The following theorems involve this hypothesis, which is not repeated below.

†† See FRÉCHET, loc. cit., § 8.

An immediate consequence of this definition is the following theorem:

THEOREM 1. *If an element A is interior to a set (E), all but a finite number of elements of any sequence whose limiting element is A are interior to (E).* For if not, let $A_1, A_2, \cdots, A_p, \cdots$ be the infinite set of elements not interior to (E) selected from such a sequence. Since A_i is not interior to (E), it can be approached by a sequence of elements $A_i^{(1)}, A_i^{(2)}, A_i^{(3)}, \cdots, A_i^{(k)}, \cdots$ of (D) which do not belong to (E). The derivative of the set of elements $A_i^{(k)}$ contains the elements A_i. Since this derivative is closed, A is one of its elements; hence A can be approached by a sequence of elements of the set $A_i^{(k)}$ none of which belong to (E). This is contrary to the hypothesis that A is interior to (E), and the theorem is proved.

3. The Heine-Borel Theorem. The theorem just proved enables us to show that the Heine-Borel theorem * holds for a fundamental domain (D) which has the properties of section 1.

THEOREM 2. *Let $(E_1), (E_2), \cdots, (E_n), \cdots$ be a countable set of assemblages, such that every element of a certain closed assemblage (K) is interior to at least one of the assemblages (E_i); then every element of (K) is interior to one of a finite number of the assemblages (E_i).*

For suppose the theorem untrue. Let A_1 be an element of (K) not interior to (E_1), and let (E_{n_1}) be the first of the (E_i) to which A_1 is interior. Let A_2 be an element not in $(E_j), j \leqq n_1$; and let (E_{n_2}) be the first (E) to which A_2 is interior. Continuing this process indefinitely, as we may in case the theorem is untrue, we obtain an infinite sequence of elements A_i, such that (E_{n_i}) is the first (E) to which A_i is interior, and $n_i > n_{i-1}$. The sequence A_i has a limiting element A, which is itself interior to some (E), say (E_q). We may suppose A is the only limiting element of the A_i, for, if not, a partial sequence can be selected out of the A_i which has A as its only limiting element and for which the essential properties of the A_i are preserved. Since A is then the only limiting element of the A_i, it follows from Theorem 1 that all but a finite number of the A_i are interior to (E_q). This shows that the original process is fallacious, since $n_i = q$ for all sufficiently large values of i. Hence the theorem as stated must be correct.

4. Examples: The Hilbert Space. As an instance of a fundamental domain to which the preceding theorems apply we may cite, of course, any limited region of an ordinary space of one dimension (or in fact of n dimensions), with the usual definition of limiting point. In fact, any assemblage of points in n

* The theorem is restricted to the case in which sets E_i form a *countable* assemblage; this was the original form of the BOREL statement, afterwards generalized, for real variables, by LEBESGUE and others. FRÉCHET gives (loc. cit., §§36, 42) proofs for any domain of type (V) (*ibid.*, § 27, et seq.), under restrictions on the definition of distance (voisinage) and of limiting element which include the conditions of this paper (*ibid.*, § 28). The more general form of the theorem occurs in this paper as part of Theorem 16.

dimensions which is itself closed may be chosen as the fundamental domain (D). Such a domain evidently has the property that the first derived assemblage of any subset (E) is closed. A more characteristic example, in which the definition of distance has nothing to do with the definition of limiting object is the Hilbert space of an infinite number of dimensions;* the objects used are the points $(x_1, x_2, \cdots, x_n, \cdots)$ with the restriction $\Sigma x_1^2 \leqq 1$. The definition of distance between two points is $[\Sigma (x_1^{(1)} - x_2^{(2)})^2]^{\frac{1}{2}}$; and the points $(x^{(1)}), (x^{(2)}), \cdots, (x^{(n)}), \cdots$ approach the point (x) as a limit if $\lim_{n=\infty} x_i^{(n)} = x_i$ for every value of i.† This space evidently possesses the properties of § 1; yet the set of points $(x^{(n)})$ which *approach* (x) may all lie *at unit distance* from (x), as in the example $x_i^{(n)} = 0$ if $n \neq i$, $x_n^{(n)} = 1$; $x_i = 0$. FRÉCHET has given a definition of distance (voisinage) for which the preceding definition of limit points makes the distance between $(x^{(n)})$ and (x) approach zero.‡ The preceding work shows that it is not always necessary to set up a definition of distance for the Hilbert space; for other domains of objects to do so might be very difficult or even impossible.

5. Other Theorems. Another property of sets of real numbers which holds for a fundamental domain (D) having the properties of section 1 is the following:§

THEOREM 3. *The set of elements at which the oscillation of any limited function $f(x)$ is not less than a given number k is closed.* Let S be any sequence of elements $A_1, A_2, A_3, \cdots, A_i, \cdots$ whose only limiting element is A; let $M_s(A, n)$ be the upper limit of the values of $f(x)$ at the points $A_{n+p} (p = 1, 2, \cdots)$; then $M_s(A) = \lim_{n=\infty} M_s(A, n)$ surely exists and is called the *maximum* of $f(x)$ at A with respect to S. The *maximum* of $f(x)$ at an element A is defined to be the upper limit of the maxima at A with respect to all possible sequences of elements different from A whose only limiting element is A. The *minimum* at A is similarly defined. The *oscillation* at A is the difference between the maximum and the minimum.‖

Let L be a limiting element of a sequence $A_1, A_2, \cdots, A_n, \cdots$ such that the oscillation of $f(x)$ at A_n, for all values of n, is not less than k. If the maxima M_n of $f(x)$ at A_n are not less than a fixed number H for an infinite

* HILBERT, *Wesen and Ziele einer Analysis der unendlichvielen unabhängigen Variabeln*, Rendiconti del Circolo Matematico di Palermo, vol. 27 (1909). Also see HILBERT's *Mitteilungen über die Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen* (vierte und fünfte) Göttinger Nachrichten, 1906.

† HILBERT, Palermo Rendiconti, loc. cit., p. 3.

‡ FRÉCHET, Nouvelles Annales de Mathématiques, 4th series, vol. 8, March and July, 1908. Strictly, by the wording of pages 1 and 23 (July), it would appear that the Heine-Borel theorem does not hold for the Hilbert space; this is however only due to a change of nomenclature, as a closer inspection enables the reader to prove.

§ Compare, for example, BAIRE, *Leçons sur les fonctions discontinues*, p. 73.

‖ Compare BAIRE, loc. cit., p. 70; and a paper by the writer, Bulletin of the American Mathematical Society, vol. 13 (1907), p. 378.

number of values of n, it follows that the maximum M_L of $f(x)$ at L is not less than H; for if $n_1, n_2, \cdots, n_i, \cdots$ are the values of n for which $M_{n_i} \geqq H$, we can select a set of elements $A_{n_i}^{(1)}, A_{n_i}^{(2)}, \cdots, A_{n_i}^{(j)}, \cdots$ which approach A_{n_i}, and for which $f(x)$ exceeds $H - \epsilon$, where ϵ is an arbitrarily small fixed number independent of i. Since L is, by section 1, in the first derivative of $A_{n_i}^{(j)}$, the maximum at L is not less than H.

Likewise if the minima m_n of $f(x)$ at A_n are not greater than K for an infinite number of values of n; the minimum m_L of $f(x)$ at L is not greater than K. Suppose that the maximum M_L and the minimum m_L of $f(x)$ at L differ by $k - 4\delta$. Then there are surely only a finite number of values of n for which M_n exceeds $M_L + \delta$ and only a finite number of values of n for which m_n is less than $m_L - \delta$; hence there are only a finite number of values of n for which $M_n - m_n$ exceeds $k - 2\delta$, which is contrary to hypothesis. Therefore the oscillation, $M_L - m_L$, of $f(x)$ at L is not less than k; and the theorem is proved. It results also from the proof just given that*:

THEOREM 4. *The maximum of $f(x)$ is itself a function which is semi-continuous above; and the minimum is semi-continuous below.*

Another theorem which remains true is:

THEOREM 5. *The frontier of any set is closed.*

The *frontier* of a set (E) of (D) is defined to be the elements of (E) which are limiting elements of elements not in (E) together with the elements not in (E) which are limiting elements of elements of (E). Let A be any limiting element of elements of the frontier. If A is a limiting element of frontier elements $f_1, f_2, \cdots, f_n, \cdots$ which belong to (E), each f_i is a limiting element of elements not in E; hence A is a limiting element of elements not in E. Hence A is a limiting element of elements of (E) and also of elements not in (E); it is therefore a frontier element. A similar argument applies in case A is a limiting element of frontier elements not in (E).

That the several theorems proved above hold true without any restrictions on the definition of distance, or any connection between that definition and the definition of limiting element, is especially remarkable because the proofs of the analogous theorems for real numbers apparently make essential use of the distance notion. That such is the case, however, is demonstrated here by the entire absence of any definition of distance.†

It is also clear that the property emphasized in section 1 — that the derivative of any set be closed — may be used to characterize a type of domains (with assigned definitions of limiting elements) which have in common some fundamental properties.

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*Compare BAIRE, loc. cit., p. 72.

†See SCHOENFLIES, *Bericht*, loc. cit., p. 289. Compare, however, VEBLEN, these Transactions, vol. 6 (1905), p. 167.

Another important theorem which follows from these same assumptions occurs incidentally in the next section.

6. **Enclosable Domains.** In the theory of linear point sets the following statement may be taken as an axiom: If the interval (A_i, B_i) is interior to the interval (A_{i-1}, B_{i-1}) and if the distance $\overline{A_i B_i}$ approaches zero, there is a single point common to all these intervals.

We shall now assume the analogous property: *Corresponding to any element A of (D) there exists a countable set of closed assemblages $Q_i(A)$ which contain the single common interior element A, and which have the following properties:*

(*a*) *$Q_i(A)$ is interior to $Q_{i-1}(A)$.*

(*b*) *Corresponding to any integer m an integer n exists, such that, for any point B whatever, any Q_n which contains B is interior to $Q_m(B)$.*

When the domain D with its definition of limiting element satisfies this requirement, we shall say that D has the *enclosable** property. All of the assemblages Q are supposed to be selected now, and are held fixed in all that follows. Any Q_i is said to be of *rank i*.

Several consequences of these assumptions may be stated at once:

LEMMA 1. *The fundamental domain (D), if it contains more than a single element, is perfect.* For any isolated point of (D) is "interior" to any assemblage whatever. And (D) is closed, by its definition.

LEMMA 2. *If A_i is interior to $Q_i(B)$, for all values of i, B is the only limiting element of the elements A_i.* For since B is the only element interior to all the $Q_i(B)$, any element C distinct from B may be enclosed in $Q_j(C)$, so that $Q_j(C)$ and $Q_i(B)$ have no elements in common for sufficiently high values of i and j. Then the lemma follows from Theorem 1.

It follows also from Theorem 1, that *a necessary and sufficient condition that $\mathrm{Lim}_{i=\infty} A_i = B$ is that at least one of the A_i lies in $Q_j(B)$ for every value of j.*

LEMMA 3. *If a sequence A_i have the single limiting element B, corresponding to any fixed integer m, an integer N exists such that B is interior to $Q_m(A_{N+p})$ for all values of p.*

For a given m select n by (b); since $Q_n(B)$ contains A_{N+p}, where N is determined by Theorem 1, $Q_n(B)$, and therefore B itself, is interior to $Q_m(A_{N+p})$, by (b).

LEMMA 4. *If B is interior to (E), a number m exists such that any Q_m which contains B is interior to (E).*

* My attention is called to the similarity between this hypothesis and that given by VEBLEN for point sets, these Transactions, vol. 6 (1905), p. 168. The notion of uniformity of enclosure is essential in both cases. It is suggested that the word "enclosable" might read "uniformly enclosable"; the former term is retained here, however, for simplicity.

For let R_i be a point of $Q_i(B)$ which is not interior to (E). By Lemma 2, $\text{Lim}_{i=\infty} R_i = B$, and all but a finite number of the R_i are interior to (E) by Theorem 1. It follows from (b) that m exists as stated in the theorem.

LEMMA 5. *If B is interior to (E) and if $\text{Lim}_{i=\infty} A_i = B$, where A_i is an element interior to a certain assemblage Q'_i of the type Q_i for every value of i, then n exists such that Q'_{n+p} is interior to (E) for all values of p.*

For $Q_i(A_j)$ contains B and lies in (E) for $i = m$, $j \geqq N$ by Lemma 4 and Lemma 3. Corresponding to m, a number n exists, by (b), such that any Q_n which contains A_j is interior to $Q_m(A_j)$; hence Q'_{n+p} is interior to $Q_m(A_{n+p})$ and therefore to (E), if also $n \geqq N$.

THEOREM 6. *If any sequence Q'_i of assemblages of the types Q_i $(i = 1, 2, 3, \cdots, n, \cdots)$ are such that Q'_i is interior to Q'_{i-1}, there is one and only one point B interior to all Q'_i, and also $\text{Lim}_{i=\infty} A_i = B$ if A_i is any element interior to Q'_i.*

For if A_i is interior to Q'_i the sequence A_i has at least one limiting element, B, and B is interior to every Q'_i. Any other element C distinct from B can be enclosed by $Q_j(C)$ so that $Q_j(C)$ and $Q_i(B)$ have no elements in common. But $Q_i(B)$ contains Q'_k for *some* value of k, by Lemma 5; hence $Q_j(C)$ contains at most a finite number of the points A_i. It follows that B is the only element common to all the Q'_i, and is the only limiting element of A_i.

THEOREM 7. (*Cauchy Fundamental Theorem.*) *The necessary and sufficient condition that the sequence A_j have a single limiting element is that, corresponding to any arbitrarily assigned value of i, a number n exists such that, for all values of p, all the elements A_{n+p} are interior to a single assemblage of the type Q_i.*

The condition is necessary. For if $\text{Lim}_{j=\infty} A_j = B$, a number m exists by Lemma 3, such that $Q_i(A_m)$ contains B; hence $Q_i(A_m)$ contains A_{n+p} for sufficiently high values of n, by Theorem 1.

The condition is sufficient. For if Q_i contains A_{n+p}, the sequence A_{n+p} has at least one limiting element B; if there were another such limiting element C, $Q_k(C)$ and $Q_k(B)$ would contain no common element for sufficiently high values of k; but $Q_k(B)$ contains Q_i, by (b), for sufficiently high values of i, whence $Q_k(C)$ contains at most a finite number of the elements A_j, and is therefore not a limiting element.

THEOREM 8. *The set P of elements of condensation* of any condensed† assemblage E is either vacuous or perfect.*

It can be shown that P is *closed* without the assumption of the enclosable

* A limiting element is an element of condensation if it remains a limiting element after any countable assemblage is removed. See FRÉCHET, loc. cit., p. 6, § 8.

† A condensed set is one for which any non-countable set of elements gives rise to at least one point of condensation. See FRÉCHET, loc. cit., p. 19, footnote.

property. For let A be a limiting element of elements of P, then A belongs to (E'), since P belongs to (E'), and (E') is closed. If a countable set of elements are removed from (E), P remains intact; hence A remains a limit point of P, and A remains a limit point of the remaining (E). Hence A belongs to (P).*

Let us now show that (P) is dense on itself; let A be any point of (P). Surround A by a set of assemblages (Q_i). Then there are an infinite number of values of i, $i_1, i_2, \cdots, i_n, \cdots$, for which a non-countable set of elements of (E) are interior to (Q_{i-1}), but not to (Q_i), since otherwise a value of i exists for which only a countable set of elements of (E) are interior to Q_i, and this, by Theorem 1, contradicts the hypothesis that A belongs to (P). Since (E) is condensed, there is therefore at least one element of (P) in Q_{i_n-1}, but not in Q_{i_n+1}, for all values of i_n. Let these be $p_1, p_2, \cdots, p_n, \cdots$. Then A is the limit point of p_n, by Theorem 6.

THEOREM 9. *The set of elements (C) of any condensed assemblage (E) which are not elements of condensation is countable.*

For every element k of (C) is such that no element of condensation lies in $Q_i(k)$, where i can be found when k is given. The set of elements (C_i) for which i has the same fixed value is countable, for otherwise an element of condensation p determined by (C_i) would lie in $Q_i(k)$ for some element k of (C_i), by Lemma 3. Hence the entire assemblage (C) is countable.

It follows that *every* CLOSED *condensed assemblage (E) consists of a countable set and a set which is either vacuous or perfect.* The conclusion is still valid if (E) is not closed but contains all its points of condensation.

Finally, any condensed assemblage (E) consists of a countable set and a set of which every point is a point of condensation; the latter set is dense on itself.

With a modification of Fréchet's proof (Thesis, p. 20) similar to the modification of Lindelöf's proof† of Theorem 9 above, it can be shown that

THEOREM 10. *The elements of any assemblage (E) which do not belong to its first derived assemblage (E') form a countable set.*

Hence, if any derived assemblage is countable, the original assemblage is countable.

Precisely following the proofs of Fréchet (Thesis, pages 20–22) we may now show that:

THEOREM 11. *Any closed assemblage (E) is the first derivative of some assemblage selected from (D).*

THEOREM 12. *Any closed sub-assemblage (F) of an assemblage (E) can be formed from (E) by omitting from (E) the points interior to a countable number of the assemblages Q.*

*See FRÉCHET, Palermo Rendiconti, vol. 30 (1910), p. 23, footnote. The closure property of section 1 above is the principle needed in FRÉCHET's thesis, p. 19, § 29.

† BOREL, *Leçons*, page 5.

On account of its importance in what follows we shall repeat Fréchet's proof, which requires alteration in certain details, in the following special case:

If we select (F) to be vacuous and $(E) \equiv (D)$, it results that

THEOREM 13. *Every point of (D) is interior to at least one of a countable set of assemblages Q.**

Select some Q at random, say $Q^{(0)}$. Then any point A of (D) not belonging to $Q^{(0)}$ has the property that $Q_i(A)$ has no point in common with $Q^{(0)}$ for sufficiently high values of i. We shall show that there exists a countable set of elements of (D):

$A_1^{(1)}, \cdots, A_1^{(p_1)}; A_2^{(1)}, \cdots, A_2^{(p_2)}; \cdots A_n^{(1)}, \cdots, A_n^{(p_n)}; \cdots$ such that (a) $Q_n(A_n^{(j)})$ has no point in common with $Q^{(0)}$, and (b) any element of (D) belongs to $Q^{(0)}$ or is interior to $Q_n(A_n^{(j)})$ for some values of n and j. For suppose the process carried out to any stage, so that the points up to $A_{n-1}^{(p_{n-1})}$ are selected. Suppose A exists such that $Q_n(A)$ contains no point in common with $Q^{(0)}$ and that A is not interior to $Q_k(A_k^{(j)})$ for $k < n$. Take this A to be $A_n^{(1)}$; and select another A of (D) which has the same properties and which is not interior to $Q_n(A_n^{(1)})$. Continuing in this way, form $A_n^{(1)}, A_n^{(2)}, \cdots$. This suite is finite, for if not the limit point $A_n^{(0)}$ would exist and $Q_n(A_n)$ would contain $A_n^{(0)}$ by Lemma 3, and therefore also $A_n^{(m+p)}$, by Theorem 1, for sufficiently high values of m.

This completes the proof if we note that every point of $Q^{(0)}$ is interior to that Q which precedes $Q^{(0)}$ in the sequence of Q's which belong to the same point A_0 as does $Q^{(0)}$.

Since every $Q_n(A_n^{(j)})$ may be replaced by Q_k, where $k < n$, since $Q_n(A_n^{(j)})$ is interior to $Q_k(A_n^{(j)})$, it follows that every point of (D) is interior to at least one of a countable set of assemblages Q_k *of the same rank*, k.

But, in the preceding proof, all of the points up to and including $A_{n-1}^{(p_n-1)}$ may be omitted; hence it follows that

THEOREM 14. *Every element of the fundamental domain (D) is interior to a countable number of the assemblages Q_k, for any preassigned value of k.*

Hence also, by Theorem 2, every element of (D), or of any closed subset of D, is interior to a *finite* number of the assemblages Q_k, for any preassigned value of k.

THEOREM 15. *The domain (D) itself is the first derived assemblage of a certain countable set of its own elements; that is, (D) is "separable."*†

For let us take the set of elements

$$A_0; A_1^{(1)}, \cdots, A_1^{(p_1)}; A_2^{(1)}, \cdots, A_2^{(p_2)}; \cdots$$

defined in the proof of Theorem 13, and the elements

$$A_{0_k}; A_{(k+1)_k}^{(1)}, \cdots, A_{(k+1)_k}^{(p_{k+1})_k}; A_{(k+2)_k}^{(1)}, \cdots, A_{(k+2)_k}^{(p_{k+2})_k}; \cdots$$

*See FRÉCHET, Thesis, p. 25.

†See FRÉCHET, Thesis, p. 23.

obtained in the same proof by omitting the points $A_1^{(1)}, \cdots, A_k^{(p_k)}$ and by choosing $Q^{(0)}$ to be at least of rank $k+2$, as in the proof of Theorem 13. All of these points form a countable set; we shall show that any element of (D) is a limiting element of this set.

For let A be any element of (D). By Theorem 13, A is interior to $Q_n(A_n^{(j)})$ for definite values of n and j. With this fixed value of n, let us select m by Lemma 4 so that any Q_m which contains A lies in $Q_n(A_n^{(j)})$. Then choosing $k \geqq m$, select r so that A lies in $Q_{k+r}(A_{(k+r)_k}^{(s)})$. Proceeding in the same way, we may locate A in the interior of a sequence of assemblages $Q^{(1)}$, $Q^{(2)}, \cdots$ such that $Q^{(n)}$ is of higher rank than $Q^{(n-1)}$ and interior to it, and such that $Q^{(n)}$ contains a point $A^{(n)}$ of the countable set selected above. It follows, by Theorem 6, that $\operatorname{Lim}_{n=\infty} A^{(n)} = A$.

COROLLARY. *Any subset of (D) which is dense on itself is contained in the first derived assemblage of a countable set of its own elements.*

THEOREM 16. *Let (E) be an assemblage, each of whose elements is interior to at least one of a family (H) of assemblages I, where (H) is not necessarily countable; then the necessary and sufficient condition that every point of (E) is interior to at least one of a finite number of the assemblages I is that (E) be closed.*

The condition is sufficient. For, since every element e of (E) is interior to some assemblage I, there is an upper limit to the value of n for which $Q_n(e)$ is not interior to some I. If not, let $e_1, e_2, \cdots, e_n, \cdots$ be a set of elements of (E) such that $Q_n(e_n)$ is not interior to any I. Let e_0 be one of the limiting elements of e_n; we may suppose without loss of generality that the e_n have been already selected so that $\lim_{n=\infty} e_n = e_0$. Let I_0 be an assemblage of (H) to which e_0 is interior. Then $Q_{n+p}(e_{n+p})$ is interior to I_0 for sufficiently large values of n, by Lemma 5.

Let N be the upper limit on n for which $Q_n(e)$ is not interior to some I. Select the finite set of assemblages Q of rank $N+1$ for which every point of (D) is interior to some Q_{N+1}, by Theorem 14 and Theorem 2. Let $Q^{(1)}, Q^{(2)}, \cdots, Q^{(k)}$ be the subset of these Q's each of which contain at least one point of (E). Select $I^{(1)}, I^{(2)}, \cdots, I^{(k)}$ such that $Q^{(i)}$ is interior to $I^{(i)}$; then every point of (E) is interior to one of this finite set of I's. *

The condition is also necessary. For if (E) is not closed, let $e_1, e_2, \cdots, e_n, \cdots$ be a set of elements of e whose only limiting element e_0 does not belong to (E); and consider the family (H) of assemblages I_n whose elements belong to (E) but not to $Q_n(e_n)$. Not all the points e_i, and, *a fortiori*, not all the points of (E) can belong to a finite number of the I_n, by Theorem 7.

* This proof follows in spirit that of FRÉCHET, Thesis, p. 26, but in style it follows a proof by BAGNERA, Rendiconti del Circolo Matematico di Palermo, vol. 28 (1909), p. 244, which is essentially the same as FRÉCHET's.

THEOREM 17. *Every non-countable assemblage* (E) *of* (D) *is condensed.*

Given a non-countable set of elements of (E), at least one of the finite set of assemblages Q_k of rank k, to at least one of which each point of (E) belongs, contains a non-countable set of elements of (E). Hence a sequence $Q_1^{(1)}$, $Q_2^{(2)}$, $Q_3^{(3)}$, $\cdots$, $Q_n^{(n)}$, of the type Q exists, such that $Q_n^{(n)}$ is interior to $Q_{n-1}^{(n-1)}$ and that $Q_n^{(n)}$ contains a non-countable set of elements of (E). Hence, by Theorem 6, a point A exists which is the only point interior to all of the $Q_n^{(n)}$ and which is a point of condensation of (E).

From Theorems 9 and 17 we now conclude:

THEOREM 20. *Every closed assemblage* (E) *consists of a countable set together with a set which is perfect.*

We have now proved a set of theorems analogous to the principal fundamental theorems of the theory of bounded sets of points; for these the only assumptions made were those of § 1 and § 6. The hypothesis that the first derived assemblage of every set is closed is therefore seen to be a fundamental hypothesis of far-reaching character. The fundamental domains which possess that property include many important examples besides the original bounded sets of points, and have characteristics in common which make that class of domains one of great importance.

It should also be noticed that the enclosable property of section 6 is possessed by all ordinary domains under any of the usual definitions of distance, if $Q_i(A)$ be taken to be the spheroid about A as center, with radius $1/i$.

IMPORTANT COVARIANT CURVES AND A COMPLETE STSTEM OF INVARIANTS OF THE RATIONAL QUARTIC CURVE*

BY

J. E. ROWE

Introduction.

It is well understood that different domains of rationality are useful in discussing different properties of curves. Two domains are employed in the following, to render possible the geometric interpretation of certain invariants and covariant loci of the rational plane quartic.

Section 1 is divided into two parts: In the first part is given a straightforward proof of the covariance of curves derived from R^n by a certain translation scheme; in the second part SALMON's work on the combinants of two binary quartics is applied to those covariant curves of the R^4 which can be found as combinants. In Section 2 the most important invariants of the R^4 are discussed, and four invariants are found in terms of which any other invariant relation on the R^4 can be expressed algebraically; in this sense these four invariants constitute an *algebraically complete system.* Section 3 contains a treatment of the invariants of the R^4 when the R^4 is taken as the section of the Steiner Quartic Surface by a plane; in this scheme the invariants occur as symmetric functions of the coefficients of the cutting plane.

§ 1*a*. *Certain Covariants of Rational Curves.*

Let the rational curve of order n, which may be called R^n, be written parametrically

$$x_i = a_i t^n + b_i t^{n-1} + c_i t^{n-2} + \cdots \qquad (i = 0, 1, 2). \tag{1}$$

If (1) is cut by two lines

$$(\xi x) = \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 = 0, \tag{2}$$

and

$$(\eta x) = \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2 = 0, \tag{3}$$

the results of this operation are the two binary n-ics

$$(a\xi) t^n + (b\xi) t^{n-1} + (c\xi) t^{n-2} \cdots = 0, \tag{4}$$

* Presented to the Society, September 6, 1910.

and

$$(a\eta)t^n + (b\eta)t^{n-1} + (c\eta)t^{n-2} \cdots = 0, \tag{5}$$

which yield the n parameters of the points in which the lines (2) and (3) cut the curve.

The combinants of (4) and (5) are expressible as rational functions of the two-rowed determinants of the matrix

$$\begin{Vmatrix} (a\xi) & (b\xi) & (c\xi) & (d\xi)\cdots \\ (a\eta) & (b\eta) & (c\eta) & (d\eta)\cdots \end{Vmatrix}. \tag{6}$$

But if in any two-rowed determinant of (6), the quantities x_0, x_1, x_2 are substituted for the coordinates of the point in which (2) and (3) intersect, the result may be expressed as a three-rowed determinant. For instance,

$$\begin{vmatrix} (a\xi) & (b\xi) \\ (a\eta) & (b\eta) \end{vmatrix} = \begin{vmatrix} a_0 & b_0 & x_0 \\ a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \end{vmatrix} = |abx|; \tag{7}$$

and evidently the other determinants of (6) assume similar form. Consequently, by means of the above translation scheme * from the combinants of (4) and (5) certain loci are derived which are related to the R^n in a special manner. Certain sets of points on the R^n defined by some projective relation are connected in this way with a point of the plane, namely, the intersection of (2) and (3). This fact alone warrants the assertion that loci related to the R^n in such a manner are *covariant curves* † of the R^n. But it is important to give an actual algebraic proof of this fact.

In order to prove formally that these curves are covariants, it is sufficient to show that they are unaltered by any linear transformation of the x's or t's.

Let the transformation of the x's be

$$x_i' = l_i x_0 + m_i x_1 + n_i x_2 \qquad (i = 0, 1, 2). \tag{8}$$

After this transformation (1) assumes the form

$$x_i' = (l_i a_0 + m_i a_1 + n_i a_2)t^n + (l_i b_0 + m_i b_1 + n_i b_2)t^{n-1} \cdots, \tag{9}$$

or

$$x_i = (l_i a)t^n + (l_i b)t^{n-1} + (l_i c)t^{n-2} \cdots, \tag{10}$$

where $(l_i a) = l_i a_0 + m_i a_1 + n_i a_2$, etc.

* GRACE and YOUNG, *Algebra of Invariants*, pp. 314–318.

† GRACE and YOUNG in the place already referred to have practically proved that such loci are covariant.

From what has been stated the combinant curves of (10) are expressible as sums of terms, each term being of the same degree in determinants of the type

$$(11) \qquad \begin{vmatrix} (l_0 a) & (l_0 b) & (l_0 x) \\ (l_1 a) & (l_1 b) & (l_1 x) \\ (l_2 a) & (l_2 b) & (l_2 x) \end{vmatrix}.$$

But by the theorem for the multiplication of determinants (11) becomes

$$(12) \qquad |lmn|\,|abx|;$$

also each three-rowed determinant of the type (11) formed from (10) is equal to the product of $|lmn|$ by the corresponding determinant formed from (1). Hence, as $|lmn|$ factors out of each term of the combinants of (10) to the same degree, these curves are unaltered by a transformation of the x's. This is a property which is possessed by any homogeneous function of the determinants of the type $|abx|$; however, not every such function is a covariant. But if such a function is also unaltered by a linear fractional transformation of the parameter t, then such a function is a covariant. Therefore it remains to show that combinant curves possess also this property.

Let the transformation of the t's ($t = t_1/t_2$) be

$$(12) \qquad t_1 = m_1 t_1 + m_2 t_2, \qquad t_2 = l_1 t_1 + l_2 t_2.$$

Transforming (1) by means of (12) we obtain

$$(13) \qquad x_i = a'_i t^n + b'_i t^{n-1} + c'_i t^{n-2} \cdots \qquad (i = 0, 1, 2).$$

If we cut (13) by the two lines (2) and (3) we shall obtain two binary n-ics,

$$(14) \qquad (a'\xi) t^n + (b'\xi) t^{n-1} \cdots = 0,$$

and

$$(15) \qquad (a'\eta) t^n + (b'\eta) t^{n-1} \cdots = 0.$$

But (14) and (15) are exactly what we should have obtained by operating upon (4) and (5) with (12). However, the combinants of (4) and (5) are unaltered by any such transformation as (12); hence *the combinant curves are unaltered by a transformation of the parameter t and are covariant curves of the* R^n. All loci obtained in the manner indicated are covariants, but other covariants occur which cannot be derived as combinants.

Salmon* has discussed the combinants of two binary quartics from an algebraic standpoint and we shall apply his result with some extensions to a study of those covariants of the R^4 which can be derived in this manner.

* *Higher Algebra*, third edition, pp. 200–206.

§ 1*b*. *Covariant Curves of R^4 as Combinants.*

Let the equation of the R^4 be written parametrically

$$x_i = a_i t^4 + 4b_i t^3 + 6c_i t^2 + 4d_i t + e_i \qquad (i=0, 1, 2). \tag{16}$$

If the R^4 is cut by the two lines (2) and (3) we obtain the two binary quartics,

$$U = (a\xi)t^4 + 4(b\xi)t^3 + 6(c\xi)t^2 + 4(d\xi)t + (e\xi) = 0, \tag{17}$$

$$V = (a\eta)t^4 + 4(b\eta)t^3 + 6(c\eta)t^2 + 4(d\eta)t + (e\eta) = 0, \tag{18}$$

giving the parameters of the points in which (2) and (3), respectively, cut the curve. Considering the u and v above to be the same as those of Salmon * by reason of (5) and (7) we have

$$\begin{gathered} \alpha=(ab)=|abx|, \quad \alpha'=(de)=|dex|, \quad \beta=(ad')=|adx|, \quad \beta'=(be')=|bex|, \\ \gamma=(ae')=|aex|, \quad \delta=(bd')=|bdx|, \quad \lambda=(ac')=|acx|, \quad \lambda'=(ce')=|cex|, \\ u=(be')=|bcx|, \quad u'=(cd')=|cdx|. \end{gathered} \tag{19}$$

Consider the pencil $u + Kv$ as a single binary quartic, substitute its coefficients in its two invariants g_2 and g_3, and let the resulting quadratic and cubic in K be written

$$S'_0 K^2 + S'_1 K + S'_2 = 0 \qquad \text{and} \qquad T_0 K^3 + T_1 K^2 + T_2 K + T_3 = 0; \tag{20}$$

the quadratic gives those two values of K for which $u + Kv$ becomes a self-apolar quartic; and the cubic those values of K for which $u + Kv$ becomes a catalectic quartic.

Equating to zero the discriminant of the quadratic of (20) and making the substitutions (7), we have

$$\begin{aligned} A \equiv |aex|^2 + 16|bdx|^2 + 12|acx||cex| - 48|bex||cdx| - 8|abx||dex| \\ -8|adx||bex| = 0, \end{aligned} \tag{21}$$

which is the locus whose tangents cut R^4 in self-apolar sets of points. If the g_2 of (4) were formed it would be the envelope of lines which cut R^4 in self-apolar quartics. Hence A is the point equation of the conic g_2. A line which cuts R^4 in three consecutive points is a line of A, for the g_2 of a quartic having a triple root vanishes. Consequently the six inflexional tangents of R^4 touch A, and indeed A is often spoken of as the conic on the six inflexional tangents of R^4.

The condition for u and v to have the same apolar cubic, or that each may be expressed linearly in the same three fourth powers, leads to a second combinant which with the substitutions (7) becomes

* *Higher Algebra*, third edition, pp. 200–206.

$$B \equiv |acx||cex| - |bcx||cdx| - |adx||cdx| - |bex||bcx| + |bdx|^2 - |abx||dex| = 0. \tag{22}$$

This conic may be identified with the conic which Stahl* gives parametrically; it is the locus of the vertices of flex-triangles of all first osculants.

The eliminant of u and v expresses the condition for a common root or the condition that they intersect on the curve. Hence by reason of (7) R becomes the point equation of the R^4.

The Jacobian of u and v is

$$\alpha t^6 + 3\lambda t^5 + (3\beta + 6\mu)t^4 + (\gamma + 8\delta)t^3 + (3\beta' + 6\mu')t^2 + 3\lambda' t + \alpha' = 0. \tag{23}$$

By a well known property of the Jacobian (23) gives the six tangents from a point to R^4. The condition that these form a self-apolar set is

$$-40I_2\dagger = A + 48B. \tag{24}$$

Consequently the locus of points such that tangents from them to R^4 form self-apolar sets is the conic

$$A + 48B = 0. \tag{25}$$

Further we find that the following relation holds

$$320(I_2^2 - 100I_4) = (A - 16B)^2 - R, \tag{26}$$

where I_2 and I_4 are the well known invariants of the sextic (23). But $I_2^2 - 100I_4 = 0$ for a sextic which has either a triple root, or a double root and the other four forming a self-apolar set. Consequently such points lie on the quartic curve

$$(A - 16B)^2 - R = 0. \tag{27}$$

The points on R which behave in this manner are on the conic $A-16B=0$. Hence the six flexes and the two points q_i (which have the property that tangents drawn from them to R^4 form self-apolar sets) lie on the conic

$$A - 16B = 0. \tag{28}$$

It is worthy of notice that the above not only gives an easy way to write down the equation of the conic on the flexes but is an independent analytical proof that they do lie on a conic with the two points q_i.

Salmon's combinant equation $C = 0$, expressing the condition that a member of the pencil $u + Kv$ can have two squared factors, and from our standpoint the locus of points such that lines drawn from them cut the R^4 in quartics which

* W. Stahl, Journal für die reine und angewandte Mathematik, vol. 101 (1886).

† $I_2 = 0$ is the self-apolarity condition of (23).

have two squared factors, can be nothing else than the point equation of the four double tangents of the R^4. Since it may be written

$$128\,C = (A - 48B)^2 - R, \tag{29}$$

it is evident that the conic on the eight points of contact of the four double tangents is

$$A - 48B = 0. \tag{30}$$

Forming the Hessian of the cubic of (20) and requiring it to be apolar to the quadratic of (29) yields a combinant which may be expressed thus,

$$128I = R - (A - 16B)(A + 48B). \tag{31}$$

When $I \equiv 0$, the R^4 breaks up into two conics. If the two lines from a point which cut the R^4 self-apolarly are apolar to the Hessian pair of the three lines which cut R^4 in catalectic quartics, then I vanishes; therefore together with (7) it gives the locus of such points.

Forming the third transvectant of u and v we get a quadratic in t whose discriminant vanishes on the locus

$$A - 12B = 0. \tag{32}$$

This may be identified with Stahl's conic N. The first osculant of R^4 at a point t' is an R^3 and therefore has three flexes on a line. The equation of this flex-line is a quadratic in t' and the discriminant of this quadratic may be identified with (32). Or writing the R^4 symbolically

$$(a\xi)(\alpha t)^4 = 0 \qquad \text{and} \qquad (b\xi)(\beta t)^4 = 0, \tag{33}$$

then taking the third transvectant we have

$$|abx||\alpha\beta|^3(\alpha t)(\beta t) = 0, \tag{34}$$

which is the conic N given parametrically. Of course the letters in (33), (34) are only symbols and must not be confused with the same letters already used with different meanings.

The eliminant of the cubic and quadratic of (20) is the condition for such a K_1 to exist that $u + K_1 v = 0$ has roots that are both self-apolar and harmonic pairs. But a quadratic for which $g_2 = g_3 = 0$ has a triple root, and therefore with (7) Salmon's $D = 0$ becomes the equation of the six flex tangents of the R^4.

The combinant E, the discriminant of the cubic of (20), evidently yields the locus of point whose lines cut R^4 in catalectic quartics. It is therefore the point equation of the g_3 of equation (4).

Suppose that the polar of the quadratic of (20) as to the cubic of (20) is ϕ_1, and the polar of ϕ_1 as to the quadratic is ϕ_2, by using the cubicovariant of the

cubic instead of the cubic itself we obtain ϕ_1' and ϕ_2' by the same operations. The condition for $\phi_1 = 0$ and ϕ_2' (or ϕ_1' and ϕ_2) to be the same is the combinant M, and by using (7) we have the locus of such points.

Before leaving covariant curves we observe that from Salmon's identities

$$D - E = -2IB - B^2(A - 16B), \tag{35}$$

and

$$128I = R - (A + 48B)(A - 16B), \tag{36}$$

it follows that

$$64(D - E) = B[(A - 16B)^2 - R]. \tag{37}$$

Consequently,* *twelve of the intersections of the flex tangents of R^4 and E lie on the conic B.*

Also from (26) and (37) we have

$$320(D - E) = B[(A + 48B)^2 - 100I_4]. \tag{38}$$

If $I_4 \equiv 0$ (i. e. if tangents drawn from any point to R^4 form a catalectic set), the intersections of D and E lie on two conics; the flex tangents touch E along $A + 48B = 0$ and intersect E along B. This particular R^4 is of the lemniscate type.

§ 2. *Invariants of R^4 as Combinants of the Fundamental Involution.*

If the R^4 be referred to a special triangle of reference its equations are

$$\begin{aligned} x_0 &= at^4 + 4bt^3, \\ x_1 &= 4dt + e, \\ x_2 &= 4bt^3 + 6ct^2 + 4dt. \end{aligned} \tag{1}$$

The condition for R^4 to have a triple point,† or the condition for R^4 to have a perspective point for (1) becomes

$$B_1' \equiv 36a^2c^2e^2 + 256ab^2d^2e - 96a^2cd^2e - 96ab^2ce^2 - 16a^2bde^2 = 0. \tag{2}$$

If the g_2 of u of Section 1 is found for (1), its discriminant multiplied by eight may be written

$$A_1' = -6a^2c^2e^2 - 96b^2c^2d^2 + 8a^2bde^2 + 48abc^2de - 32ab^2d^2e = 0. \tag{3}$$

Since the six inflexional tangents are lines of the conic g_2, A' vanishes when the R^4 has three concurrent flex tangents.

*See THOMSEN, American Journal of Mathematics, vol. 32 (1910), p. 222.

† W. STAHL, Mathematische Annalen, vol. 38 (1891); GROSS, Mathematische Annalen, vol. 32 (1888).

Calculating the conic on the flexes from the preceding section for (1) and taking its discriminant, we have

$$C_1' \equiv 256(a^3b^2cd^4e^2)^* - 64(a^4c^2d^4e^2) + 32(a^4c^3d^2e^3)$$
$$(4) \qquad - 192a^3b^2c^2d^2e^3 - 512a^2b^3c^2d^3e^2 + 128(a^3bc^3d^3e^2)$$
$$- 32a^3bc^4de^3 - 4a^4c^4e^4 = 0.$$

Since it is known that if an R^4 has three flexes on a line it has a fourth, $C'=0$ is the condition for four collinear flexes of the R^4.

These three invariants are independent, as may be shown in the following manner: A' and B' are evidently independent. Further if C' were expressible in terms of A' and B', such a relation as $C' = \lambda B'^2 + \mu A'B'$ would have to hold because C' contains no term in $b^4c^4d^4$, i. e., B' would have to be a factor of C', but this is disproved by making $b=0$ in (2), (3) and (4). Similarly D' is shown to be independent of A', B', and C'.

The condition for $A - 16B$ in lines to be apolar to A in points we shall call $R_2' = 0$, and we find that

$$(5) \qquad 36R_2' + 108C_1' + B_1'^2 + 10A_1'B_1' + 16A_1'^2 = 0.$$

If $H=0$ is the apolarity condition of A in lines to $A - 16B$ in points, then

$$(6) \qquad 6H + A_1'B_1' + 8A_1'^2 = 0.$$

The discriminants of A and B are known, and therefore, the discriminant of the whole pencil

$$(7) \qquad \lambda(A - 16B) + \mu A = 0$$

is

$$(8) \qquad C_1'\lambda^3 + R_2'\lambda^2\mu + H\lambda\mu^2 - A_1'^2\mu^3 = 0.$$

But a more convenient formula than (8) is the discriminant of

$$(9) \qquad \lambda'(cA + dB) + \mu'(aA + 16bB) = 0,$$

which is

$$\begin{aligned}
&\big[-(c+d)^3A_1'^2 - d(c+d)^2H + d^2(c+d)R_2' - d^3C_1'\big]\lambda'^3 \\
&+\big[-3(a+b)(c+d)^2A_1'^2 - \big(b(c+d)^2 + 2d(a+b)(c+d)\big)H \\
&\qquad + \big(d^2(a+b) + 2bd(c+d)\big)R_2' - 3bd^2C_1'\big]\lambda'^2\mu' \\
&+\big[-3(a+b)^2(c+d)A_1'^2 - \big(2(a+b)(c+d)b + (a+b)^2d\big]H \\
&\qquad + \big(2bd(a+b) + (c+d)b^2\big)R_2' - 3b^2dC_1'\big]\lambda'\mu'^2 \\
&+\big[-(a+b)^3A_1'^2 - (a+b)^2bH + (a+b)b^2R_2' - b^3C_1'\big]\mu'^3 = 0.
\end{aligned} \qquad (10)$$

* The expressions in parentheses carry with them their conjugate expressions.

By substituting proper values in (9) and (10) any invariant of any member of the pencil of conics may be easily found. For instance, by a method to be given presently the discriminant of B^* is found to differ only by a numerical factor from the undulation condition. Hence by substitution in (9) and (10) the undulation condition is

$$R_2' - H - A_1'^2 - C_1' = 0, \tag{11}$$

or by reason of (5) and (6)

$$144\,C_1' + (B_1' + 2A_1')^2 = 0. \tag{12}$$

Also the invariant whose vanishing is the condition for three concurrent double tangents is the square root of the discriminant of the conic $A - 12B$, or at least it differs only by a factor. Hence the condition for $A - 12B$ to degenerate is

$$-(B_1' + 6A_1')^2 = 0, \tag{13}$$

and for three concurrent double tangents,

$$B_1' + 6A_1' = 0. \tag{14}$$

The conic $A - 16B$ meets R^4 in two points q_i besides the six flexes. By substituting (1) in $A - 16B$ and comparing with the sextic giving the flexes we find a quadratic giving the q's. Its discriminant is

$$B_1' + 8A_1' = 0\,; \tag{15}$$

hence (15) is the condition for the q's to unite.

Having taken the R^4 in a workable form to find relations among its invariants, we shall now give the most important invariants in their most general forms. It is well known that all line sections of the R^4 are apolar to a pencil of quartics which constitute the *Fundamental Involution*. From the theorem of Grassmann† on the proportionality of determinants it follows that the combinants of the Fundamental Involution are invariants of the R^4. An easy method to pass from the combinants of line sections of an R^4 to combinants of the Fundamental Involution is to replace each determinant by its complementary, i. e., $|acx|$ and $|bkx|$ must be replaced by $|bde|$ and $|ace|$, respectively, etc. Of course, allowance must be made for binomial coefficients. For instance, if these changes are made in the combinant B of Section 1*b* we have

$$\begin{aligned} B' \equiv\ & |bde|\,|abd| - |ade|\,|abe| - |bce|\,|abe| \\ & - |acd|\,|ade| + |ace|^2 - |cde|\,|abc| = 0, \end{aligned} \tag{16}$$

* This was proved in a different way by Dr. THOMSEN in his dissertation, Johns Hopkins University, 1909.

† W. F. MEYER, *Apolarität und Rationale Curven*, § 11.

which is the triple-point condition for R^4 written without binomial coefficients. Further, if these changes are made in A of Section 1b the result is -768 times the discriminant of A for R^4 written without binomial coefficients.

We shall summarize the results of these two sections in a table. Let (R^4) mean the R^4 written parametrically with binomial coefficients and $(\acute{R}^4)$ mean R^4 written in the same way without them. Also let A' stand for the combinant A of the Fundamental Involution which may be derived from A as already explained. In the first column is given the name of the combinant of two line sections of (R^4); in the second we have the locus resulting from the process explained in Section 1a; in the third is the same combinant of the Fundamental Involution giving an invariant of $(\acute{R}^4)$ whose meaning appears in the fourth column. Also if A' means an invariant of $(\acute{R}^4)$ we shall understand that A_1' stands for the same invariant of (R^4), possibly differing by a factor. Evidently any such invariant as A_1' can be obtained from A' by substitution of binomial coefficients, i. e., $|cde|$ and $|bcd|$ of A' would be replaced by $24|cde|$ and $96|bcd|$, respectively, to obtain a multiple of A_1.

Table of Related Invariants and Covariants.

Combinant of two line sections of (R^4).	Covariant Curve of (R^4).	Combinant of F. I. giving an invariant of $(\acute{R}^4)$.	Condition for $(\acute{R}^4)$ to have
A	Locus whose lines cut (R^4) in self-apolar sets.	A'	3 concurrent flex-tangents.
B	Locus of vertices of flex Δ's of 1st osculants.	B'	Triple point.
C	Product of 4 double tangents.	C'	4 collinear flexes.
D	Product of 6 flex tangents.	D'	Cusp.
R	Point equation of (R^4).	R'	Undulation.
E	Locus whose lines cut (R^4) in catalectic sets.	E'	Tac-node.
$A-12B$	Locus of flex lines of 1st osculants.	$A'-12B'$	Its 2 q's unite.
M	Remote meaning.	M'	Skew invariant.

For the purpose of deriving these invariants a special form of the R^4 has been used; and after they have been derived, to identify them with the proper combinant of the Fundamental Involution is a comparatively easy matter. This was the case with $A'-12B'$, which as a combinant of the Fundamental Involution one would hardly have expected to give the condition for three concurrent

double tangents. On the other hand, forming the cubic of (8), Section 1*a*, of the Fundamental Involution, we have three catalectic sets associated with the double points; if two of these are the same we have a tac-node, the condition for which is $E' = 0$, and this would probably be a difficult result to obtain directly from the equations of the curve. $D' = 0$ is the condition for (R^4) to have a doubly perspective quintic, and therefore the condition for a cusp.

Let the invariants A', B', C', and D' be called I_2, I'_2, I_4, and I_6, respectively. *These four invariants constitute a complete system in the sense that any other invariant relation on the R^4 can be expressed in terms of them*, i. e. any other invariant of the R^4 is connected with these four by an algebraic relation. The proof of this fact depends upon a theorem due to Stroh,* which applied here says that every combinant of two binary quartics multiplied by the proper power of an invariant (which in this case is the undulation condition) is an invariant of a binary sextic.† In this case $I_2(R'^2)^2$, $I'_2(R'^2)^2$, $I_4(R'^2)^4$, $I_6(R'^2)^6$, and $I_k(R'^2)^k$ (where k is the degree of an invariant) become J_6, J'_6, J_{12}, J_{18}, and J_{3k} of a binary sextic. But I_2, I'_2, I_4, and I_6 are independent, hence J_0, J_6, J_{12}, and J_{18} are independent. Suppose then that there is an invariant I_k of the R^4 which gives rise to an I_{3k} of the binary sextic. Then we should have five invariants of a binary sextic, four of which are independent, and these are always connected by an algebraic relation. Consequently any I_k of the R^4 is algebraically expressible in terms of the four invariants I_2, I'_2, I_4, and I_6.

§ 3. *The R^4 as Plane Sections of the Steiner Quartic Surface.*

The invariants of the R^4 may be treated very neatly by considering the R^4 as plane sections of the Steiner Quartic Surface‡ which we shall call S^4.

The point equation of the S^4 referred to its trope planes is

$$\sqrt{x_i} = \sqrt{x_0} + \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} = 0; \tag{1}$$

we propose to cut this by the plane

$$(\alpha x) = \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0; \tag{2}$$

in this manner any relation among the α's is of (2) yields a special R^4, or the invariants of the R^4 may be expressed in terms of the symmetric functions of the α's of equation (2).

For instance, any plane through the point (1111) cuts out an R^4 with a triple point and therefore the condition for an R^4 with a triple point is that (2) be on

* E. Stroh, Mathematische Annalen, vol. 34 (1889), pp 321–323.

† W. Stahl, Crelle's Journal, vol 104 (1889), p. 302.

‡ The Steiner Quartic Surface may be obtained as the polar of a plane as to a tetrahedron. (1) is the polar of the plane (1111) as to $\xi_0\xi_1\xi_2\xi_3 = 0$. It is of the third class, fourth order, with an enveloping cone of order six. Hence a plane section is a curve of order four and class six — an R^4.

the point (1111), which is $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0$, and this as a symmetric function of the α's is

$$S_1 = 0. \tag{3}$$

Any plane on a pinch point cuts out an R^4 with a cusp; the six pinch points have the coordinates (1100), $\cdots$, (0011), hence we have an R^4 with a cusp when any one of the six factors of

$$(\alpha_0 + \alpha_1)(\alpha_0 + \alpha_2)(\alpha_0 + \alpha_3)(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_3)(\alpha_2 + \alpha_3) = 0 \tag{4}$$

vanishes. Expressed in terms of symmetric functions, (4) becomes

$$S_1 S_2 S_3 - S_3^2 - S_1^2 S_4 = 0, \tag{5}$$

which is the general condition for a cusp. Similarly when any two factors of (4) vanish we have two cusps, and by taking the product of the factors of (4) five at a time we obtain the condition for a second cusp after (5) has vanished. I find this to be

$$2S_1 S_2^2 + S_2 S_3 + S_1^2 S_3 - S_1 S_4 = 0. \tag{6}$$

By the same argument, after (5) and (6) have vanished the condition for a third cusp is

$$S_1 S_3 + 32 S_4 + S_2^2 + 2S_1^2 S_2 = 0. \tag{7}$$

If (2) is a tangent plane of (1), the point of tangency is a double point in the curve of the section. Nodes also occur where (2) cuts the double lines of the S^4. The condition for an extra node is that (2) be a plane of (1). The plane equation of (1) is

$$(1/\xi_i) = 1/\xi_0 + 1/\xi_1 + 1/\xi_2 + 1/\xi_3 = 0. \tag{8}$$

and hence the condition for an extra node, or for R^4 to degenerate, is

$$S_3 = 0. \tag{9}$$

The points where (2) cuts the trope conics are points of contact of the double tangents in the curve of section or the double tangent are the intersections of (2) and trope planes. Hence if (2) is on a vertex of the reference tetrahedron the corresponding R^4 has three concurrent double tangents. As this can occur in four ways, the condition for three concurrent double tangents is

$$S_4 = 0. \tag{10}$$

Evidently this could not occur twice unless (9) holds, in which case R_4 degenerates.

The line joining the two points where (2) cuts a trope conic is a double tan-

gent; if these two points come together the intersection of (2) and a trope plane becomes a line of the corresponding trope conic and in the section this point is an undulation. Hence the condition for an undulation is that the intersection of (2) and a trope plane be a line of the corresponding trope conic. This may occur in four ways and the general undulation condition is the product

$$\prod^{4}(\alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_1\alpha_2) = 0,$$

or

$$S_2 S_3^2 - S_1 S_3 S_4 + S_4^2 = 0. \tag{11}$$

By the same argument as in case of cusps, the condition for a second undulation after (11) has vanished is found to be

$$S_3^2 + S_1 S_2 S_3 - S_1^2 S_4 + 7 S_2 S_4 = 0. \tag{12}$$

Also for a third, after (11) and (12) have vanished we have

$$S_2^2 + S_1 S_3 + 2 S_4 = 0. \tag{13}$$

By the use of a result of Richmond and Stuart * the quadric associated with the S^4 containing all those points which in plane sections are flexes is found in our notation,

$$8(\alpha_0^3 x_0^2 + \alpha_1^3 x_1^2 + \alpha_2^3 x_2^2 + \alpha_3^3 x_3^2) - S_3(\Sigma x_0^2 - 2\Sigma x_0 x_1) = 0. \tag{14}$$

By its form the quadric in the second parenthesis touches the edges of the tetrahedron at their mid-points, and in sections gives the points of contact of the double tangents. Knowing the meaning of (14) and of the quadric just mentioned and recalling that they (or the corresponding conics in Section 1*b*) are proportional to $A - 16B$ and $48\ B - A$, respectively, we see that the quadric $\Sigma\alpha_0^3 x_0^2 = 0$ is the locus of points which in plane sections yields the conic which is the envelope of flex-lines of all first osculants, as it is proportional to $A - 12B$.

It is desirable to form the discriminant of the pencil of conics as has been done in Section 2. In order to do this in a way that involves the α's symmetrically, we find the plane equation of (14) and substitute the α's for ξ's. Thus by using a result of Salmon † after dividing out by S_3, the discriminant of

$$\mu\Sigma\alpha_0^3 x_0^2 - \lambda S_3(\Sigma x_0^2 - 2\Sigma x_0 x_1) = 0 \tag{15}$$

is exhibited as follows,

$$\begin{aligned} S_4^2\mu^3 - (S_2 S_3^2 - 5 S_1 S_3 S_4 + 8 S_4^2)\mu^2\lambda + 4(S_1^2 S_3^2 - 2 S_2 S_3^2 - S_1 S_3 S_4)\mu\lambda^2 \\ + 4 S_3^2(S_1^2 - 4 S_4)\lambda^3 = 0; \end{aligned} \tag{16}$$

* Proceedings of the London Mathematical Society, series 2, vol. 1 (1904), pp. 129–132.

† *Geometry of Three Dimensions;* p. 50, § 67, p. 58, § 79.

and (16) in the notation of Section 1*b* is the discriminant of

$$\mu(A - 16B) + \lambda(A - 48B) = 0. \tag{17}$$

Equating (17) to A and substituting in (16) we find the discriminant of A to be

$$A_2 \equiv [\tfrac{2}{3}(S_1 S_3 - 4S_4)]^2 = 0, \tag{18}$$

therefore three flex tangents are concurrent if

$$S_1 S_3 - 4S_4 = 0. \tag{19}$$

Or by equating (17) to B we find the discriminant of B

$$\frac{1}{4.36^2}(S_4^2 + S_2 S_3^2 - S_1 S_3 S_4) = 0, \tag{20}$$

but (20) is only a multiple of (11), showing that B degenerates when R^4 has an undulation.

If we identify (17) with

$$\nu A + K(A - 16B) = 0 \tag{21}$$

we find that the discriminant of (21) is

$$\begin{aligned}[\tfrac{2}{3}(S_1 S_3 - 4S_4)]^2\nu^3 + \tfrac{4}{27}(S_1^2 S_3^2 - 20 S_1 S_3 S_4 + 64 S_4^2)\nu^2 K \\ + \tfrac{4}{81}(64 S_4^2 - 5 S_1^2 S_3^2 + 12 S_2 S_3^2 + 4 S_1 S_2 S_4)\nu K^2 \\ + \tfrac{4}{81}(8 S_1 S_3 S_4 + S_1^2 S_3^2 - 4 S_2 S_3^2) K^3 = 0.\end{aligned} \tag{22}$$

Let us write equation (22) briefly as

$$A_2^2\nu^3 + H_2\nu^2 K + R_3\nu K^2 + C_2 K^3 = 0. \tag{23}$$

The importance of (23) will appear later.

If a point where a double line of S^4 cuts (2) is moved up to be a point of (14) we have brought a node and a flex together and have a flecnode; except that at the two pinch points which are on each double line we get cusps.

The points where the double lines of S^4 cut (2) are

$$\begin{aligned} x_0 = x_1 = \alpha_2 + \alpha_3, \quad & x_2 = x_3 = -(\alpha_0 + \alpha_1); \\ x_0 = x_2 = \alpha_1 + \alpha_3, \quad & x_1 = x_3 = -(\alpha_0 + \alpha_2); \\ x_0 = x_3 = \alpha_1 + \alpha_2, \quad & x_1 = x_2 = -(\alpha_0 + \alpha_3). \end{aligned} \tag{24}$$

Substituting these in equation (14) and factoring the cusp condition twice out each time, I find that the conditions under which a flecnode can occur are three expressions of the type

$$(\alpha_0 - \alpha_1)^2(\alpha_2 + \alpha_3) + (\alpha_2 - \alpha_3)^2(\alpha_0 + \alpha_1) = 0; \tag{25}$$

i. e., the vanishing of any one of the three expressions of the type of (25) insures a flecnode.

By arguments similar to those already used the general condition for a flecnode is shown to be

$$32S_1S_2S_3^2 - 4S_1^2S_2^2S_3 - 64S_3^3 + S_1^4S_2S_3 - 5S_1^3S_2^2 + 16S_1^2S_3S_4 - S_1^5S_4 = 0. \tag{26}$$

The condition for a second flecnode after (26) is satisfied is

$$S_1^2S_2^2 - 16S_1S_2S_3 + 48S_3^2 + S_1^3S_3 - 48S_1^2S_4 = 0. \tag{27}$$

Also, for a third, in addition to (26) and (27) we find

$$S_1S_2 - 6S_3 = 0. \tag{28}$$

Obviously an R^4 cannot have three flecnodes without one of them being a biflecnode. The R^4 is cut in eight points by $A - 16B = 0$; six of these points are flexes and the other two are the points q. For a flecnode, a q and a flex must come together; this could occur only twice and a third flecnode would require two flexes of the R^4 to cross, thus forming a biflecnode.

The matter of biflecnode conditions is somewhat unsatisfactory. To find them we introduce a conic of fundamental importance. The conic

$$\lambda(x_0^2 + x_1^2 + x_2^2) + 2fx_1x_2 + 2gx_0x_2 + 2hx_0x_1 = 0 \tag{29}$$

may be transformed by $x_i = 1/x_i$ into the R^4 referred to its nodal triangle, and its equation is

$$\lambda(x_1^2x_2^2 + x_0^2x_2^2 + x_0^2x_1^2) + 2fx_0^2x_1x_2 + 2gx_1^2x_0x_2 + 2hx_2^2x_0x_1 = 0. \tag{30}$$

It may be shown that the invariants of the R^4 calculated directly from equation (30) are connected with the invariants we have found from the S^4 by the following relations

$$\begin{aligned} 2\lambda &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = S_1, \\ 2f &= \alpha_0 + \alpha_1 - \alpha_2 - \alpha_3, \\ 2g &= \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3, \\ 2h &= \alpha_0 - \alpha_1 - \alpha_2 + \alpha_3. \end{aligned} \tag{31}$$

The equation of a pair of nodal tangents may be written down from (30). If we impose upon each of these tangents the condition that it cut the R^4 in three consecutive points the results are the conditions for a biflecnode. These operations yield three pairs of equations of the type

$$h\lambda - gf + g\gamma = 0, \qquad gf - h\lambda + g\gamma = 0, \tag{32}$$

where $\lambda = 0$ is the triple-point condition and $\gamma = 0$ is the cusp condition. Ruling out the vanishing of λ or γ, one such pair as (32) will vanish if only any two of the quantities (f, g, h) vanish. A pair of conditions expressible in symmetric functions of the α's which are equivalent to these two conditions are

$$\sum^{3} f^2 g^2 = 0 \qquad \text{and} \qquad f^2 g^2 h^2 = 0, \tag{33}$$

and these are expressible as

$$3S_1^4 - 16S_1^2 S_2 + 16S_2^2 + 16S_1 S_3 - 64S_4 = 0, \tag{34}$$

and

$$S_1^6 - 8S_1^4 S_2 + 16S_1^2 S_2^2 + 48S_2^3 + 16S_1^3 S_3 - 64S_1 S_2 S_3 + 64S_3^2 = 0. \tag{35}$$

It may be easily verified that the conic on the flexes degenerates when (34) and (35) vanish simultaneously.

AN APPLICATION OF MOORE'S CROSS-RATIO GROUP TO THE SOLUTION OF THE SEXTIC EQUATION*

BY

ARTHUR B. COBLE†

Introduction.

By making use of the fact that all the double ratios of n things can be expressed rationally in terms of a properly chosen set of $n-3$ double ratios, E. H. MOORE‡ has developed the "cross-ratio group," $C_{n!}$, a Cremona group in S_{n-3} of order $n!$ which is isomorphic with the permutation group of n things. H. E. SLAUGHT§ has discussed the $C_{5!}$ in considerable detail. Both $C_{5!}$ and $C_{6!}$ have been listed by S. KANTOR.‖

I have already pointed out¶ that $C_{n!}$ defines a form-problem which I shall denote by $PC_{n!}$. The solution of $PC_{n!}$ carries with it the solution of the n-ic equation, and I have worked out in detail the application to the quintic equation. After the adjunction of the square root of the discriminant of the n-ic, a new form-problem, $PC_{n!/2}$, can be enunciated whose underlying group is the even subgroup $C_{n!/2}$ of $C_{n!}$. It appears in the particular case, $n=5$, that the solution of $PC_{5!/2}$ can be expressed rationally in terms of the solution of KLEIN'S A-problem.** The transition from the one problem to the other is accomplished by a very direct process which is suggested by simple geometric considerations. Explicit formulæ are always desirable, and these have been lacking hitherto in the solutions of the quintic equation in terms of the A-problem. By introducing $PC_{5!/2}$ as an intermediate stage I have obtained such formulæ from the long known invariant theory of the binary quintic.

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† This and allied investigations are being carried on under the auspices of the Carnegie Institution of Washington.

‡ American Journal of Mathematics, vol. 22 (1900), p. 279.

§ American Journal of Mathematics, vol. 22 (1900), p. 343; and vol. 23 (1901), p. 99.

‖ See his *Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene*, Berlin (1895), p. 105, Type XIV; also *Theorie der Transformationen im* R_3, etc., Acta Mathematica, vol. 21 (1897), p. 1, in particular, p. 77, Type LI.

¶ These Transactions, vol. 9 (1908), p. 396, cited hereafter as c1.

**An explicit solution of the A-problem in terms of the icosahedral irrationality is found in c1, § 5.

It is my object in this paper to determine whether any material advantage is gained in the solution of the sextic equation by the introduction of the form-problem of the cross-ratio group of six things, $PC_{6!}$ or $PC_{6!/2}$. The main result in this direction is found in § 2, namely, that after the adjunction of the square root of the discriminant Δ of the given sextic S, the $PC_{6!/2}$ and therefore S itself has a rational resolvent sextic Σ. This resolvent Σ is of the general diagonal type, i. e., the fifth and third powers of the unknown are missing. This resolvent is not unknown, * and the main point of novelty here is the very natural way in which it is suggested by the $C_{6!}$.

In the first paragraph, the equations of generating substitutions of $C_{6!}$ are derived, particular sets of conjugate points are considered, the invariant spreads of the group are obtained, and the form-problems of $C_{6!}$ and $C_{6!/2}$ are set forth. The solution of the sextic S in terms of the solution of either of these form-problems is exhibited. In the second paragraph the solution of $PC_{6!/2}$ is expressed rationally in terms of the solution of the resolvent Σ. Some data concerning the history of Σ and its connection with a particular sextic Φ of MASCHKE are reviewed. These data are correlated to some extent by a geometric interpretation. A solution of $PC_{6!}$ by means of hyperelliptic modular functions also appears. In the third paragraph the reduction of the solution of Σ to the solution of the form-problem PG_{360} of the VALENTINER collineation group G_{360} in S_2 is outlined. The method there used applies equally well to the given sextic S,† so that the advantage which arises from the introduction of Σ is due only to the absence of two of its coefficients. But the rational functions of the coefficients required for the reduction to PG_{360} are of necessity so complicated ‡ that this advantage is important.

§ 1. *The Invariants and Form Problems of* $C_{6!}$.

A sextic S with given roots $z_1, z_2, \cdots, z_6$ can be transformed linearly in such a way that z_6 becomes ∞ and the other roots become $y_1, y_2, \cdots, y_5$ respectively, where $\sum_1^5 y_i = 0$. The transformed roots y are determined to within a factor of proportionality. Their values in terms of the differences of the roots z are

$$
\begin{aligned}
\rho y_1 &= \overline{21}\ \overline{36}\ \overline{46}\ \overline{56} + \overline{31}\ \overline{26}\ \overline{46}\ \overline{56} + \overline{41}\ \overline{26}\ \overline{36}\ \overline{56} + \overline{51}\ \overline{26}\ \overline{36}\ \overline{46}, \quad \overline{ik} = z_i - z_k, \\
\rho y_2 &= \overline{12}\ \overline{36}\ \overline{46}\ \overline{56} + \overline{32}\ \overline{16}\ \overline{46}\ \overline{56} + \overline{42}\ \overline{16}\ \overline{36}\ \overline{56} + \overline{52}\ \overline{16}\ \overline{36}\ \overline{46}, \\
\rho y_3 &= \overline{13}\ \overline{26}\ \overline{46}\ \overline{56} + \overline{23}\ \overline{16}\ \overline{46}\ \overline{56} + \overline{43}\ \overline{16}\ \overline{26}\ \overline{56} + \overline{53}\ \overline{16}\ \overline{26}\ \overline{46}, \\
\rho y_4 &= \overline{14}\ \overline{26}\ \overline{36}\ \overline{56} + \overline{24}\ \overline{16}\ \overline{36}\ \overline{56} + \overline{34}\ \overline{16}\ \overline{26}\ \overline{56} + \overline{54}\ \overline{16}\ \overline{26}\ \overline{36}, \\
\rho y_5 &= \overline{15}\ \overline{26}\ \overline{36}\ \overline{46} + \overline{25}\ \overline{16}\ \overline{36}\ \overline{46} + \overline{35}\ \overline{16}\ \overline{26}\ \overline{46} + \overline{45}\ \overline{16}\ \overline{26}\ \overline{36}.
\end{aligned} \tag{1}
$$

* See the references at the end of § 2.

† COBLE, Mathematische Annalen, vol. 70 (1911), p. 337; cited hereafter as C2.

‡ An estimate of the number and order of these functions is found in C2.

The differences of the transformed roots y are

$$\rho(y_i - y_k) = -5 \cdot ik\ \overline{l6}\ \overline{m6}\ \overline{n6} \quad (i, k, l, m, n = 1, 2, \cdots, 5). \tag{2}$$

The system of values y subject to the relation, $\sum y_i = 0$, can be interpreted as the homogeneous coördinates of a point y in S_3. In this system of coördinates there is associated with the five reference planes, $y_i = 0$, a set of five points, each the polar point of one plane as to the tetrahedron of the other four. Denote these five points by $p_1, p_2, \cdots, p_5$; their coördinates are $p_1(-4, 1, 1, 1, 1), \cdots, p_5(1, 1, 1, 1, -4)$.

In the formulæ (1) let z_6 be a variable parameter, t. Then the point y describes a rational twisted cubic. As t takes the values $z_1, z_2, \cdots, z_6$, y takes the positions, $p_1, p_2, \cdots, p_5, y$. Thus to construct the point y which corresponds to a given sextic S, we associate five of the roots of S to the points p and pass through these points a rational cubic such that their parameters p on the cubic are projective to the five roots of S. In this projectivity, the sixth root of S determines a sixth point y on the cubic curve. Since five roots of S can be associated with the five points in 720 ways, a sextic S determines a set of 720 points y. The change in y due to a permutation of the roots of S from some given order furnishes a transformation of Moore's group, $C_{6!}$. The 720 points y determined by S are a set of conjugate points under $C_{6!}$.

Let $T_{(\ldots)\ldots(\ldots)}$ denote a permutation written in cycle form from the order $z_1, z_2, \cdots, z_6$ to the order $z_1', z_2', \cdots, z_6'$. The same symbol will be used for the transformation which carries y into y'. Clearly the permutations of $z_1, \cdots, z_5$ alone give rise to the permutation group of the five coördinates y, i. e., to the collineation group, C_{120}, which transforms the system of five planes y_i or the system of five points p_i into itself. The $C_{6!}$ can be generated by this group and an additional transformation, say $T_{(16)}$. To obtain this transformation we make use of formulæ (2).

$$\begin{aligned}
\rho'(y_1' - y_2') &= -5 \cdot \overline{1'2'}\ \overline{3'6'}\ \overline{4'6'}\ \overline{5'6'} = -5 \cdot \overline{62}\ \overline{31}\ \overline{41}\ \overline{51} = \sigma \frac{1}{y_1 - y_2}, \\
\rho'(y_1' - y_3') &= -5 \cdot \overline{1'3'}\ \overline{2'6'}\ \overline{4'6'}\ \overline{5'6'} = -5 \cdot \overline{63}\ \overline{21}\ \overline{41}\ \overline{51} = \sigma \frac{1}{y_1 - y_3}, \\
\rho'(y_1' - y_4') &= -5 \cdot \overline{1'4'}\ \overline{2'6'}\ \overline{3'6'}\ \overline{5'6'} = -5 \cdot \overline{64}\ \overline{21}\ \overline{31}\ \overline{51} = \sigma \cdot \frac{1}{y_1 - y_4}, \\
\rho'(y_1' - y_5') &= -5 \cdot \overline{1'5'}\ \overline{2'6'}\ \overline{3'6'}\ \overline{4'6'} = -5 \cdot \overline{65}\ \overline{21}\ \overline{31}\ \overline{41} = \sigma \frac{1}{y_1 - y_5},
\end{aligned} \tag{3}$$

where

$$\sigma = 25\rho \cdot \overline{12}\ \overline{13}\ \overline{14}\ \overline{15}\ \overline{62}\ \overline{63}\ \overline{64}\ \overline{65}.$$

Thus $T_{(16)}$ is an involutory cubic transformation with the singular tetrahedron p_2, p_3, p_4, p_5 and the fixed point p_1. Moreover any permutation which affects

z_6 gives rise to a cubic transformation with a singular tetrahedron whose vertices are found among the points p and with the same period as the permutation. The equations of the transformation can be obtained in precisely the same way as for $T_{(16)}$.

A point in general position is one of a set of 720 points which form a conjugate set under $C_{6!}$. If a point takes less than 720 positions under $C_{6!}$, it must be unaltered by a certain subgroup G of $C_{6!}$. To such a point there must correspond a class of projective binary sextics, each of which must be self-projective under a binary transformation group Γ which is isomorphic with G. If all the roots of the sextic are distinct, any permutation of them which can be effected by a binary projectivity defines the projectivity. If however equalities among the roots exist—to fix ideas let $z_1 = z_2$—then to the identical projectivity there corresponds the identical permutation and also the permutation (z_1, z_2). From the formulae (2) we find that the points y which correspond to sextics for which $z_i = z_k\,(i, k = 1, 2, \cdots, 5)$ lie in the plane $y_i - y_k = 0$; those which correspond to sextics for which $z_i = z_6$ are directions about the point p_i. These five points and ten planes are a conjugate system of manifolds under $C_{6!}$, the locus of points for which the invariant Δ of S vanishes. Since we are interested primarily in the solution of S we shall assume that $\Delta \neq 0$. Then the groups G and Γ are simply isomorphic. The types of sextics S invariant under a group Γ have been tabulated by Bolza.* From these types the systems of points in question are easily derived.

Because of the projective character of the transition from sextic S to point y it is clear that the locus of points y which map sextics S for which a certain invariant I of S vanishes is a surface, $M = 0$, invariant under $C_{6!}$. Conversely to every surface, $M = 0$, invariant under $C_{6!}$ there corresponds an invariant I of S. For if $M = 0$ is invariant under $C_{6!}$, M must be a symmetric function of $y_1, y_2, \cdots, y_5$. Since $\sum y_i = 0$, M is the leading coefficient of a covariant of the quintic, $(t_1 - y_1 t_2) \cdots (t_1 - y_5 t_2)$. Then $M = 0$ expresses that ∞ is a certain covariant point of $y_1, \cdots, y_5$, or that z_6 is a certain covariant point of $z_1, \cdots, z_5$. The condition that z_i be a similar covariant point of the other z's is that $M = 0$ for all points conjugate to y, and this is satisfied, since $M = 0$ is invariant under $C_{6!}$. But this condition implies that the roots z satisfy a rational invariant relation, $I = 0$.

The invariant surfaces therefore are merely the invariants of the sextic

$$6t_2(t_1 - y_1 t_2) \cdots (t_1 - y_5 t_2), \qquad \sum y_i = 0,$$

a sextic in a typical form employed by Clebsch (*Binäre Formen*, p. 351); the invariants of such forms are calculated there up to the sixth order. Another

*Mathematische Annalen, vol. 30 (1887), p. 546; American Journal of Mathematics, vol. 10 (1888), p. 47.

mode of procedure is to express the invariants of S in terms of the differences of the roots z. By identifying z_6 with ∞ and $z_1, z_2, \cdots, z_5$ with $y_1, y_2, \cdots, y_5$ we get the corresponding invariant surface from the coefficient of the highest power of z_6. The index of this power is the degree n of I in the coefficients of S. The weight $3n$ of I is the order in the differences. But n differences in each term of I contain z_6. Thus the order of the corresponding invariant surface M in y is $2n$. Since M must contain all the points to the same multiplicity k and also must be invariant under $T_{(16)}$, we find that $k = n$. Hence

All surfaces invariant under $C_{6!}$ are rational integral functions of a set of surfaces M_4, M_8, M_{12}, M_{20}, M_{30} of orders 4, 8, 12, 20, 30 *respectively and of multiplicities* 2, 4, 6, 10, 15 *respectively at the points p. These surfaces correspond to the members of a complete system of invariants of S of degrees* 2, 4, 6, 10, 15 *respectively.*

The square of M_{30} can be expressed by means of the others; whence the general invariant surface M either is of order $4k$ with $2k$-fold points at p_i or is the aggregate of such a surface and M_{30}. The discriminant Δ of degree ten corresponds to a surface of order twenty, which as we have seen must be the square of the ten planes on p. This surface has 12-fold points at p_i instead of the normal 10-fold points. This is accounted for the fact that the ∞^2 directions about p_i also map sextics for which $\Delta = 0$. In general if the multiplicity at p_i is greater than the normal multiplicity, $2k$, the ten planes are part of the invariant surface.

The simplest linear system of surfaces which is unaltered by $C_{6!}$ is the system of ∞^4 quadrics on the five points p. These quadrics map S_3 on a cubic surface in S_4, whence there is among five linearly independent quadrics on p only a single cubic relation. There are

$$(n+1)(n+2)(n+3)(n+4)/4! - (n-2)(n-1)n(n+1)/4! = \frac{(n+1)(n^2+2n+2)}{2}$$

linearly independent forms of order n in these five quadratic forms. This is exactly the number.

$$\frac{1}{3!}(2n+1)(2n+2)(2n+3) - \frac{5}{3!}n(n+1)(n+2) = \frac{n+1}{2}(n^2+2n+2),$$

of linearly independent forms of order $2n$ with n-fold points at p_i. Hence every surface invariant under $C_{6!}$ can be expressed as an algebraic form with the five quadrics as variables. For purposes of manipulation however it is convenient to have a symmetrical set of quadrics on the points p_i. Since the five points are permuted by C_{120}, each member of such a set would be unaltered by a subgroup of C_{120} whose index is the number of members. This number should be five at least and we look for quadrics invariant under a metacyclic subgroup C_{20} of C_{120} whose index is 6. Selecting the particular subgroup generated by $T_{(12345)}$ and $T_{(2354)}$ we find that the quadric required is

$$(y_1-y_2)(y_5-y_3)+(y_2-y_3)(y_1-y_4)+(y_3-y_4)(y_2-y_5)$$
$$+(y_4-y_5)(y_3-y_1)+(y_5-y_1)(y_4-y_2).$$

By using the even permutations the following set of six conjugate quadrics is obtained:

$$\begin{aligned}
2A &= (25)(13)+(51)(42)+(14)(35)+(43)(21)+(32)(54), \quad (ik)=(y_i-y_k),\\
2B &= (12)(53)+(23)(14)+(34)(25)+(45)(31)+(51)(42),\\
2C &= (53)(41)+(34)(25)+(42)(13)+(21)(54)+(15)(32),\\
2D &= (31)(45)+(24)(53)+(25)(41)+(15)(32)+(43)(21),\\
2E &= (31)(24)+(12)(53)+(25)(41)+(54)(32)+(43)(15),\\
2F &= (42)(35)+(23)(14)+(31)(52)+(15)(43)+(54)(21).
\end{aligned} \tag{4}$$

There must exist among these quadrics a linear relation (due to the identities of the type $(12)(34)+(13)(42)+(14)(23)=0$; this is necessarily symmetrical,

$$A+B+C+D+E+F\equiv 0. \tag{5}$$

The even permutation group of the y's, a C_{60}, permutes the six quadrics as the icosahedral group permutes the diagonals of the icosahedron; the odd permutations permute the quadrics oddly into the quadrics with changed sign. The Cremona transformation $T_{(16)}$ permutes the quadrics in the order $(AE)(BC)(DF)$ to within a common outstanding factor. Thus to within common factors the quadrics are permuted in all possible ways by the $C_{6!}$. Each quadric is unaltered by one of a second set of six conjugate subgroups of $C_{6!}$ of order 120. The cubic identity satisfied by these quadrics must be unaltered by $C_{6!}$ and therefore must be symmetric in $A, \cdots, F$. Since $\Sigma A=0$, this identity is

$$A^3+B^3+C^3+D^3+E^3+F^3\equiv 0. \tag{6}$$

Every invariant surface can be expressed in terms of these quadrics, and being invariant must be a symmetric or alternating rational integral function of them. Conversely every such function determines a surface invariant under $C_{6!}$. For example, ΣA^5 does not vanish, since the quadrics satisfy only the relations, $\Sigma A=0$ and $\Sigma A^3=0$. The odd permutations of C_{120} change the sign of ΣA^5, hence it is an alternating invariant S of degree 5, i. e., to within a numerical factor, $\sqrt{\Delta}$. Also $\Pi(A-B)$ does not vanish and is unaltered by both the even and the odd permutations of C_{120}. It must be then the skew invariant of S of degree 15. The symmetric functions, ΣA^2, ΣA^4, and ΠA correspond to invariants of S of degrees 2, 4, and 6 respectively.

To identify the preceding results with known systems let us define, following

Clebsch and Gordan, some invariants and covariants of the sextic. Let

$$
\begin{gathered}
S = a_x^6, \qquad k = k_x^4 = (ab)^4 a_x^2 b_x^2, \\
l = l_x^2 = (ak)^4 a_x^2, \qquad m = m_x^2 = (kl)^2 k_x^2, \qquad n = n_x^2 = (km)^2 k_x^2, \\
A = (ab)^6, \qquad B = (kk_1)^4, \qquad C = (kk_1)^2 (kk_2)^2 (k_1 k_2)^2, \\
D = (ln)^2, \qquad R = (lm)(ln)(mn), \qquad \sqrt{\Delta} = \Pi (z_i - z_k) \qquad (i<k).
\end{gathered} \tag{7}
$$

Adjusting the numerical coefficient in the formula for Δ given by Gordan (*Invarianten-Theorie*, p. 295) we have

$$
\begin{aligned}
-\frac{1}{2\cdot 3^4}\Delta = {} & 3\cdot 2^7 A^5 - 3\cdot 2^4\cdot 5^3 A^3 B - 2^4\cdot 5^4 A^2 C \\
& + 3\cdot 2\cdot 5^5 (AB^2 + BC) + 3^2\cdot 5^5 D.
\end{aligned} \tag{8}
$$

Let the invariants A, B, C, D, and R be calculated for the sextic,

$$6t_2 \prod_1^5 (t_1 - y_i t_2),$$

and expressed symmetrically in terms of the quadrics $A, \cdots, F$. These quadrics satisfy a sextic equation, $\Sigma = 0$, where

$$\Sigma = (Q-A)(Q-B)\cdots(Q-F) = Q^6 + 15q_2 Q^4 + 15q_4 Q^2 + 6q_5 Q + q_6. \tag{9}$$

The coefficients of Σ are integral in A, B, C, $\sqrt{\Delta}$, and the square root of the discriminant of Σ is a numerical multiple of R. The result of the comparison is

$$
\begin{gathered}
15q_2 = \frac{5}{3} A, \qquad 15q_4 = \frac{5}{3} A^2 - \frac{5^3}{2\cdot 3^2} B, \\
q_6 = -\frac{5}{3^3} A^3 + \frac{5^3}{2\cdot 3^3} AB + \frac{5^4}{3^4} C, \qquad 6q_5 = \frac{1}{3^5}\sqrt{\Delta}.
\end{gathered} \tag{10}
$$

Another set of fundamental invariants is defined by the equations,

$$
\begin{gathered}
\bar{A} = \frac{5}{9} A, \qquad \bar{B} = \frac{1}{2\cdot 9^2}(8A^2 - 75B), \\
\bar{C} = \frac{5}{2\cdot 9^3}(-2^3\cdot 13 A^3 + 3^2\cdot 5^2 AB + 2\cdot 3^2\cdot 5^3 C),
\end{gathered} \tag{11}
$$

where $\bar{B}$ and $\bar{C}$ both vanish if S has a triple root. Then the sextic Σ has the particularly simple form

$$\Sigma = (Q^2 + \bar{A})^3 + 15\bar{B}Q^2 + \frac{1}{3^5}\sqrt{\Delta}\, Q + C = 0. \tag{12}$$

If we let

$$\sqrt{\Delta_\Sigma} = \Pi (A-B)\cdots(A-F)(B-C)\cdots(B-F)\cdots(E-F), \tag{13}$$

then

$$\sqrt{\Delta_\Sigma} = -\frac{2^2\cdot 5^{10}}{3^6} R. \tag{14}$$

The form problem of the $C_{6!}$ reads as follows:

(15) *Given the numerical values of* A, B, C, Δ, *invariant under* $C_{6!}$, *to find the ratios of the coördinates of a point* y *for which these quantities take the assigned values.*

The given values determine the numerical values of B/A^2, C/A^3, and Δ/A^5, whence y is a meeting point of invariant surfaces of orders 8, 12 and 20. Apart from the points p of multiplicities 4, 6, and 10 respectively, these surfaces meet in $8\cdot12\cdot20 - 5\cdot4\cdot6\cdot10 = 720$ points which form a conjugate set under $C_{6!}$. Thus all the solutions of $PC_{6!}$ are obtained rationally from one solution by means of the transformations of $C_{6!}$.

If the numerical value which the product of the ten planes, $\sqrt{\Delta}$, takes for the point y be given it is not possible to determine a set of 360 points conjugate under $C_{6!2}$ by means of three absolute invariants alone. As before, the known ratios B/A^2, C/A^3, and Δ/A^5 determine *two* sets of 360 conjugate points. If also the value R be given, the known ratios B/A^2, C/A^3, $R/(\sqrt{\Delta})^3$ determine *three* sets of 360 conjugate points, since the surfaces are of orders 8, 12, and 30 with 4-, 6-, and 15-fold points respectively at p_i. But there is an identity (Gordan, *Invarianten-theorie*, pp. 290–1) which expresses the square of $R/(\sqrt{\Delta})^3$ in terms of B/A^2, C/A^3 and Δ/A^5. Since this identity is cubic in Δ/A^5, only one of the above three sets is contained in the two sets. Thus the form-problem of $PC_{6!2}$ reads:

(16) *Given the numerical values of the invariants* A, B, C, $\sqrt{\Delta}$, *and* R *subject to the identity which connects them, to find the ratios of the coördinates* y *of one of the associated set of* 360 *points conjugate under* $C_{6!2}$.

We have to show finally how the roots z of the given sextic S can be expressed rationally in terms of a solution, y, of $PC_{6!2}$. First let us assume that $\Delta \neq 0$ and $R \neq 0$, for otherwise S can be solved by means of radicals. The invariants A, B, C, Δ, and R of S are calculated and a value of $\sqrt{\Delta}$ is adjoined. These values are identified with the numerical values in $PC_{6!2}$. The solution of $PC_{6!2}$ furnishes a sextic

$$S' = 6t_2(t_1 - y_1t_2)\cdots(t_1 - y_5t_2)$$

which is projective to S. If we denote the invariants and covariants of S' by primed letters, then

$$(17)\quad I_2^2(lx)^2 = \rho I_2'^2(l't)^2, \qquad I_2(mx)^2 = \rho I_2'(m't)^2, \qquad (nx)^2 = \rho(n't)^2.$$

Here I_2 is a non-vanishing invariant of weight 6 which can always be formed from A, B, C, and $\sqrt{\Delta}$ unless $A = B = C = 0$, in which case S is solvable by radicals. ρ is a power of the determinant of the projectivity that transforms S' into S. The determinant of the system (17) in the variables x_1^2, $2x_1x_2$, and x_2^2 is I_2^3R and is not zero. From this system we can find the

values of x_1^2/ρ, $2x_1x_2/\rho$, and x_2^2/ρ in terms of the coefficients of S, of the coördinates $y_1, \cdots, y_5$ and of t_1 and t_2. Then from

$$z = \frac{\frac{1}{\rho}x_1^2}{\frac{1}{\rho}x_1x_2} \qquad \text{or} \qquad z = \frac{\frac{1}{\rho}x_1x_2}{\frac{1}{\rho}x_2^2}$$

we get the six roots $z_1, z_2, \cdots, z_6$ by assigning the values $y_1, y_2, \cdots, y_5, \infty$ to the ratio t_1/t_2.

Hence the solution of S and that of $PC_{6!}$ or the solution of S after the adjunction of $\sqrt{\Delta}$ and that of $PC_{6!/2}$ are rationally equivalent problems.

§ 2. *The Sextic Resolvent* Σ *of* $PC_{6!/2}$.

The given quantities in $PC_{6!/2}$ as formulated in (16) determine rationally the coefficients of the sextic Σ in (9). Also, according to (14), $\sqrt{\Delta_\Sigma}$ is rationally known from $PC_{6!/2}$. The Galois group of Σ is of order 360 and is simply isomorphic with $C_{6!/2}$. Since the roots of Σ are rational functions of the required coordinates y in $PC_{6!/2}$, Σ is in a sense a rational resolvent of $PC_{6!/2}$. In order to complete the connection between Σ and $PC_{6!/2}$ we require formulae which give the ratios of the coordinates y in terms of the roots $A, \cdots, F$ of Σ. From the formulae (4) we get at once

$$\begin{aligned}
&\tfrac{1}{2}(A+B) = (51)(42), \quad \tfrac{1}{2}(B+C) = (34)(25), \quad \tfrac{1}{2}(C+E) = (13)(42),\\
&\tfrac{1}{2}(A+C) = (53)(41), \quad \tfrac{1}{2}(B+D) = (31)(45), \quad \tfrac{1}{2}(C+F) = (54)(21),\\
(18)\quad &\tfrac{1}{2}(A+D) = (43)(21), \quad \tfrac{1}{2}(B+E) = (12)(53), \quad \tfrac{1}{2}(D+E) = (25)(41),\\
&\tfrac{1}{2}(A+E) = (32)(54), \quad \tfrac{1}{2}(B+F) = (23)(14), \quad \tfrac{1}{2}(D+F) = (24)(53),\\
&\tfrac{1}{2}(A+F) = (25)(13), \quad \tfrac{1}{2}(C+D) = (15)(32), \quad \tfrac{1}{2}(E+F) = (15)(43).
\end{aligned}$$

From these pairs of differences the ratios of four linearly independent differences can be derived in a variety of ways. One verifies at once that

$$\begin{aligned}
\rho(12) &= -(A+D)(B+E)(C+F),\\
\rho(13) &= (A+D)(B+E)(B+D),\\
\rho(14) &= (A+D)(A+C)(C+F),\\
\rho(15) &= (E+F)(B+E)(C+F),
\end{aligned} \tag{19}$$

where

$$\rho = 8\cdot(12)^2(34)(35)(45).$$

From these ratios of the differences we find the required ratios of the coördinates y,

$$
\begin{aligned}
5y_1 &= (12) + (13) + (14) + (15) \\
5y_2 &= -4(12) + (13) + (14) + (15) \\
&\cdots\cdots\cdots\cdots
\end{aligned}
\tag{20}
$$

Hence the solution of Σ and the solution of $PC_{6\,1/2}$ are rationally equivalent problems.

Since the solutions of $PC_{6\,1/2}$ and the solution of S after the adjunction of $\sqrt{\Delta}$ are also rationally equivalent, it must be possible to exhibit Σ directly as a rational resolvent of S. This is accomplished by passing from the quadrics $A, \cdots, F$ to functions of the differences $(z_i - z_k)$ by means of formulae (2). Dropping the factor, $25\Pi_{i=1}^{5}(i6)$, from the six we obtain functions, $A', \cdots, F'$, of weight 3 and of order 1 in a particular root. They arise from $A, \cdots, F$ by replacing each term $(ik)(lm)$ by the term $(ik)(lm)(n6)$. These new functions $A', \cdots, F'$ are the roots of a rational resolvent of S after adjunction of $\sqrt{\Delta}$. They have been used by previous writers on this subject, but in these earlier accounts the introduction of the functions seems to be somewhat fortuitous. Here their existence and properties have been shown to be direct consequences of quite elementary geometric group-theory. Moreover the inverse process of expressing the roots of S in terms of the roots of $\Sigma[(19), (20), (17)]$ is as readily suggested and accomplished by the use of familiar formulae.

The functions, $A', \cdots, F'$ of the six roots z seem to have been first used by Joubert,* who has noted their most striking properties and has calculated the equation which they satisfy. A similar study is found in a paper by H. W. Richmond.† Joubert's main interest in this sextic apparently lay in its utility as a resolvent of the quintic.‡

From a quite different point of view, H. Maschke § developed another particular sextic,

$$\Phi = y^6 - 6F_8 y^4 + 4F_{12} y^3 + 9F_8^2 y^2 - 12F_{20} y + 4F_{24} = 0. \tag{21}$$

The coefficients F_i are quaternary forms of order i in the variables z_1, z_2, z_3, z_4 which are invariant (or which at most change in sign if $i \not\equiv 0 \bmod 8$) under a group of linear substitutions of order 64.720. The group is obtained by effecting transformations of the periods on the Borchardt moduli, z_i, for the hyperelliptic thetas of genus 2, where

$$
\begin{aligned}
z_1 &= \theta_5 = \vartheta_5(0, 0;\ 2\tau_{11}, 2\tau_{12}, 2\tau_{22}), \qquad & z_3 &= \theta_{23} = \vartheta_{23}(0, 0;\ 2\tau_{11}, 2\tau_{12}, 2\tau_{22}), \\
z_2 &= \theta_4 = \vartheta_4(0, 0;\ 2\tau_{11}, 2\tau_{12}, 2\tau_{22}), \qquad & z_4 &= \theta_{01} = \vartheta_{01}(0, 0;\ 2\tau_{11}, 2\tau_{12}, 2\tau_{22}),
\end{aligned}
\tag{22}
$$

*Comptes Rendus (1867), p. 1025 and p. 1080.

†Quarterly Journal of Mathematics, vol. 31 (1899), p. 57.

‡See the historical account by KLEIN, *Ikosaeder*, II, 1, §§ 4–5, p. 148.

§Mathematische Annalen, vol. 30 (1887), p. 496; in particular p. 506, (14).

The roots of $\Phi = 0$ are quartic forms in the z's which I shall denote by Φ_A, $\cdots$, Φ_F [see Maschke, l. c. (7) and (11)], each of which is unaltered by the same group, Γ_{64}, of linear substitutions or by an isomorphic group, Γ_{16}, of collineations. They satisfy the two identities

$$\sum \Phi_A \equiv 0, \qquad \sum \Phi_A \Phi_B \Phi_C \Phi_D = \tfrac{1}{4}(\sum \Phi_A \Phi_B)^2. \tag{23}$$

On the other hand, Bolza* has obtained the values of the invariants of the binary sextic in terms of the zero values of the hyperelliptic thetas, $\vartheta_{ik}(0, 0; \tau_{11}, \tau_{12}, \tau_{22})$, associated with the sextic. The squares of the ten even thetas can be expressed as quadratic functions of the z's and therefrom the invariants can be derived as functions of the forms F_i [Bolza, loc. cit., (30), (32); Maschke, loc. cit., (21) and (23)],

$$A^0 = \frac{\rho^2}{3\cdot 5\cdot 2^7}\frac{F_{12}^2 - F_{24}}{F_8 F_{12} - F_{20}}, \qquad B^0 = \frac{\rho^4}{3^4\cdot 5} F_8, \qquad C^0 = -\frac{\rho^6}{3^6\cdot 5} F_{12},\dagger$$
$$\Delta^0 = \frac{2^8\rho^{10}}{3^4}(F_8 F_{12} - F_{20}), \qquad \rho = \frac{(2\pi i)^2}{\omega_{11}\omega_{22} - \omega_{21}\omega_{12}}. \tag{24}$$

Maschke ‡ has remarked further that by means of a Tschirnhausen transformation, $y = x^2 + \lambda x + \mu$, the general sextic can be transformed into $\Phi = 0$ by solving a quartic equation. Then the known values of F_i determine the absolute invariants formed from A^0, B^0, C^0, Δ^0. If therefore the periods and the thetas be calculated for a normal hyperelliptic integral associated with a binary sextic which has these absolute invariants, values of $\sqrt{\rho} z_i$ in (24) are obtained and these in turn determine the roots Φ_A of $\Phi = 0$. Thus the general sextic is solved by means of hyperelliptic modular functions.

Commenting on this paper, F. Brioschi § notes that the sextic Φ can be written as

$$(y^3 - 3F_8 y + 2F_{12})^2 + 12(F_8 F_{12} - F_{20})y - 4(F_{12}^2 - F_{24}) = 0. \tag{25}$$

If we use Bolza's invariants and set $\xi = \rho^2 y/3$, this becomes

$$(\xi^3 - 5\cdot 3^3 B^0 \xi - 10\cdot 3^3 C^0)^2 + \frac{\Delta^0}{4^3}(\xi - 5\cdot 2^7 A^0) = 0. \tag{26}$$

If

$$\xi = 5\cdot 2^7 A^0 - t^2,$$
$$T = t^6 - 3\cdot 5\cdot 2^7 A^0 t^4 + 3\cdot 5(5\cdot 4^7 A^{0^2} - 9B^0)t^2 + \frac{\sqrt{\Delta}}{8} t - 10(5^2\cdot 4^{10} A^{0^3} - 5\cdot 12^3 A^0 B^0 - 3^3 C^0) = 0. \tag{27}$$

* Mathematische Annalen, vol. 30 (1887), p. 478.

† The A^0, B^0, C^0, Δ^0, here refer to Bolza's A, B^*, C^*, Δ.

‡ Rendiconti della R. Accademia dei Lincei, series 4 (1888), p. 181.

§ Rendiconti della R. Accademia dei Lincei, series 4 (1888), p. 183.

In later papers Brioschi * develops more fully the connection between these sextics and the theta moduli.

The sextic T is in Joubert's normal form. By comparing it with Σ in (12), the relations between the invariants of Bolza and those of § 1 appear. Clearly the solution of T or Σ implies the solution of Maschke's sextic Φ and conversely. We shall find farther on [see (32)]* the rational transformation inverse to (27).

The formulæ given above are illuminated by some geometric considerations allied with the papers of H. W. Richmond.† Denote the quadrics A by $A(y^2)$, their polarized forms by $A(yy')$; the quartics Φ_A by $\Phi_A(z^4)$, their polarized forms by $\Phi_A(z^3z')$. If we polarize the identity (6), $\Sigma A^3(y^2) \equiv 0$, we get the identity, $\Sigma A^2(y^2)A(yy') \equiv 0$. But this expresses that, for a given point y, the quadric, $\Sigma A^2(y^2)A(y'^2) = 0$, has a double point at y. For given y' we have the condition that a quadric on the six points p_i and y' may have a double point at y. Hence

(28) *The equation,* $\Sigma A^2(y^2)A(y'^2) = 0$, *is, for given* y, *a quadric cone on the points* p *with vertex at* y; *for given* y', *the Weddle quartic surface with double points at the points* p *and* y'.

The quartics Φ_A are unaltered by a collineation Γ_{16} determined by a Kummer surface, and every such quartic is a linear combination of the Φ_A's, say $\Sigma\lambda_A\Phi_A = 0$. Since $\Sigma\Phi_A = 0$ we can suppose that $\Sigma\lambda_A = 0$. The condition that $\Sigma\lambda_A\Phi_A = 0$ be a Kummer surface is then easily shown to be $\Sigma\lambda_A^3 = 0$. Hence $\Sigma A(y'^2)\Phi_A = 0$ is for given point y' a Kummer quartic. It is easily verified ‡ that the fundamental sextic of this Kummer quartic and the sextic $y_1', \cdots, y_5', \infty$ are projective; whence

(29) *The Weddle quartic surface,* $\Sigma A(y'^2)A^2(y^2) = 0$, *and the Kummer quartic surface,* $\Sigma A(y'^2)\Phi_A(z^4) = 0$, *are birationally equivalent.*

If on the other hand we polarize the quartic identity (23) we get the new identity

$$\Sigma\Phi_A(z^3z')\Phi_B(z^4)\Phi_C(z^4)\Phi_D(z^4) - \tfrac{1}{2}[\Sigma\Phi_A(z^4)\Phi_B(z^4)][\Sigma\Phi_A(z^3z')\Phi_B(z^4)] \equiv 0.$$

Interchanging z and z' we have the condition that the quartic surface in the variables z,

$$\Sigma\Phi_A(z^4)\Phi_B(z'^4)\Phi_C(z'^4)\Phi_D(z'^4) - \tfrac{1}{2}[\Sigma\Phi_A(z'^4)\Phi_B(z'^4)][\Sigma\Phi_A(z^4)\Phi_B(z'^4)] = 0, \tag{30}$$

may have a double point at z'. Being unaltered by Γ_{16} it must have 16 double points, whence

* Acta Mathematica, vol. 12 (1889), p. 83; Annales de l'Ecole Normale Supérieure, (3), vol. 12 (1895), p. 343.

† Quarterly Journal of Mathematics, vol. 31 (1899), p. 125; vol. 34 (1902), p. 117.

‡ The necessary formulæ are given by HUDSON, *Kummer's Quartic Surface*, pp. 81–2.

(31) *The quartic surface* (30) *is a Kummer quartic surface with double points at the* 16 *points conjugate to* z' *under* Γ_{16}.

The coefficients of $\Phi_A(z^4), \cdots,$ in (30) after being modified so that their sum is zero can be identified with the coefficients of $\Phi_A(z^4)$ in (29). Expressing each coefficient in terms of the symmetric functions of the Φ_A's, i. e., the F_i, and of a particular Φ_A we find with little trouble that

$$A = -\Phi_A^3 + 3F_8\Phi_A - 2F_{12}. \tag{32}$$

This formula furnishes the Tschirnhausen transformation of the sextic Φ into the sextic Σ.

If we interpret the A's as homogeneous point coordinates, and the Φ_A's as space coordinates in S_4, then $\Sigma A^3 = 0$ is the equation of a diagonal cubic surface, K_3, whose reciprocal is a quartic envelope, R_4. The equation of R_4 is precisely the quartic identity (23). Thus we have the particularly simple interpretation of (27) and (32):

(33) *The inverse Tschirnhausen transformations* (27) *and* (32) *from the Joubert to the Maschke normal form* (*and vice versa*) *represent the process of passing from point to tangent space* (*and vice versa*) *of the reciprocal loci* K_3 *and* R_4 *in the space* S_4.

If the Maschke sextic be solved as already indicated, then the roots of Σ are obtained from (32). Consequently also the solution of $PC_{6!}$ in terms of hyperelliptic modular functions of genus two is known. Ultimately then the roots of the given sextic S are expressed in terms of such functions, i. e. in terms of transcendental functions of three variables, the moduli $\tau_{11}, \tau_{12}, \tau_{22}$. This however can hardly be accepted as a final solution. The PG_{360} contains only two independent variables and by the use of it in place of the modular functions an essential simplification is obtained.

§ 3. *The Reduction of* $PC_{6!2}$ *to the* Valentiner *Form-Problem.*

The even permutations which do not alter z_6 give rise to a collineation subgroup C_{60} of $C_{6!2}$. This subgroup is one of a set of six conjugate subgroups, $G_{60}^{(1)}, \cdots, G_{60}^{(6)}$, the root z_i being associated with $G_{60}^{(i)}$. The quadrics $A, \cdots, F$ also are permuted in all possible even ways by $C_{6!2}$. Therefore $C_{6!2}$ contains a second set of six conjugate subgroups, $G_{60}^{(A)}, \cdots, G_{60}^{(F)}$, the quadric A being unaltered by $G_{60}^{(A)}$, etc. The same set of subgroups $G_{60}^{(A)}$ is defined by the functions $A', \cdots, F'$ of the roots z. If then we begin with the quadrics $A, \cdots, F$ and form from them six functions $A'', \cdots, F''$ in the same way that the A''s are formed from z_i, these new functions will be unaltered respectively by a set of conjugate subgroups G_{60} which are different from $G_{60}^{(A)}, \cdots, G_{60}^{(F)}$. The groups $G_{60}^{(A'')}, \cdots, G_{60}^{(F'')}$ must be identical to within their order with the groups $G_{60}^{(1)}, \cdots, G_{60}^{(6)}$, since $C_{6!2}$ is known to contain only two

systems of conjugate icosahedral subgroups. Let us then name these functions $H^{(1)}, \cdots, H^{(6)}$. Each H is of order 3 in the differences of $A, \cdots, F$, of order 6 in the differences of the y's and of maximum order 3 in a particular y. Also since H_6 is unaltered by $G_{60}^{(6)}$ or C_{60} it must be a seminvariant of the quintic with roots y of degree 3 and weight 6. The quintic has only one such covariant—its canonizant of order 3. Thus $H_6 = 0$ must be the condition that ∞ be a canonizant point of $y_1, y_2, \cdots, y_5$. Similarly $H_5 = 0$ is the condition that y_5 be a canonizant point of $y_1, \cdots, y_4, \infty$; etc. The surfaces, $H_i = 0$, are sextic surfaces with triple points at p_i which satisfy relations similar to (5) and (6). Adjusting the numerical factor so that

$$\sum_{i=1}^{6} H_i \equiv 0, \qquad \sum_{i=1}^{6} H_i^3 \equiv 0,$$

we find that

$$H_i = y_i^6 - 3y_i^4 s_2^{(i)} + 2y_i^3 s_3^{(i)} - y_i^2 [(s_2^{(i)})^2 - 70 s_4^{(i)}] + 8 y_i s_2^{(i)} s_3^{(i)}$$

$$+ (s_2^{(i)})^3 + 8(s_3^{(i)})^2 - 20 s_2^{(i)} s_4^{(i)} \quad (i = 1, 2, 3, 4, 5); \tag{36}$$

$$H_6 = -s_2^3 - 10 s_3^2 + 20 s_2 s_4,$$

where

$$s_k = \Sigma y_1 y_2 \cdots y_k, \qquad s_k^{(i)} = s_k - y_i s_{k-1}^{(i)}, \qquad s_0^{(i)} = 1.$$

The Valentiner group of 360 ternary collineations, G_{360}, is isomorphic with $C_{6!/2}$ and has also two sets of conjugate icosahedral subgroups, $G_{60}^{(1)}, \cdots, G_{60}^{(6)}$; and $G_{60}^{(A)}, \cdots, G_{60}^{(F)}$. As invariants under the subgroups we have two sets of six conics, $(k_1 x)^2, \cdots, (k_6 x)^2$; and $(ax)^2, \cdots, (fx)^2$ [See $C2$, § 1]. If each conic be multiplied by its reciprocal, the two sets of six products are permuted by G_{360} without extraneous factors. Hence the forms

$$\sum_{1}^{6} (k_1 x)^2 \cdot (k_1 k_1' u)^2 \cdot H_1$$

and

$$\sum_{a}^{f} (ax)^2 \cdot (aa' u)^2 \cdot A \tag{37}$$

are *Kleinian forms*, i. e., algebraic forms in the three sets of variables x_1, x_2, x_3; u_1, u_2, u_3; $y_1, y_2, \cdots, y_5$, which are merely multiplied by a factor when the variables x and u are contragrediently transformed by the collineations of G_{360} and simultaneously the variables y are transformed by the corresponding substitutions of $C_{6!/2}$. The correspondence between G_{360} and $C_{6!/2}$ is established by associating H_i with $(k_i x)^2$. From these two forms we derive the covariant connex (1, 1)

$$\sum_{1}^{6} \sum_{a}^{f} H_1 \cdot A \cdot (k_1 k_1' a)^2 \cdot (k_1 aa')(k_1 x)(aa' u) \tag{38}$$

which is also a Kleinian form. This connex viewed as a collineation has three fixed points which can be separated by means of an "accessory" cubic equation. One of these fixed points is a so-called "covariant point" by means of which the solution of $PC_{6\,1/2}$ is reduced to PG_{360}.

Simpler algebraically than this direct transition from $PC_{6\,1/2}$ to PG_{360} is the solution of the resolvent Σ in terms of PG_{360}. This is accomplished exactly as outlined for the general sextic in $C\,2$ (in particular § 4), the expressions being considerably shorter owing to the fact that in the case of Σ, $a_1 = a_3 = 0$. For example, the accessory cubic equation [$C\,2$, § 4, (1), (2), (3), (4)] is

$$\sigma^3 + J\sigma - K = 0,$$

where

(39)
$$J = \tfrac{1}{2}[\,15q_2^2q_4 - \tfrac{1}{5}q_2q_6 - 2q_4^2\,],$$

$$K = \tfrac{1}{2}[-45q_2^2q_4^2 + \tfrac{6}{5}q_2q_4q_6 + 8q_4^3 - \tfrac{1}{2}\tfrac{1}{5}q_6^2\,].$$

From the preceding considerations I conclude that *as a matter of practical convenience* it seems advisable to effect the solution of the diagonal sextic resolvent Σ in terms of PG_{360} rather than the given sextic S. As the natural bond between S and Σ, the Cremona $C_{6\,1/2}$ must be introduced.

BALTIMORE, *Oct.* 1, 1910.

ON THE USE OF THE CO-SETS OF A GROUP*

BY

G. A. MILLER

§ 1. *Introduction.*

ABBATI proved in a letter addressed to RUFFINI and dated September 30, 1802, that it is possible to divide all the operators of a group G into sets with respect to an arbitrary subgroup H so that no two of the sets have a common operator. That is, every subgroup of G may be regarded as a modulus with respect to either right or left multiplication. For any subgroup H a set of operators S_2, S_3, $\cdots$, S_ρ may be selected so that this division can be effected in each of the following four ways:†

$$\begin{aligned} G &= H + HS_2 + HS_3 + \cdots + HS_\rho, \\ &= H + S_2H + S_3H + \cdots + S_\rho H, \\ &= H + S_2^{-1}H + S_3^{-1}H + \cdots + S_\rho^{-1}H, \\ &= H + HS_2^{-1} + HS_3^{-1} + \cdots + HS_\rho^{-1}. \end{aligned}$$

The sets HS_α, $S_\alpha H$ $(\alpha = 2, 3, \cdots, \rho)$ have been called Nebengruppen‡ or *co-sets* of G as regards H, the former being called the right co-sets and the latter the left co-sets. When H is added to these co-sets they are called the augmented right co-sets and the augmented left co-sets, respectively.

It has been observed that the operators of each right co-set are evenly distributed among a certain number of the left co-sets, or among all of them, and that the operators of each left co-set are distributed in the same way among the right co-sets. When a right co-set involves an operator which transforms H into itself all of its operators have this property and this right co-set involves exactly the same operators as some left co-set, and vice versa. A necessary and sufficient condition that H gives rise to a multiply transitive substitution group K, when G or one of its quotient groups is represented transitively as regards H in the usual manner,§ is that the operators of each right co-set of G as regards H are distributed among all the left co-sets as regards H, ρ being greater than 2.‖ The same condition is expressed by saying that whenever S

* Presented to the Society (Chicago) December 28, 1910.

† Quarterly Journal of Mathematics, vol. 41 (1910), p. 384.

‡ WEBER, *Lehrbuch der Algebra*, 2d edition, 1899, vol. 2, p. 8.

§ DYCK, Mathematische Annalen, vol. 22 (1883), p. 91.

‖ Proceedings of the American Philosophical Society, vol. 49 (1910), p. 307.

is any operator of G which is not also in H every operator of G is included in the two sets H and HSH, and that ρ exceeds 2.

GALOIS called attention to the importance of the special case when each right co-set of G as regards H is identical with some left co-set. That this case expresses the necessary and sufficient condition that H is invariant under G follows from the more general theorem noted above that a necessary and sufficient condition that H is transformed into itself by the operators of a right co-set is that this co-set is identical with some left co-set, and vice versa. This theorem is, in turn, included in the still more general statement that a necessary and sufficient condition that the operators of a co-set transform H into a group having exactly γ operators in common with H is that this co-set has exactly γ operators in common with some other co-set with respect to H. This statement may again be regarded as a special case of the more general theorem which affirms that the total number of operators in G which transform H into a group having exactly ρ operators in common with some other subgroup K is a multiple of the product of the orders of H and K divided by ρ.*

When H_1 and H_2 are any two groups the product H_1H_2 represents the totality of distinct operators obtained by multiplying on the left each operator of H_2 by each operator of H_1. The number of distinct operators in this totality is evidently the product of the orders of H_1 and H_2 divided by the number of their common operators. A necessary and sufficient condition that this totality constitutes a group is that $H_1H_2 = H_2H_1$. This is a special case of a more general theorem which may be stated as follows: A necessary and sufficient condition that the continued product of the λ groups $H_1, H_2, \cdots, H_\lambda$ is a group is that this product is not affected by the cyclic permutations of its factors. That is, $H_1H_2 \cdots H_\lambda$ is a group whenever

$$H_1H_2 \cdots H_\lambda = H_2 \cdots H_\lambda H_1 = \cdots = H_\lambda H_1 H_2 \cdots,$$

as may easily be verified.

The given method of arranging the operators of G in distinct sets, called augmented co-sets, can easily be generalized by observing that the co-sets are completely determined by H and do not depend upon the choice of the operators $S_2, S_3, \cdots, S_\rho$. Hence it follows directly that the *double co-set* $HS_\alpha H_1$, H_1 being an arbitrary subgroup of G, remains unchanged when S_α is replaced by any other operator of this double co-set. That is, two double co-sets $HS_\alpha H_1$, $HS_\beta H_1$ either have all their operators in common or they have no operator in common.† Each operator of G may therefore be uniquely represented by such double co-sets and HH_1; that is, the group may be represented by a system of augmented double co-sets, as follows:

* Bulletin of the American Mathematical Society, vol. 16 (1910), p. 510.

† FROBENIUS, Journal für reine und angewandte Mathematik, vol. 101 (1887), p. 274.

$$G = HH_1 + HS_2H_1 + \cdots + HS_\lambda H_1.$$

When $H_1 = H$ and G or one of its quotient groups is represented as a transitive substitution group with respect to H in the ordinary way, the number of transitive constituents in the subgroup composed of all the substitutions which omit one letter of this substitution group is λ, if the omitted letter or letters are counted as constituents; and the numbers of distinct operators in these various double co-sets are equal to the products of the order of H and the degrees of the various transitive constituents of this subgroup.* When $H_1 = 1$ these double co-sets reduce to right co-sets and when $H = 1$ they reduce to left co-sets.

The general properties of double co-sets were first studied by FROBENIUS in an article entitled *Ueber die Congruenz nach einem aus zwei endlichen Gruppen gebildeten Doppelmodul*, Journal für die reine und angewandte Mathematik, volume 101 (1887), page 273. Since then the concept of equivalence of operators as regards a double modulus has been employed by various writers. The object of the present article is to extend the abstract theory as regards co-sets with a view to a greater usefulness of this concept. This article is practically a continuation of the one entitled, *Some relations between substitution group properties and abstract groups*, Proceedings of the American Philosophical Society, volume 49 (1910), page 307. The theorem of § 3 which exhibits a dual relation between the degrees of the transitive constituents of various subgroups when the group is represented with respect to these subgroups is perhaps the most useful result of the present paper.

§ 2. *Double co-sets with respect to a single subgroup.*

The double co-set $HS_\alpha H$ involves all the right co-sets which are conjugate under H with HS_α. It also includes all the conjugates under H of the left co-set $S_\alpha H$. In fact, each of these two complete sets of conjugates involves exactly the same operators, and is identical with $HS_\alpha H$. The number of distinct conjugates in each of these two sets may be obtained by dividing the order of H by the number of operators common to H and $S_\alpha^{-1}HS_\alpha$ since this is also the number of operators common to the two co-sets HS_α and $S_\alpha H$, and hence it represents the number of left co-sets among which all the operators of HS_α are evenly distributed as well as the number of right co-sets among which the operators of $S_\alpha H$ are evenly distributed. We shall prove in the next paragraph that this same number is also the degree of a transitive constituent of the subgroup composed of all the substitutions omitting one letter in the transitive substitution group K which corresponds to the permutations of the rows when all the operators of G are arranged in the ordinary rectangular form and the operators of H constitute the first row.†

* Proceedings of the American Philosophical Society, vol. 49 (1910), p. 311.

† DYCK, Mathematische Annalen, vol. 22 (1883), p. 91.

To prove this theorem we shall at first assume that K is simply isomorphic with G. This is equivalent to assuming that H contains no invariant subgroup of G besides the identity, or that K represents G. Let K_1 represent the subgroup of K which corresponds to H in this simple isomorphism. Hence K_1 is composed of all the substitutions of K which omit a given letter (a) and HS_a corresponds to all the substitutions of K which replace a by a given other letter. The conjugates of HS_a are then all the operators of G which correspond to those substitutions of K which replace a by the different letters in a transitive constituent of K_1. That is, *if the operators of G are arranged into double co-sets with respect to a single subgroup in the following manner:*

$$G = H + HS_2H + HS_3H + \cdots + HS_\lambda H,$$

and if H does not involve any invariant subgroup of G besides the identity, so that G may be represented as a transitive substitution group K with respect to H, then the operators in each of these double co-sets correspond to all the substitutions of K which replace the letter omitted in the subgroup corresponding to H by the letters in a transitive constituent of this subgroup.

When H includes an invariant subgroup of G or is itself invariant under G, it gives rise to the transitive representation of a quotient group of G and the preceding theorem applies to this quotient group. Hence λ represents the number of transitive constituents (each omitted letter being also regarded as a transitive constituent) in the subgroup composed of all the substitutions of this quotient group which omit a given letter, and we may establish a $(1, 1)$ correspondence between the double co-sets and the transitive constituents of this subgroup in such a way that the number of distinct operators in each double co-set is equal to the product of the order of H into the degree of the corresponding transitive constituent. In particular when H is invariant this quotient group is regular and the number of these double co-sets is equal to the degree of this quotient group, as each of the transitive constituents of the given subgroup is of degree one in this special case. Hence we have as a corollary of the given theorem the known theorem that necessary and sufficient conditions that H gives rise to a multiply transitive group are that H contains less than half the operators of G and that $\lambda = 2$.

A necessary and sufficient condition that the transitive substitution group K to which H gives rise is primitive is that G is generated by each double co-set as regards H, since this condition is equivalent to the condition that H is a maximal subgroup. A necessary and sufficient condition that the subgroup of K composed of all its substitutions which omit one letter omits exactly α letters is that there are $\alpha - 1$ double co-sets as to H each involving the same number of distinct operators as H does. This results directly from the more general theorem that all the operators of such a double co-set transform H into

groups all having the same number of operators in common with H and that the number of the distinct operators in this co-set is equal to the square of the order of H divided by the number of these common operators. This is, in turn, a special case of a theorem which will be proved in the following section.

§ 3. *Double co-sets with respect to two distinct subgroups.*

A special case of a double co-set with respect to two distinct subgroups was developed by Cauchy to prove the fundamental theorem that every group whose order is divisible by a given prime number involves a subgroup whose order is this prime.* This theorem was announced by Galois, who however did not give any proof in his published papers. Frobenius extended, in the article cited at the close of § 1, the development of the theory of double co-sets and employed this theory to re-establish Sylow's theorem as well as to prove other fundamental related theorems. Recently the present writer pointed out that Sylow's theorem may be proved directly by a slight extension of Cauchy's theorem relating to double co-sets and then following the steps which Cauchy took to establish the special but fundamental case noted above.† These facts may suffice to indicate that the concept of double co-sets relates to fundamental matters in group theory and seems to deserve more attention than it has recently received.

If H_1 and H_2 are any two subgroups of G, all the operators of G may be represented uniquely in either of the following two forms in accord with the results stated above:

$$G = H_1 H_2 + H_1 S_2 H_2 + \cdots + H_1 S_\lambda H_2,$$
$$= H_2 H_1 + H_2 S_2^{-1} H_1 + \cdots + H_2 S_\lambda^{-1} H_1.$$

The second augmented system of double co-sets is evidently composed of the inverses of the operators in first system. To obtain a concrete illustration of some of the properties of these augmented double co-sets it may be observed that when G can be represented as a transitive substitution group K with respect to H_1, the substitutions corresponding to $H_1 H_2$ will be composed of all those of K which replace the letter a omitted in all the substitutions of the subgroup K_1 corresponding to H_1 by the letters of the transitive constituent to which a belongs in the subgroup K_2 corresponding to H_2. In a similar manner we may observe that all the operators of the double co-sets $H_1 S_\alpha H_2$ ($\alpha = 2, 3, \cdots, \lambda$) correspond to all the substitutions which replace the letter by which a is followed in S_α by all the letters of a transitive constituent in K_2. The number of the distinct operators in the double co-set $H_1 S_\alpha H_2$ is therefore equal to the order of

* CAUCHY, *Oeuvres complètes*, 1st series, vol. 9 (1896), p. 358.

† Bulletin of the American Mathematical Society, vol. 16 (1910), p. 510.

H_1 multiplied by the degree of the corresponding transitive constituent of K_2. In particular, this number is the order of H_1 whenever the letter which follows a in S_a does not occur in K_2. As a special case we have also the theorem that a necessary and sufficient condition that $H_1 H_2 = G$ is that K_2 be transitive.

From the preceding results we may deduce a general theorem as regards the transitive substitution groups which are simply isomorphic with G. This theorem may be stated as follows: *If G is represented as a transitive substitution group K and if K_1 is the subgroup composed of all the substitutions of K which omit a given letter, then any subgroup K' of K has the same number of transitive constituents as the subgroup which corresponds to K_1 has when K is represented as a transitive group with respect to K'. Moreover, the transitive constituents in the two given subgroups have the same relative degrees in the two different representations of G.*

The significance of this theorem may be inferred from the following illustrations. When the icosahedral group is represented as the alternating group of degree 5, K_1 is the alternating group of degree 4 and all the subgroups of order 12 in K are conjugate. If we let K' represent one of the subgroups of order 4 in K it is evident that K' has two transitive constituents of degrees 4 and 1 respectively. Hence when K is represented as a transitive group of degree 15 with respect to K' its subgroups of order 12 will have two transitive constituents of degrees 12 and 3 respectively. On the other hand, if we let K' represent a subgroup of order 6 in K, it is clear that it has two transitive constituents of degrees 3 and 2 respectively. Hence when K is represented as a transitive group of degree 10 its subgroups of order 12 will have two transitive constituents of degrees 6 and 4 respectively. Finally, if we let K' be a subgroup whose order is either 5 or 10, it is transitive, and hence the subgroups of order 12 in the icosahedral group are transitive when this group is represented transitively on either 6 or 12 letters.

It should be observed that the given theorem remains true even when K does not represent G but represents merely a quotient group of G; that is when H_1 either involves an invariant subgroup of G or is itself invariant under G. In these cases it is evidently necessary to consider the groups which correspond to the subgroups K_1 and K' in the transitive representation instead of these subgroups themselves. The given development exhibits not only an interesting dual relation between the forms of the various subgroups when a group is represented transitively in the various possible ways, but it also exhibits a dual relation between certain abstract group properties and substitution groups, since not only does λ express the number of transitive constituents in the group corresponding to H_2 when G is represented transitively as regards H_1, but the number of distinct operators in each of the double co-sets is equal to the product of the order of H_1 and the degree of the corresponding transitive constituent.

The number of distinct operators in each one of the augmented double co-sets is a multiple of the orders of the two subgroups H_1, H_2. That is, *if K' is any subgroup of any transitive group whatever of order g and of degree n, and if n_1 is the degree of a transitive constituent of K', then the order of K' is a divisor of $n_1 g/n$.* This limits the orders of the intransitive subgroups of any transitive substitution group and it reduces to Lagrange's theorem when the subgroups are transitive. It may therefore be regarded as a generalization of Lagrange's theorem. It should be observed that this theorem may also be proved directly by employing the fact that the order of the subgroup composed of all the substitutions of K' which omit a given letter is a divisor of g/n.

To illustrate the usefulness of the theorem in the last paragraph we may again make use of transitive representations of the icosahedral group. When this group is represented as the alternating group of degree 5, K' can have a transitive constituent of degree 2 only when the order of K' is a divisor of 24, and it can have a transitive constituent of degree 3 only when this order is a divisor of 36, in accord with the given theorem. The only non-cyclic subgroups which have constituents of degree 2 or 3 are those of order 6, and as 6 is a divisor of both the numbers 24 and 36 this agrees with the theorem. On the other hand, when the icosahedral group is represented as a transitive group of degree 6, K' can have a transitive constituent of degree 2 only when the order of K' is a divisor of 20. As a subgroup whose order is divisible by 5 in a transitive group of degree 6 could clearly not have a transitive constituent of degree 2, it results that the order of K' is a divisor of 4 whenever it has a transitive constituent of degree 2. In fact, it is evident that the subgroups of orders 2 and 4 are the only ones which have transitive constituents of degree 2 in the present case. Similarly we observe that the order of K' is a divisor of 6 whenever it has a transitive constituent of degree 3, and that in the present case the subgroups of order 6 have two constituents of this degree. These illustrations may suffice to exhibit how the theorem under consideration may be employed to gain an insight into the structure of transitive substitution groups.

It is known that a necessary and sufficient condition that the product $H_1 H_2$ constitutes a group is that it is equal to its inverse $H_2 H_1$. We proceed to inquire what is implied by the condition that each double co-set is identical with the inverse of a double co-set as regards the same subgroups. That is, we suppose that the two series of augmented double co-sets

$$\begin{aligned} G &= H_1 H_2 + H_1 S_2 H_2 + \cdots + H_1 S_\lambda H_2 \\ &= H_2 H_1 + H_2 S_2^{-1} H_1 + \cdots + H_2 S_\lambda^{-1} H_1 \end{aligned}$$

are identical if the order is not considered. Since each of the two products $H_1 H_2$ and $H_2 H_1$ involves the identity it results that these must be identical when the given condition is satisfied, and hence a first result is that $H_1 H_2$ must be a sub-

group of G. The assumption that $H_1 S_\alpha H_2 = H_2 S_\alpha H_1$ clearly implies that $H_1 H_2$ is invariant under S_α. That is, *whenever a double co-set as regards two permutable subgroups coincides with the inverse of a double co-set as regards the same two subgroups then the product of these subgroups is invariant under each of these double co-sets.* In particular, a necessary and sufficient condition that $H_1 H_2$ is an invariant subgroup under G is that each double co-set as regards H_1, H_2 is identical with the inverse of some double co-set as regards these subgroups. If either H_1 or H_2 is assumed to be the identity we obtain as a special case the theorem noted in the third paragraph of the Introduction.

When G or a quotient group of G is represented as a substitution group as regards H_1, the preceding results may also be employed to establish a useful criterion to determine when $H_1 H_2$ constitutes a subgroup of G. In fact, since $H_1 H_2$ is composed of all the operators of G corresponding to the substitutions which replace the letter (a) omitted in the subgroup corresponding to H_1 in this substitution group by the letters of a transitive constituent which involves a in the subgroup corresponding to H_2, it results that *a necessary and sufficient condition that $H_1 H_2$ is a group is that the transitive constituent which involves a in the subgroup corresponding to H_2 involves all the letters of one or more transitive constituents in the group corresponding to H_1 when G, or one of its quotient groups, is represented transitively as regards H_1 and a is the letter omitted in the group corresponding to H_1.*

The preceding theorem is a special case of another theorem which we proceed to establish. Since $H_1 H_2$ is composed of all the operators of G which correspond to the substitutions, in the transitive group to which H_1 gives rise, which replace the letter a omitted in all the substitutions corresponding to H_1 by all the letters of the transitive constituent involving a in the subgroup corresponding to H_2, the number of operators common to $H_1 H_2$ and $H_2 H_1$ can be directly determined from this substitution group. In fact, a necessary and sufficient condition that $H_1 H_2$ and $H_2 H_1$ have in common all the operators which correspond to the total number of substitutions which replace a by the letter c is that all the letters in the constituent to which c belongs in the subgroup corresponding to H_1 are found in the transitive constituent to which a and c belong in the subgroup corresponding to H_2. In particular, if all the letters in the latter constituent constitute the letters of two or more constituents in the group corresponding to H_1 it results that $H_1 H_2$ and $H_2 H_1$ are identical, as was proved in the preceding paragraph. This proves also that $H_1 H_2$ *cannot involve the inverses of all the operators of a double co-set with respect to* H_1, H_2.

The co-sets which contain the inverse of at least one of the operators in $H_1 H_2$ can readily be determined when G or a quotient group of G is represented with respect to H_1. In fact, if a letter in the transitive constituent involving a in the subgroup corresponding to H_2 is replaced, in the subgroup

corresponding to H_1, by a letter of another transitive constituent of the former of these two subgroups the double co-set corresponding to the latter constituent involves the inverse of an operator of $H_1 H_2$ and vice versa. To determine the double co-sets which involve the inverse of at least one of the operators of $H_1 H_2$ it is therefore only necessary to observe which of the transitive constituents of the subgroup corresponding to H_2 involves at least one letter that succeeds in the subgroup corresponding to H_1 a letter of the constituent involving a in the former subgroup. When a set of constituents, including the one involving a, of this subgroup involves all the letters of a set of constituents of the group corresponding to H_1, the letters of these constituents clearly represent a system of imprimitivity of the substitution group with respect to H_1. In particular this group is always imprimitive when $H_1 H_2 \equiv H_2 H_1 < G$.

When G is written as the sum of the subgroup H and a series of corresponding co-sets, as follows:

$$G = H + HS_2 + HS_3 + \cdots + HS_\rho ,$$

the series of operators $S_2, S_3, \cdots, S_\rho$ may be so selected that the same totality of co-sets is obtained by replacing each of these operators by its inverse, as was observed in the opening paragraph. This can evidently also be done as regards double co-sets with respect to a single subgroup, but it cannot always be done as regards double co-sets with respect to two distinct subgroups. The truth of this statement may be verified if we let G be the transitive substitution group of order 32 and of degree 8 generated by the substitution $aebf \cdot cgdh$ and the two subgroups

$$H_1 = 1,\ eg \cdot fh,\ bd \cdot eg,\ bd \cdot fh, \qquad H_2 = 1,\ ac \cdot eg .$$

Hence it results that G must have special properties when it is possible to select the operators $S_2, \cdots, S_\lambda$ so that both of the following arrangements are possible when every operator of G is uniquely represented by the former:

$$\begin{aligned} G &= H_1 H_2 + H_1 S_2 H_2 + \cdots + H_1 S_\lambda H_2 , \\ &= H_1 H_2 + H_1 S_2^{-1} H_2 + \cdots + H_1 S_\lambda^{-1} H_2 . \end{aligned}$$

From the theorem that $H_1 H_2$ cannot involve the inverses of all the operators of a double co-set with respect to $H_1 H_2$ it results that G may always be represented in these two ways if $S_2, \cdots, S_\lambda$ are properly chosen and $\lambda < 4$. It is also clear that this is always possible when H_1 gives rise to a doubly transitive substitution group, since all the double co-sets involve operators of order 2 in this special case.

THE SOUTHERLY DEVIATION OF FALLING BODIES*

BY

WILLIAM H. ROEVER

Introduction

In the work of previous writers on the subject of this paper the potential function used for that of the earth's gravitational field of force has, in general, not been more than a first approximation, i. e., a development, in the neighborhood of the initial point of the falling body, which includes terms of only the first order in the independent variables. The circle of reference (parallel of latitude) used has also, in general, been different from that used in experiments for the determination of the same quantity.

From equations (1) and (2) which follow it appears that the southerly deviation is proportional to the square of the height h through which the body falls (at least for sufficiently small values of h) and that the constant of proportionality involves the first *and also the second derivatives of the potential function* f_1.† Hence the truth of the following statement:

I. The potential function to be used for that of the earth's gravitational field of force should be *a second approximation*, i. e., a development in the neighborhood of the initial point of the falling body, which includes terms of at least the second order in the independent variables.

In experiments for the determination of the southerly deviation of falling bodies, a plumb-line P_0R is supported at the point P_0 (Fig. 5), from which the body falls, and the deviation is measured on the level (equipotential) surface which passes through the plumb-bob R, from the circle of latitude which passes through R. The direction of gravity at R is RP_0, and that at P_0 is the limiting direction which is approached by the line RP_0 as the length P_0R approaches zero. Let us denote by t the limiting position of P_0R. It is easy to prove the following statement:

II. In a field of force in which the lines of force are not rectilinear, plumb-

* Presented to the Society (Southwestern Section), November 27, 1909, and November 26, 1910. See also footnote to formula (3).

† In this paper, as in the work of previous writers, a distribution of revolution is assumed, i. e., the earth's field of force is assumed to be the same in every plane through the axis of rotation.

lines* of different lengths which are supported at the same point (in particular a plumb-line P_0R and the line t) do not coincide.

In order to be able to compare the results of experiment with those of theory, the same circle of reference should be used in the computation as in the experiment. This, however, has not been done by previous writers. In their computations they used as their circle of reference on the level surface of R, not the circle of latitude of R, but that of the point T in which the line t pierces the level surface of R. That a great error has thus been made, appears in Assumption 4, below, in which it is shown that for a second approximation to the potential function the distance (measured along a meridian) between the circles of latitude of R and T (the part which has been neglected) is eighteen times as great as that between T and S (the part which has been computed), S being the point in which the body, which falls from P_0, strikes the level surface of R. For a first approximation to the potential function, however, the distance between the circles of latitude of R and T may be small in comparison to that between T and S.

Now let us compare the assumptions and the results of previous writers, who disregarded either one or both of the facts set forth in I and II, with those of this paper, in which these facts are not disregarded. In order to do this it is desirable first to give a brief description of the method of this paper.

Statement and Comparison of Results.

After the falling body is released from its initial position, P_0, of rest with respect to the rotating earth, it moves under the influence of a field of force F_1 which is fixed in space.† A plumb-line P_0R at rest is in equilibrium under the influence of a field of force F_2 which is fixed with respect to the rotating earth. If $U_1 = f_1(r, z)$ and $U_2 = f_2(r, z)$ (where r represents the distance of a general point from the earth's axis of rotation OZ, and z that from a plane π perpendicular to OZ at O, and where O is taken at the earth's center, although this is not necessary) are potential functions of the fields F_1 and F_2 respectively, there exists between them the relation

$$U_1 + \tfrac{1}{2}\omega^2 r^2 = U_2,$$

in which ω denotes the angular velocity of the earth's rotation.

The path of the falling body, which has received an initial velocity from the rotating earth, is a curve c in the fixed field F_1. The curve c lies on a surface of revolution whose axis coincides with that of the earth. Let us denote by c_1

* It is assumed that the plumb-bob is a heavy particle and that the line is weightless and perfectly flexible.

† That is, fixed with respect to a set of axes whose origin is the center of the earth and whose directions are fixed directions in space.

that meridian curve of this surface of revolution which lies in the plane of OZ and P_0. The locus of the plumb-bobs R of all plumb-lines which are supported at P_0, is a curve c_2 which also lies in the plane of OZ and P_0. It will be shown later that the curves c_1 and c_2 are tangent at P_0. The line t defined above is their common tangent.

Let us now take the common tangent and the common normal to these curves as a pair of rectangular axes and refer the equations of curves c_1 and c_2 to them. The coördinate which is measured along the tangent from the point P_0 we will denote by ξ and regard as positive when measured in the direction of the earth's center. The other coördinate we will denote by η and regard as positive when measured in the direction of the north pole. (See Fig. 4.) The equations of c_1 and c_2 then become

$$\eta_1 = -\frac{1}{6}\frac{C + B_1\omega^2 + A_1\omega^4}{D^{\frac{3}{2}}}\xi^2 + \cdots, \tag{1}$$

$$\eta_2 = -\frac{C + B_2\omega^2 + A_2\omega^4}{D^{\frac{3}{2}}}\xi^2 + \cdots, \tag{2}$$

where

$$C = \left(\frac{\partial^2 f_1}{\partial r \partial z}\right)_0\left[\left(\frac{\partial f_1}{\partial r}\right)_0^2 - \left(\frac{\partial f_1}{\partial z}\right)_0^2\right] + \left[\left(\frac{\partial^2 f_1}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f_1}{\partial r^2}\right)_0\right]\left(\frac{\partial f_1}{\partial r}\right)_0\left(\frac{\partial f_1}{\partial z}\right)_0,$$

$$B = 2\left(\frac{\partial^2 f_1}{\partial r \partial z}\right)_0\left(\frac{\partial f_1}{\partial r}\right)_0 r_0 + \left[\left(\frac{\partial^2 f_1}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f_1}{\partial r^2}\right)_0\right]\left(\frac{\partial f_1}{\partial z}\right)_0 r_0,$$

$$A = \left(\frac{\partial^2 f_1}{\partial r \partial z}\right)_0 r_0^2,$$

$$B_1 = B + 3\left(\frac{\partial f_1}{\partial r}\right)_0\left(\frac{\partial f_1}{\partial z}\right)_0, \qquad B_2 = B - \left(\frac{\partial f_1}{\partial r}\right)_0\left(\frac{\partial f_1}{\partial z}\right)_0,$$

$$A_1 = A + 3\left(\frac{\partial f_1}{\partial z}\right)_0 r_0, \qquad A_2 = A - \left(\frac{\partial f_1}{\partial z}\right)_0 r_0,$$

$$D = \left(\frac{\partial f_1}{\partial r}\right)_0^2 + \left(\frac{\partial f_1}{\partial z}\right)_0^2 + 2\left(\frac{\partial f_1}{\partial r}\right)_0 r_0\omega^2 + r_0^2\omega^4,$$

the subscript $(_0)$ indicating that the derivatives of f_1 have been computed for the point $P_0(r = r_0, z = z_0)$.

The southerly deviation is the difference $\eta_2 - \eta_1$, when we substitute for ξ the height $h = P_0R$ through which the body falls, and for f_1 a sufficiently close approximation to the potential function of the earth's gravitational field of force.

The southerly deviation can also be expressed in the following very simple form *

* This formula was presented to the Society (Chicago Section) April 29, 1911.

$$\frac{4\omega^2 \sin\phi\cos\phi + 5\left(\frac{\partial g}{\partial \eta}\right)_0}{6g_0} h^2, \tag{3}$$

in which g_0 and ϕ denote the acceleration and the astronomical latitude, respectively, at the point P_0. For,

$$g_0 = \sqrt{\left(\frac{\partial f_2}{\partial z}\right)_0^2 + \left(\frac{\partial f_2}{\partial r}\right)_0^2} = \sqrt{D}, \qquad g_0 \sin\phi = \left(\frac{\partial f_2}{\partial z}\right)_0 = \left(\frac{\partial f_1}{\partial z}\right)_0,$$

$$g_0 \cos\phi = \left(\frac{\partial f_2}{\partial r}\right)_0 = \left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0.$$

By the equation of transformation in § 7,

$$U_1 = f_1(r, z) = f_1(-\xi\cos\phi - \eta\sin\phi + r_0, \ -\xi\sin\phi + \eta\cos\phi + z_0),$$

and therefore

$$\frac{\partial^2 U_1}{\partial\eta\,\partial\xi} = (\sin^2\phi - \cos^2\phi)\frac{\partial^2 f_1}{\partial r\,\partial z} + \sin\phi\cos\phi\left[\frac{\partial^2 f_1}{\partial r^2} - \frac{\partial^2 f_1}{\partial z^2}\right].$$

On the other hand,

$$\frac{\partial^2 U_1}{\partial\eta\,\partial\xi} = \frac{\partial^2 U_2}{\partial\eta\,\partial\xi} - \omega^2\sin\phi\cos\phi.$$

Hence equations (1) and (2) assume the following forms:

$$\eta_1 = \frac{1}{6g_0}\left[\left(\frac{\partial^2 U_2}{\partial\eta\,\partial\xi}\right)_0 - 4\omega^2\sin\phi\cos\phi\right]\xi^2, \qquad \eta_2 = \frac{1}{g_0}\left(\frac{\partial^2 U_2}{\partial\eta\,\partial\xi}\right)_0 \xi^2,$$

and

$$\eta_2 - \eta_1 = \frac{1}{6g_0}\left[4\omega^2\sin\phi\cos\phi + 5\left(\frac{\partial^2 U_2}{\partial\eta\,\partial\xi}\right)_0\right]\xi^2.$$

But

$$\left(\frac{\partial^2 U_2}{\partial\eta\,\partial\xi}\right)_0 = \left[\frac{\partial}{\partial\eta}\left(\frac{\partial U_2}{\partial\xi}\right)\right]_0 = \left(\frac{\partial g}{\partial\eta}\right)_0,$$

and hence formula (3).

It will be observed that the coefficient of ξ^2 in each of the equations (1) and (2) contains the second as well as the first derivatives of f_1. It is for this reason that the potential function must be developed to terms of the second order inclusive in the neighborhood of $P_0(r_0, z_0)$.

In the discussion which follows we shall call η_1 and η_2 the parts of the southerly deviation which are contributed by the trajectory and the plumb-bob locus respectively. It is the part η_2 which has been neglected, and the part η_1 which has been computed, by previous writers. We shall compare these for each of several forms of f_1, we shall also express the results corresponding to different forms of f_1 in terms of the same unit.

1. *The Assumption of Gauss.*

Gauss assumed that the force to which a body, at rest with respect to the rotating earth, is subjected is constant in magnitude and direction. That is,*

$$f_2(r, z) = -g\cos\phi \cdot r - g\sin\phi \cdot z,$$

and therefore

$$f_1(r, z) = f_2(r, z) - \tfrac{1}{2}\omega^2 r^2,$$

in which ϕ and g are the astronomical latitude and the acceleration at P_0. For this potential function equations (1) and (2) become †

$$\eta_1 = -\tfrac{2}{3}\sin\phi\cos\phi \cdot \frac{\omega^2}{g} \cdot \xi^2, \qquad \eta_2 = 0 \cdot \xi^2,$$

and

$$\eta_2 - \eta_1 = -\eta_1 = \tfrac{2}{3}\sin\phi\cos\phi \cdot \frac{\omega^2}{g} \cdot \xi^2.$$

Thus we see that in this case the *whole southerly deviation is due to the trajectory,* since the plumb-bob locus coincides with the tangent t. (See Fig. 1.) The curve c_1 lies to the south of the tangent t.

The potential function used by Gauss is a first approximation to that of the earth's field. By statement I this is not a sufficiently close approximation.

Before making the next assumption let us express the result just obtained in a form in which it may be compared with the results which follow.

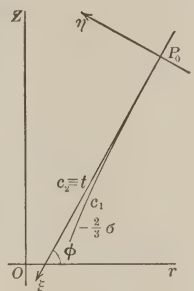

FIG. 1.

With sufficient approximation

$$\sin\phi = \frac{z_0}{\rho_0}, \qquad \cos\phi = \frac{r_0}{\rho_0}, \qquad \frac{\omega^2}{g} = \frac{\sigma}{\rho_0},$$

where ‡

$$\sigma = \frac{\omega^2\rho_0}{M/\rho_0^2}, \qquad \rho_0 = \sqrt{r_0^2 + z_0^2},$$

and M represents the mass of the earth. Then

*Strictly speaking this function does not quite represent Gauss' assumption. Under Gauss' assumption the equipotential surfaces of the field F_2 are parallel planes, while the equipotential surfaces corresponding to the function f_2 here assumed are parallel cones of revolution which have as a common axis that of the earth.

†The formula given by GAUSS is

$$\tfrac{1}{6}\sin\phi\cos\phi\, g\omega^2 t^4.$$

But since

$$\xi = \tfrac{1}{2}gt^2 + \cdots,$$

we see that the result is the same as the one just found. See GAUSS' *Werke*, vol. 5 (1867), p. 502. GAUSS' formula can also be obtained by putting $(\partial g/\partial\eta)_0 = 0$ in formula (3).

‡The value of σ is about $\frac{1}{289}$.

$$\eta_2 - \eta_1 = -\eta_1 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0} \cdot \tfrac{2}{3}\sigma \cdot \xi^2.$$

2. *A Second Assumption.*

Let us now assume that the field of force F_1' is central, the center being at the earth's center and the law of attraction that of Newton. Then *

$$f_1(r, z) = \frac{M}{\rho}, \qquad \rho = \sqrt{r^2 + z^2},$$

where M represents the mass of the earth. For this potential function equations (1) and (2) become †

$$\eta_1 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0}\left[0 + 0 + \frac{1}{2}\frac{z_0^2}{\rho_0^2}\sigma^2 + \cdots \right]\xi^2,$$

$$\eta_2 = \frac{r_0 z_0}{\rho_0} \cdot \frac{1}{\rho_0}\left[0 + 4\sigma + \frac{8r_0^2 - z_0^2}{\rho_0^2}\sigma^2 + \cdots \right]\xi^2,$$

and

$$\eta_2 - \eta_1 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0}\left[0 + 4\sigma + \frac{16r_0^2 - 3z_0^2}{2\rho_0^2}\sigma^2 + \cdots \right]\xi^2 + \cdots.$$

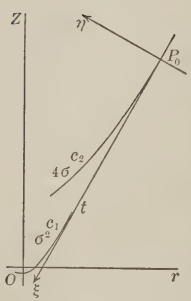

FIG. 2.

In this case *practically the whole southerly deviation is due to the plumb-bob locus*, since in the equation of the curve c_1 the coefficient of ξ^2 is an infinitesimal of second order with respect to the coefficient of ξ^2 in the equation of c_2. (See Fig. 2.) Both of the curves c_1 and c_2 lie to the north of the tangent t.

Since the curve c_1 lies to the north of t, it follows that those computers who neglect the part η_2 are forced to say that, for this potential function, the falling body has a *northerly*, and not a southerly, deviation.

3. *A Third Assumption.*

Let us assume, as above, that the field of force F_1 is central, the center being at the earth's center. Here we will assume, not the law of Newton, but that the

* For this potential function the level (equipotential) surface of the field F_2 which passes through the plumb-bob R, is an ellipsoid of oblateness $\frac{1}{2}\sigma$. [Put $\varepsilon = 0$ in the assumption 4.]

† In this case the curve c_1 is a conic of which $O\,(r=0,\ z=0)$ is a focus. This follows from the fact that the curve c is an ellipse with focus at O.

magnitude of the force varies directly as the distance from the center. Then*

$$f_1(r, z) = -\frac{K}{2}\rho^2, \qquad \rho^2 = r^2 + z^2,$$

where K is a positive constant. For this potental function equations (1) and (2) become†

$$\eta_1 = -\tfrac{1}{2}\frac{r_0 z_0}{\rho_0^2}\cdot\frac{1}{\rho_0}\left[0 + \frac{\omega^2}{K} + \frac{2r_0^2 - z_0^2}{\rho_0^2}\frac{\omega^4}{K^2}\cdots\right]\xi^2 + \cdots,$$

$$\eta_2 = -2\eta_1 \qquad \text{and} \qquad \eta_2 - \eta_1 = -3\eta_1.$$

In this case c_1 falls to the south, and c_2 to the north, of t. Hence the trajectory contributes $\frac{1}{3}$, and the plumb-bob locus $\frac{2}{3}$, of the southerly deviation. (See Fig. 3.)

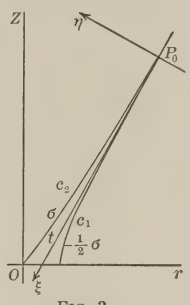

FIG. 3.

If, at the initial point, the forces due to assumptions 2 and 3 be the same, $K\rho_0 = M/\rho_0^2$ and therefore $\omega^2/K = \sigma$.

The General Differential Equation of Gauss.

Before making the supposition stated in Assumption 1 above, Gauss found the differential equations of motion (relative to a set of axes fixed with respect to the rotating earth) of a falling body, on the supposition that the field of force F_1 is central,‡ the center being the point in which the earth's axis is pierced by the vector which represents the force of the field F_1 at the initial point of the falling body. Gauss does not solve these differential equations, in which he denotes by p the magnitude of the force of the field F_1 and by M a constant due to air resistance. If we put this M equal to zero (i. e., neglect air resistance) and assume that p varies

* For this potential function all of the level (equipotential) surfaces of the field F_1 are ellipsoids of oblateness $\frac{1}{2}\sigma$. Their equations are

$$K(r^2 + z^2) - \omega^2 r^2 = C.$$

Their oblateness is

$$1 - \frac{\rho_p}{\rho_e} = 1 - \frac{\sqrt{C}}{\sqrt{K}} = 1 - \sqrt{1 - \frac{\omega^2}{K}} = \tfrac{1}{2}\frac{\omega^2}{K} = \tfrac{1}{2}\sigma.$$

† In this case the curve c_1 is an hyperbola whose center is at O and whose foci are on the r-axis; the curve c_2 is an equilateral hyperbola which passes through O and whose asymptotes are parallel to the axes of r and z. The truth of the first part of this statement follows from the fact that the curve c is an ellipse whose center is at O. The curve c_2 is of the form mentioned since it is the locus of the feet of perpendiculars dropped from a point to a family of similar and concentric ellipses.

‡ At least if q is constant in his equations [3].

inversely as the square of the distance (or directly as the distance) from the center ($X=0$, $Y=0$, $Z=q$) of Gauss' field F_1', we obtain a set of differential equations, the solution of which should yield for the x of Gauss the same value as that obtained by replacing z_0 by $z_0+\alpha$ in the expression given for η_1 under Assumption 2 (or 3), α being the distance of the center of Gauss' field F_1' to the south from the earth's center O.

4. *A Fourth Assumption.*

Now let us assume that the potential function f_1 has the form *:

$$f_1(r,z)=\frac{M}{\rho}+\frac{\epsilon}{3}M\rho_1^2\frac{r^2-2z^2}{\rho^5},$$

in which ρ and M have the same meaning as in assumption 2. The symbol ρ_1 represents the mean radius of the Standard Spheroid, and

$$\epsilon=e_1-\tfrac{1}{2}\sigma_1,$$

where e_1 represents the ellipticity (or oblateness) of the Standard Spheroid, and

$$\sigma_1=\frac{\omega^2\rho_1}{M/\rho_1^2}.$$

The values of the constants are †

$$\sigma_1=\frac{1}{288.38}=.003468 \qquad \text{and} \qquad e_1=\frac{1}{293.5}=.003407,$$
$$\epsilon=.00167.$$

For this form of f_1, equations (1) and (2) become

$$\eta_1=-\frac{1}{6}\frac{r_0z_0}{\rho_0^2}\cdot\frac{1}{\rho_0}\left[2\left(\frac{\rho_1}{\rho_0}\right)^2\epsilon+\left(\frac{\rho_1}{\rho_0}\right)^4\frac{16z_0^2-10r_0^2}{\rho_0^2}\epsilon^2-2\left(\frac{\rho_1}{\rho_0}\right)^2\frac{r_0^2}{\rho_0^2}\epsilon\sigma-3\frac{z_0^2}{\rho_0^2}\sigma^2\cdots\right]\xi^2,$$

$$\eta_2=-\frac{r_0z_0}{\rho_0^2}\cdot\frac{1}{\rho_0}\left[2\left(\frac{\rho_1}{\rho_0}\right)^2\epsilon-4\sigma+\left(\frac{\rho_1}{\rho_0}\right)^4\frac{16z_0^2-10r_0^2}{\rho_0^2}\epsilon^2-6\left(\frac{\rho_1}{\rho_0}\right)^2\frac{r_0^2}{\rho_0^2}\epsilon\sigma+\frac{z_0^2-8r_0^2}{\rho_0^2}\sigma^2\cdots\right]\xi^2,$$

and

$$\eta_2-\eta_1=\frac{r_0z_0}{\rho_0^2}\cdot\frac{1}{\rho_0}\left[-\frac{5}{6}\left(\frac{\rho_1}{\rho_0}\right)^2 2\epsilon+4\sigma-\frac{5}{6}\left(\frac{\rho_1}{\rho_0}\right)^4\frac{16z_0^2-10r_0^2}{\rho_0^2}\epsilon^2\right.$$
$$\left.+\frac{17}{3}\left(\frac{\rho_1}{\rho_0}\right)^2\frac{r_0^2}{\rho_0^2}\epsilon\sigma+\frac{16r_0^2-3z_0^2}{2\rho_0^2}\sigma^2\cdots\right]\xi^2.$$

The ratio ρ_1/ρ_0 is nearly equal to unity and $2\epsilon(=.00334)$ is nearly equal to $\sigma(=.00345)$. Hence, approximately, the above equations become

* This function may be regarded as a sufficiently close approximation to the potential function for which the Standard Spheroid is an equipotential surface.

† For the derivation of this function and the values of e_1 and σ_1 see POINCARÉ, *Figures d'Equilibre d'une Masse Fluide* (1902), Chap. V.

$$\eta_1 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0} \left[-\tfrac{1}{6}\sigma \right] \xi^2, \qquad \eta_2 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0} \left[3\sigma \right] \xi^2,$$

and

$$\eta_2 - \eta_1 = \frac{r_0 z_0}{\rho_0^2} \frac{1}{\rho_0} \left[\tfrac{19}{6}\sigma \right] \xi^2.$$

In this case c_1 falls to the south, and c_2 to the north, of t. The trajectory contributes $\frac{1}{19}$, and the plumb-bob locus $\frac{18}{19}$, to the southerly deviation. (See Fig. 4.)

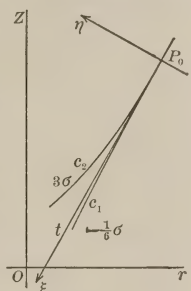

FIG. 4.

It will be observed that the potential function and the results under the second assumption may be obtained from those under this assumption by setting ϵ equal to zero.

The preceding results and figures 1, 2, 3 and 4 clearly show that the curves c_1 (or c_2) which correspond to the different assumptions are not even approximations to one another. The four potential functions assumed above may be regarded as having equal first derivatives* at $P_0(r_0, z_0)$. But in order that the curves c_1 (or c_2) which correspond to two potential functions f_1, shall osculate at P_0, both the first and second derivatives of these two potential functions must be equal at P_0. This follows from the fact that the coefficient of ξ^2 in each of equations (1) and (2) involves both the first and second derivatives of f_1.

The Work of M. le Comte de Sparre.†

M. le Comte de Sparre computes the portion of the southerly deviation which is denoted by η_1 in this paper. In his first paper he assumes potential functions which are practically identical with those of Assumptions 2 and 3. His results, under these assumptions, are

$$x = -\frac{1}{2} \frac{R\omega^4 h^2}{g^2} \sin^3 \lambda \cos \lambda \quad \text{and} \quad x = \frac{1}{2} \frac{\omega^2 h^2}{g} \sin \lambda \cos \lambda,$$

respectively, and those of this paper are

$$\eta_1 = \frac{1}{2} \frac{r_0 z_0}{\rho_0^2} \cdot \frac{z_0^2}{\rho_0^2} \cdot \frac{1}{\rho_0} \sigma^2 \xi^2 \quad \text{and} \quad \eta_1 = -\frac{1}{2} \frac{r_0 z_0}{\rho_0^2} \frac{1}{\rho_0} \left[\sigma + (\ ) \sigma^2 \right] \xi^2,$$

respectively. Since x is positive for southerly, and η_1 is positive for northerly deviations, and since

* That is, the vector which represents the force of the field F_1 at the initial point of the falling body in F_1, may be taken to be the same for all the assumptions. As a consequence of this the line t will be the same for all the assumptions.

† Bulletin de la Société Mathématique de France, vol. 33 (1905). First paper, page 65, second paper, page 146.

$$R = \rho_0, \qquad \sin\lambda = \frac{z_0}{\rho_0}, \qquad \cos\lambda = \frac{r_0}{\rho_0}, \qquad h^2 = \xi^2, \qquad R\frac{\omega^2}{g} = \sigma,$$

it follows that the two sets of formulas are identical.

In the second paper a potential function is chosen which is practically equivalent to that of Assumption 4. The result of M. le Compte de Sparre is

$$x = \frac{\omega^2}{2\beta}\sin\lambda\cos\lambda\, gt^4$$

and that of this paper is

$$\eta_1 = \frac{r_0 z_0}{\rho_0^2}\frac{1}{\rho_0}(-\tfrac{1}{6}\sigma\cdots)\xi^2.$$

Since $h = \xi = \frac{1}{2}gt^2 + \cdots$, and by the above substitution, these formulas may be written in the following forms:

$$x = \tfrac{4}{3}\sin\lambda\cos\lambda\frac{\sigma}{R}h^2 \qquad \text{and} \qquad -\eta_1 = +\tfrac{1}{6}\sin\lambda\cos\lambda\frac{\sigma}{R}h^2.$$

Therefore these expressions are nearly the same.

Conclusion.

Of the facts set forth in statements I and II of the introduction, the assumption of Gauss disregarded the first but not the second, and the assumption of the second paper of M. le Comte de Sparre disregarded the second but not the first, while the two assumptions of the first paper of M. le Comte de Sparre disregarded both the first and the second. On the other hand the diference $\eta_2 - \eta_1$ for assumption 4 was obtained with due regard for the facts set forth in Statements I and II. This difference yields results for the southerly deviation which are several times as great as those obtained from the formulas of Gauss and M. le Compte de Sparre. It is not necessary to retain all the terms which are included in the expression for this difference, since σ^2, ϵ^2 and $\epsilon\sigma$ are small in comparison with σ and ϵ. Since the ratio ρ_1/ρ_0 is nearly equal to unity, and therefore

$$\sigma\left(=\frac{\omega^2\rho_0}{M/\rho_0^2}\right) \qquad \text{and} \qquad \sigma_1\left(=\frac{\omega^2\rho_1}{M/\rho_1^2}\right)$$

are nearly equal, the expression for the southerly deviation of a falling body assumes the form:

$$\tfrac{1}{2}\sin 2\phi_0(4\sigma_1 - \tfrac{5}{3}\epsilon)\frac{h^2}{\rho_0},$$

where ϕ_0 denotes the geocentric latitude of the initial point P_0, ρ_0 the distance from the earth's center to P_0, h the height through which the body falls, and σ_1 and ϵ the numerical constants given in Assumption 4.

A Numerical Example.

For the data of Benzenberg's experiment in St. Michael's Tower, namely $\phi = 53° 33'$, $h = 235$ feet, the formula of Gauss ($\frac{2}{3} \sin \phi \cos \phi \omega^2 h^2/g$) yields the value .00046 lines *, and the formula of this paper $[\frac{1}{2} \sin 2\phi_0 (4\sigma_1 - \frac{5}{3}\epsilon) h^2/\rho_0$, where ϕ_0 denotes the geocentric latitude of P_0 and equals 53° 22′ if the astronomical latitude is $\phi = 53° 33']$ yields the value .00202 lines. The latter result is about four and one-half times as great as the former, as we knew it should be from a comparison of the expressions for $\eta_2 - \eta_1$ under assumptions 1 and 4.

The Results of Experiments.

In experiments for the determination of the southerly deviation of falling bodies, results are obtained which are many times greater than those accounted for by any of the formulas given above. †

Since the greater part of the southerly deviation is due to the plumb-bob locus, it occurs to the author that experiments for the determination of the deviation between a very long plumb-line and a short plumb-line, which are supported at the same point, might give results of greater value than those for the determination of the southerly deviation of a falling body. The annoying factor of air resistance occurs in the latter, but not in the former. By enclosing the plumb-lines in an air-tight tube and making the observations through glass windows in the tube, air currents could also be eliminated.

The Establishment of General Equations (1) and (2).

§ 1. *The curve c.* The curve c has already been defined as the path of the falling body in the fixed field F_1. In order to get its equations let us choose a set of rectangular axes in F_1 and denote by x_1, y_1, z_1 the coördinates of a general point referred to these axes, the origin of which is taken at the earth's center O. The axis of z_1 is taken coincident with the earth's axis of rotation and is regarded as positive in the direction of the earth's north pole. The axes of x_1 and y_1 then lie in the earth's equatorial plane. The direction of the axis of x_1 is so chosen that the plane $x_1 z_1$ contains the position $\bar{P}_0$ which the falling body occupied at the instant when its connection with the rotating earth was severed, and its sense is such that the coördinate of this initial point is positive. The axis of y_1, which is perpendicular to that of x_1, is positive in the direction in which the earth rotates at the initial point. (See Fig. 5.)

* 144 lines = 1 foot.

† See the following papers: Professor Florian Cajori; *The Unexplained Southerly Deviation of Falling Bodies*, Science, Vol. 14 (1901), p. 853. Professor Edwin H. Hall, *Do Falling Bodies Move South?* Physical Review, Vol. 17 (1903), Nos. 3, 4.

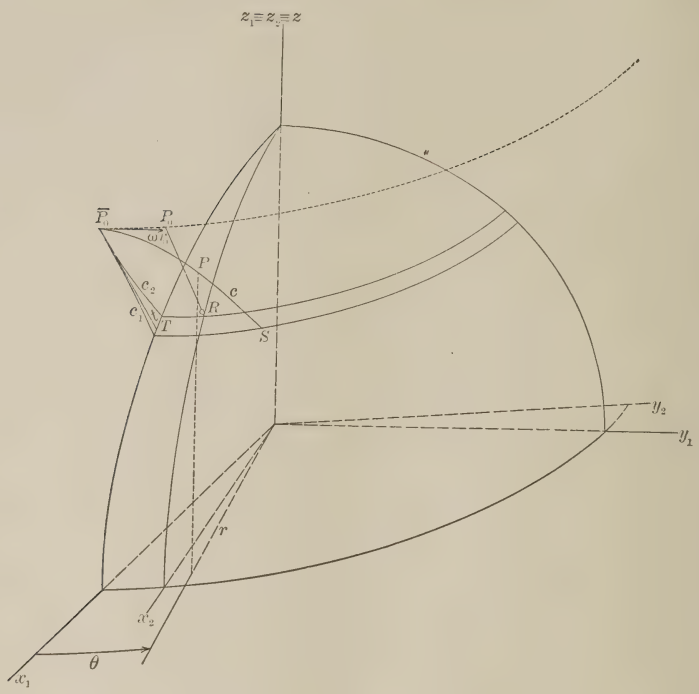

FIG. 5.

If, as above,

$$U_1(x_1, y_1, z_1) = f_1(r, z)$$

represents the potential function of the field F_1, and ω the angular velocity of the earth's rotation, then the curve c is that integral curve of the differential equations

$$\frac{d^2x_1}{dt^2} = \frac{\partial U_1}{\partial x_1}, \qquad \frac{d^2y_1}{dt^2} = \frac{\partial U_1}{\partial y_1}, \qquad \frac{d^2z_1}{dt^2} = \frac{\partial U_1}{\partial z_1},$$

which, at $t = 0$, is subject to the conditions

$$x_1 = r_0, \qquad y_1 = 0, \qquad z_1 = z_0,$$

$$\frac{dx_1}{dt} = 0, \qquad \frac{dy_1}{dt} = \omega r_0, \qquad \frac{dz_1}{dt} = 0.$$

§ 2. *The curve* c_1. The curve c_1 has already been defined as the meridian curve of the surface of revolution, of axis z_1, on which the curve c lies. It may also be defined as the path of the falling body in the moving plane which is determined by the body and the axis of z_1. In order to get its equation let us make the following transformation of coördinates:

$$x_1 = r\cos\theta, \qquad y_1 = r\sin\theta, \qquad z_1 = z,$$

in which θ denotes the angle which the plane determined by the falling body and the axis of z_1 makes with the plane $x_1 z_1$. (See Fig. 5.) The preceding differential equations are then transformed into the following differential equations:

$$r'' - r\theta'^2 = \frac{\partial f_1}{\partial r},$$

$$r\theta'' + 2r'\theta' = 0,$$

$$z'' = \frac{\partial f_1}{\partial z},$$

where the primes (′) and seconds (″) denote the first and second derivatives, respectively, with regard to the time t. The second of these equations on integration becomes

$$\theta' = \frac{k}{r^2},$$

where k is a constant of integration, and hence reduces the first to

$$r'' = \frac{k^2}{r^3} + \frac{\partial f_1}{\partial r}.$$

The initial conditions assume the following form: when $t = 0$,

$$r = r_0, \qquad \theta = 0, \qquad z = z_0,$$

$$r' = 0, \qquad \theta' = \omega, \qquad z' = 0.$$

Hence the constant k is given by the relation

$$\omega = k/r_0^2.$$

Therefore the curve c_1 is that integral curve of the differential equations:

$$r'' = \frac{\omega^2 r_0^4}{r^3} + \frac{\partial f_1}{\partial r}, \qquad z'' = \frac{\partial f_1}{\partial z},$$

which, at $t = 0$, is subject to the conditions

$$r = r_0, \qquad z = z_0, \qquad r' = 0, \qquad z' = 0.$$

If we put

$$V(r, z) = -\frac{\omega^2 r_0^4}{2r^2} + f_1(r, z),$$

the last differential equations may be written in the following form:

$$r'' = \frac{\partial V}{\partial r}, \qquad z'' = \frac{\partial V}{\partial z}.$$

§ 3. *The curve* c_2. The curve c_2 has already been defined as the locus of plumb-bobs of all plumb-lines which are supported at the initial point, P_0, of the falling body in the field F_2. It is therefore the locus of the points of tangency of tangent lines which are drawn from P_0 to the lines of force of the field F_2. In order to get its equation let us choose a set of rectangular axes in F_2 and denote by x_2, y_2, z_2 the coördinates of a general point referred to these axes, the origin of which is also taken at the earth's center O. The axis of z_2 is taken, just as that of z_1, coincident with the earth's axis of rotation and is regarded as positive in the direction of the earth's north pole. The axes of x_2 and y_2 lie in the earth's equatorial plane, and when $t = 0$, they coincide with the axes of x_1 and y_1, respectively. (See Fig. 5.) The point P_0 lies in the plane $x_2 z_2$. If, as above,

$$U_2(x_2, y_2, z_2) = f_2(r, z),$$

where

$$r = \sqrt{x_2^2 + y_2^2} \qquad \text{and} \qquad z = z_2,$$

represents the potential function of the field F_2, the equations of c_2 are

$$\frac{x_2 - r_0}{\frac{\partial U_2}{\partial x_2}} = \frac{y_2 - 0}{\frac{\partial U_2}{\partial y_2}} = \frac{z_2 - z_0}{\frac{\partial U_2}{\partial z_2}}.$$

But since

$$\frac{\partial U_2}{\partial x_2} = \frac{\partial f_2}{\partial r} \cdot \frac{x_2}{r}, \qquad \frac{\partial U_2}{\partial y_2} = \frac{\partial f_2}{\partial r} \cdot \frac{y_2}{r}, \qquad \frac{\partial U_2}{\partial z_2} = \frac{\partial f_2}{\partial z},$$

the curve lies in the plane $y_2 = 0$, in which it has the equation

$$\frac{r - r_0}{\frac{\partial f_2}{\partial r}} = \frac{z - z_0}{\frac{\partial f_2}{\partial z}}.$$

§ 4. *The equation of curve* c_1 *in powers of* $r - r_0$.

That solution of the differential equations

$$r'' = \partial V/\partial r, \qquad z'' = \partial V/\partial z$$

which, at $t = 0$, is subject to the conditions $r = r_0$, $z = z_0$, $r' = 0$, $z' = 0$, is

of the form

$$r - r_0 = \alpha_2 t^2 + \alpha_4 t^4 + \cdots, \qquad z - z_0 = \beta_2 t^2 + \beta_4 t^4 + \cdots,$$

where

$$\alpha_2 = \frac{1}{2}\left(\frac{\partial V}{\partial r}\right)_0, \qquad \beta_2 = \frac{1}{2}\left(\frac{\partial V}{\partial z}\right)_0,$$

$$\alpha_4 = \frac{1}{4!}\left\{\left(\frac{\partial^2 V}{\partial r^2}\right)_0\left(\frac{\partial V}{\partial r}\right)_0 + \left(\frac{\partial^2 V}{\partial r \partial z}\right)_0\left(\frac{\partial V}{\partial z}\right)_0\right\},$$

$$\beta_4 = \frac{1}{4!}\left\{\left(\frac{\partial^2 V}{\partial r \partial z}\right)_0\left(\frac{\partial V}{\partial r}\right)_0 + \left(\frac{\partial^2 V}{\partial z^2}\right)_0\left(\frac{\partial V}{\partial z}\right)_0\right\},$$

the subscript $(_0)$ indicating that the derivatives of V have been computed for the point $P_0(r_0, z_0)$. By the elimination of t from these equations, we obtain the equation

$$z - z_0 = a_1(r - r_0) + a_2(r - r_0)^2 + \cdots,$$

where

$$a_1 = \frac{\left(\frac{\partial V}{\partial z}\right)_0}{\left(\frac{\partial V}{\partial r}\right)_0},$$

$$a_2 = \frac{1}{6}\frac{\left(\frac{\partial^2 V}{\partial r \partial z}\right)_0\left\{\left(\frac{\partial V}{\partial r}\right)_0^2 - \left(\frac{\partial V}{\partial z}\right)_0^2\right\} + \left\{\left(\frac{\partial^2 V}{\partial z^2}\right)_0 - \left(\frac{\partial^2 V}{\partial r^2}\right)_0\right\}\left(\frac{\partial V}{\partial r}\right)_0\left(\frac{\partial V}{\partial z}\right)_0}{\left(\frac{\partial V}{\partial r}\right)_0^3}.$$

§ 5. *The equation of curve c_2 in powers of $r - r_0$.*

The equation of curve c_2, whose finite equation has already been found in § 3, may also be written in the following form:

$$z - z_0 = b_1(r - r_0) + b_2(r - r_0)^2 + \cdots,$$

where

$$b_1 = \left(\frac{z}{dr}\right)_0, \qquad b_2 = \frac{1}{2}\left(\frac{d^2 z}{dr^2}\right)_0.$$

In order to do this let us first write its equation in the form

$$F(r, z) = 0,$$

where

$$F(r, z) = \frac{\partial f_2}{\partial z}(r - r_0) - \frac{\partial f_2}{\partial r}(z - z_0).$$

The derivatives dz/dr and d^2z/dr^2 are expressible in terms of the derivatives of F by means of the relations:

$$\frac{\partial F}{\partial r}+\frac{\partial F}{\partial z}\cdot\frac{dz}{dr}=0,$$

$$\frac{\partial^2 F}{\partial r^2}+2\frac{\partial^2 F}{\partial r\,\partial z}\cdot\frac{dz}{dr}+\frac{\partial^2 F}{\partial z^2}\left(\frac{dz}{dr}\right)^2+\frac{\partial F}{\partial z}\cdot\frac{d^2z}{dr^2}=0,$$

and the derivatives of F, in terms of those of f_2'', by means of the relations:

$$\frac{\partial F}{\partial r}=\frac{\partial^2 f_2}{\partial r\,\partial z}(r-r_0)-\frac{\partial^2 f_2}{\partial r^2}(z-z_0)+\frac{\partial f_2}{\partial z},$$

$$\frac{\partial F}{\partial z}=\frac{\partial^2 f_2}{\partial z^2}(r-r_0)-\frac{\partial^2 f_2}{\partial r\,\partial z}(z-z_0)-\frac{\partial f_2}{\partial r},$$

$$\frac{\partial^2 F}{\partial r^2}=\frac{\partial^3 f_2}{\partial r^2\partial z}(r-r_0)-\frac{\partial^3 f_2}{\partial r^3}(z-z_0)+2\frac{\partial^2 f_2}{\partial r\,\partial z},$$

$$\frac{\partial^2 F}{\partial r\,\partial z}=\frac{\partial^3 f_2}{\partial r\,\partial z^2}(r-r_0)-\frac{\partial^3 f_2}{\partial r^2\partial z}(z-z_0)+\frac{\partial^2 f_2}{\partial z^2}-\frac{\partial^2 f_2}{\partial r^2},$$

$$\frac{\partial^2 F}{\partial z^2}=\frac{\partial^3 f_2}{\partial z^3}(r-r_0)-\frac{\partial^3 f_2}{\partial r\,\partial z^2}(z-z_0)-2\frac{\partial^2 f_2}{\partial r\,\partial z},$$

and therefore

$$\left(\frac{\partial F}{\partial r}\right)_0=\left(\frac{\partial f_2}{\partial z}\right)_0,$$

$$\left(\frac{\partial F}{\partial z}\right)_0=-\left(\frac{\partial f_2}{\partial r}\right)_0,$$

$$\left(\frac{\partial^2 F}{\partial r^2}\right)_0=2\left(\frac{\partial^2 f_2}{\partial r\,\partial z}\right)_0,$$

$$\left(\frac{\partial^2 F}{\partial r\,\partial z}\right)_0=\left(\frac{\partial^2 f_2}{\partial z^2}\right)_0-\left(\frac{\partial^2 f_2}{\partial r^2}\right)_0,$$

$$\left(\frac{\partial^2 F}{\partial z^2}\right)_0=-2\left(\frac{\partial^2 f_2}{\partial r\,\partial z}\right).$$

Hence we find

$$b_1=\left(\frac{dz}{dr}\right)_0=-\frac{\left(\frac{\partial F}{\partial r}\right)_0}{\left(\frac{\partial F}{\partial z}\right)_0}=\frac{\left(\frac{\partial f_2}{\partial z}\right)_0}{\left(\frac{\partial f_2}{\partial r}\right)_0},$$

$$b_2=\tfrac{1}{2}\left(\frac{d^2z}{dr^2}\right)_0=\tfrac{1}{2}\,\frac{\left(\frac{\partial^2 F}{\partial r^2}\right)_0+2\left(\frac{\partial^2 F}{\partial r\,\partial z}\right)_0\left(\frac{dz}{dr}\right)_0+\left(\frac{\partial^2 F}{\partial z^2}\right)_0\left(\frac{dz}{dr}\right)_0^2}{-\left(\frac{\partial F}{\partial z}\right)_0}$$

$$= \frac{\left(\frac{\partial^2 f_2}{\partial r\,\partial z}\right)_0 \left\{\left(\frac{\partial f_2}{\partial r}\right)_0^2 - \left(\frac{\partial f_2}{\partial z}\right)_0^2\right\} + \left\{\left(\frac{\partial^2 f_2}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f_2}{\partial r^2}\right)_0\right\}\left(\frac{\partial f_2}{\partial r}\right)_0\left(\frac{\partial f_2}{\partial z}\right)_0}{\left(\frac{\partial f_2}{\partial r}\right)_0^3}.$$

§ 6. *The relation between the curves c_1 and c_2.* In order to find a relation between the curves c_1 and c_2 let us express the coefficients a_1, a_2, b_1, b_2 in terms of the derivatives of f_1. We already have the relations:

$$V = f_1 - \frac{\omega^2 r_0^4}{2r^2} \qquad \text{and} \qquad f_2 = f_1 + \frac{\omega^2}{2} r^2,$$

from which the following relations are found:

$$\frac{\partial V}{\partial r} = \frac{\partial f_1}{\partial r} + \frac{\omega^2 r_0^4}{r^3}, \qquad \frac{\partial f_2}{\partial r} = \frac{\partial f_1}{\partial r} + \omega^2 r,$$

$$\frac{\partial V}{\partial z} = \frac{\partial f_1}{\partial z}, \qquad \frac{\partial f_2}{\partial z} = \frac{\partial f_1}{\partial z},$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{\partial^2 f_1}{\partial r^2} - \frac{3\omega^2 r_0^4}{r^4}, \qquad \frac{\partial^2 f_2}{\partial r^2} = \frac{\partial^2 f_1}{\partial r^2} + \omega^2,$$

$$\frac{\partial^2 V}{\partial r\,\partial z} = \frac{\partial^2 f_1}{\partial r\,\partial z}, \qquad \frac{\partial^2 f_2}{\partial r\,\partial z} = \frac{\partial^2 f_1}{\partial r\,\partial z},$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 f_1}{\partial z^2}, \qquad \frac{\partial^2 f_2}{\partial z^2} = \frac{\partial^2 f_1}{\partial z^2}.$$

Therefore:

$$\left(\frac{\partial V}{\partial r}\right)_0 = \left(\frac{\partial f_2}{\partial r}\right)_0 = \left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0, \qquad \left(\frac{\partial V}{\partial z}\right)_0 = \left(\frac{\partial f_2}{\partial z}\right)_0 = \left(\frac{\partial f_1}{\partial z}\right)_0,$$

$$\left(\frac{\partial^2 V}{\partial r^2}\right)_0 = \left(\frac{\partial^2 f_1}{\partial r^2}\right)_0 - 3\omega^2, \qquad \left(\frac{\partial^2 f_2}{\partial r^2}\right)_0 = \left(\frac{\partial^2 f_1}{\partial r^2}\right)_0 + \omega^2,$$

$$\left(\frac{\partial^2 V}{\partial r\,\partial z}\right)_0 = \left(\frac{\partial^2 f_2}{\partial r\,\partial z}\right)_0 = \left(\frac{\partial^2 f_1}{\partial r\,\partial z}\right)_0, \qquad \left(\frac{\partial^2 V}{\partial z^2}\right)_0 = \left(\frac{\partial^2 f_2}{\partial z^2}\right)_0 = \left(\frac{\partial^2 f_1}{\partial z^2}\right)_0.$$

Hence the relations

$$a_1 = b_1 = \frac{\left(\frac{\partial f_1}{\partial z}\right)_0}{\left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0},$$

$$a_2 = \frac{1}{6}\,\frac{C + B_1\omega^2 + A_1\omega^4}{\left\{\left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0\right\}^3}, \qquad b_2 = \frac{C + B_2\omega^2 + A_2\omega^4}{\left\{\left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0\right\}^3},$$

where

$$C=\left(\frac{\partial^2 f_1}{\partial r\partial z}\right)_0\left\{\left(\frac{\partial f_1}{\partial r}\right)_0^2-\left(\frac{\partial f_1}{\partial z}\right)_0^2\right\}+\left\{\left(\frac{\partial^2 f_1}{\partial z^2}\right)_0-\left(\frac{\partial^2 f_1}{\partial r^2}\right)_0\right\}\left(\frac{\partial f_1}{\partial r}\right)_0\left(\frac{\partial f_1}{\partial z}\right)_0,$$

$$B=2\left(\frac{\partial^2 f_1}{\partial r\partial z}\right)_0\left(\frac{\partial f_1}{\partial r}\right)_0 r_0+\left\{\left(\frac{\partial^2 f_1}{\partial z^2}\right)_0-\left(\frac{\partial^2 f_1}{\partial r^2}\right)_0\right\}\left(\frac{\partial f_1}{\partial z}\right)_0 r_0,$$

$$B_1=B+3\left(\frac{\partial f_1}{\partial r}\right)_0\left(\frac{\partial f_1}{\partial z}\right)_0,\qquad B_2=B-\left(\frac{\partial f_1}{\partial r}\right)_0\left(\frac{\partial f_1}{\partial z}\right)_0,$$

$$A=\left(\frac{\partial^2 f_1}{\partial r\partial z}\right)_0 r_0^2,\qquad A_1=A+3\left(\frac{\partial f_1}{\partial z}\right)_0 r_0,\qquad A_2=A-\left(\frac{\partial f_1}{\partial z}\right)_0 r_0.$$

Since $a_1=b_1$, it follows that the curves c_1 and c_2, which lie in the meridian plane of $P_0(r_0, z_0)$ (i. e., in the plane $y_2=0$), have a common tangent * and a common normal at P_0.

§ 7. *The General Equations* (1) *and* (2). *The Southerly Deviation.*

Let us now take the common tangent and the common normal to the curves c_1 and c_2 as a pair of rectangular axes, and obtain the equations of c_1 and c_2 referred to them. The coördinate which is measured along the tangent from P_0 we shall denote by ξ, and regard as positive when measured in the direction of the earth's center. The other coördinate we shall denote by η and regard as positive when measured in the direction of the north pole. The equations of transformation from the axes of r and z to those of ξ and η are

$$r-r_0=-\cos\phi\cdot\xi-\sin\phi\cdot\eta,\qquad z-z_0=-\sin\phi\cdot\xi+\cos\phi\cdot\eta,$$

where

$$\sin\phi=\frac{b_1}{\sqrt{1+b_1^2}}=\frac{a_1}{\sqrt{1+a_1^2}},\qquad \cos\phi=\frac{1}{\sqrt{1+b_1^2}}=\frac{1}{\sqrt{1+a_1^2}}.$$

By this transformation the equation

$$z-z_0=a_1(r-r_0)+a_2(r-r_0)^2+\cdots$$

* This common tangent is also tangent at P_0 to the line of force of the field F_2 which passes through P_0. For the differential equation of the lines of force of F_2 which line in the plane $y_2=0$ is

$$\frac{dz}{dr}=\frac{\dfrac{\partial f_2}{\partial z}}{\dfrac{\partial f_2}{\partial r}},$$

and therefore the slope at P_0 of the line of force which passes through P_0 is

$$\frac{\left(\dfrac{\partial f_2}{\partial z}\right)_0}{\left(\dfrac{\partial f_2}{\partial r}\right)_0}=\frac{\left(\dfrac{\partial f_1}{\partial z}\right)_0}{\left(\dfrac{\partial f_1}{\partial r}\right)_0+\omega^2 r_0}=a_1=b_1.$$

of the curve c_1 assumes the form

$$+\frac{a_2}{1+a_1^2}\xi^2+\frac{2a_1a_2}{1+a_1^2}\xi\eta+\frac{a_1^2a_2}{1+a_1^2}\eta^2-\sqrt{1+a_1^2}\,\eta=0,$$

which when solved for η, becomes*:

$$\eta=\frac{a_2}{[1+a_1^2]^{\frac{3}{2}}}\xi^2+\cdots.$$

By the same transformation the equation

$$z-z_0=b_1(r-r_0)+b_2(r-r_0)^2+\cdots,$$

of the curve c_2, becomes

$$\eta=\frac{b_2}{[1+b_1^2]^{\frac{3}{2}}}\xi^2+\cdots.$$

From the relations given in § 6 we find†

$$\frac{1}{[1+a_1^2]^{\frac{3}{2}}}=\frac{1}{[1+b_1^2]^{\frac{3}{2}}}=-\frac{\left[\left(\frac{\partial f_1}{\partial r}\right)_0+\omega^2r_0\right]^3}{D^3},$$

where

$$D=\left[\left(\frac{\partial f_1}{\partial r}\right)_0+\omega^2r_0\right]^2+\left(\frac{\partial f_1}{\partial z}\right)_0^2=\left(\frac{\partial f_1}{\partial r}\right)_0^2+\left(\frac{\partial f_1}{\partial z}\right)_0^2+2\left(\frac{\partial f_1}{\partial r}\right)_0r_0\omega+r_0^2\omega^2.$$

Hence, if we denote the η's of the curves c_1 and c_2 by η_1 and η_2, respectively, we obtain the equations

$$\eta_1=-\frac{1}{6}\frac{C+B_1\omega^2+A_1\omega^4}{D^3}\xi^2+\cdots,\qquad \eta_2=-\frac{C+B_2\omega^2+A_2\omega^4}{D^3}\xi^2+\cdots,$$

which are the equations (1) and (2) already given. As has been stated, the southerly deviation of a falling body is the difference $\eta_2-\eta_1$, when ξ is replaced by the height h‡ through which the body falls, and f_1 is a sufficiently close approximation to the potential function of the earth's gravitational field of force.

* The equation

$$0=Ey+Ax^2+Bxy+Cy^2+Lx^3+Mx^2y+Nxy^2+\cdots$$

when solved for y becomes

$$y=-\frac{A}{E}x^2+\cdots.$$

† The negative sign is taken because $[1+b_1^2]^{\frac{3}{2}}$ and D^3 are positive and $(\partial f_1/\partial r)_0+\omega^2r_0$ is negative.

‡ As already remarked, the circle of reference must be determined by a plumb-line of length h supported at the initial point of the falling body.

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AN APPLICATION OF A (1, 2) QUATERNARY CORRESPONDENCE TO THE WEDDLE AND KUMMER SURFACES*

BY

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Introduction.

The three-dimensional non-linear involution determined by a system of quadrics having three linear parameters has been extensively discussed. In the case in which all the quadrics have six basis points the system furnishes a convenient method of mapping the Weddle surface upon the Kummer surface. In the following paper I develop the details of this transformation analytically, obtaining a number of results already known, but by new methods, and apply the transformation to obtain a number of series of transformations which leave the surfaces invariant. A number of new transformations appear when the basis points are in simple or multiple involution. The relations which must exist among the coefficients of the equations to insure involution, the positions of the new lines lying on the Weddle surface, and the corresponding specializations of the Kummer surface are obtained.

§ 1. *The Involution I.*

1. Through six arbitrary points in space can be passed a linear system of quadric surfaces having three parameters. The equation of the system has the form

$$H \equiv \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 + \lambda_4 H_4 = 0 . \tag{1}$$

If the coördinates (ξ') of a point P' are substituted in H, between $H(x)$ and $H(\xi')$ one parameter, as λ_4, may be eliminated, and thus the net

$$\lambda_1(H_1 \cdot H_4' - H_1' \cdot H_4) + \lambda_2(H_2 \cdot H_4' - H_2' \cdot H_4) + \lambda_3(H_3 \cdot H_4' - H_3' \cdot H_4) = 0$$

is obtained. The coefficients of the λ_i define three quadrics which intersect in eight points; as seven of these points are known the coördinates of the eighth, $P'' \equiv (\xi'')$, can be expressed as rational functions of the coördinates of the six fixed points and (ξ'). In other words, the equations

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$$\frac{H_1''}{H_1'} = \frac{H_2''}{H_2'} = \frac{H_3''}{H_3'} = \frac{H_4''}{H_4'} \tag{2}$$

yield the rational solutions

$$\xi_i'' = \phi_i(\xi_1', \xi_2', \xi_3', \xi_4').$$

*The seventh and eighth basis points of any net of quadrics contained in a linear system having six fixed basis points define an involutorial birational transformation.**

Without restricting the problem, the basis points may be taken at the vertices of the tetrahedron of reference, the unit point and the point (a). It will be convenient to designate the points by number, as follows:

$$(1,0,0,0) \equiv 1, \quad (a_1 a_2 a_3 a_4) \equiv 2, \quad (0,1,0,0) \equiv 3, \quad (1,1,1,1) \equiv 4,$$
$$(0,0,1,0) \equiv 5, \quad (0,0,0,1) \equiv 6.$$

In the net defined by $1, 2, \cdots, 6, (\xi'), \xi'')$, the line $(\xi'\xi'')$ determines a pencil of quadrics whose basis curve consists of this line and the cubic curve γ_3 passing through $1, 2, \cdots, 6$. The line $(\xi'\xi'')$ is a bisecant of the cubic curve.

Any two associated points (ξ'), (ξ'') *lie on a bisecant of the cubic curve* γ_3 *defined by the six basis points.* There are fifteen special bundles, each defined by the points $1, 2, \cdots, 6$ and a line joining some two of them. These bundles will be designated by the symbols $\{ik\}$. Similarly, if P' is on γ_3, the net will contain γ_3. It will be called $\{\gamma_3\}$.

2. A line l (or any three points upon it) will uniquely define a quadric H which contains it. When (ξ') describes l, (ξ'') will describe some curve upon H_l, always lying on the bisecant of γ_3 through (ξ'). The bisecants of γ_3 from points on l define a ruled surface R_4 of order four, having l for simple directrix and γ_3 for double curve. R_4 and H_l intersect in a curve of order eight, composed of l and a curve c_7, of order seven, the locus of (ξ''). A plane through l cuts H_l in a straight line and cuts R_4 in three straight lines. The points of intersection are on c_7, hence l cuts c_7 in four points. *By the involutorial transformation I straight lines go into space curves of order seven, each cutting its image line in four points. Planes go into surfaces of order seven.*

3. Since R_4 has $1, 2, \cdots, 6$ for double points and H_l passes through each, c_7 has six double points which are principal points of the transformation, the image of each being a quadric cone having its vertex at its associated point and containing γ_3 and five lines (ik). These cones, each counted twice, constitute

*GEISER: *Ueber zwei geometrische Probleme*, Crelle's Journal, vol. 67 (1867), pp. 78–95; EBERHARDT: *Ueber eine räumlich involutorische Verwandtschaft 7. Grades und ihre Kernfläche 4. Ordnung*, Breslau dissertation (1885), 61 pp.; STURM: *Die Lehre von den geometrischen Verwandtschaften*, vol. 4 (1909), pp. 414–418. For the literature on the system of surfaces see also STAUDE: Encyklopädie der mathematischen Wissenschaften, III C2, § 141 (p. 252).

the entire Jacobian of the system ϕ. The curve γ_3 and the lines (ik) are principal curves of the second kind, such that the image of a point on any one of them is the whole line on which it lies.

4. A plane π goes into a surface F_7 of order seven. Since π has three points on γ_3, F_7 has γ_3 as a threefold curve, and similarly, F_7 contains each of the lines (ik). Two planes intersect in l; their image surfaces intersect in c_7 the image of l, in γ_3 counted nine times and in the fifteen lines (ik). Hence there are no further principal points in the transformation.

5. If l passes through a basis point i, the principal cone $K_2^{(i)}$ is part of its image. If l is the intersection of two planes π, π', each passing through i, the image c_7 will be the intersection of F_7, F_7'. Both these surfaces contain $k_2(i)$ as factor. Each of the residual quintics contains γ_3 as double curve and the ten lines (lk) not passing through i. Thus the proper image of l is a cubic curve passing through the basis points other than i, and having l for bisecant.

§ 2. *Locus of Coincident Points.*

6. If $P'' = P'$, all the surfaces of the associated net touch each other at P', and one surface has a double point, or P' is the vertex of a cone belonging to (1). The equation of the locus of P' is expressed by the vanishing of the Jacobian of (1). It can be reduced to the form

$$S_4 \equiv |a_1 x_2 x_3 x_4 \; x_1 \; a_1 \; 1| = 0.$$

The locus is the Weddle quartic surface. It contains $1, 2, \cdots, 6$ as double points, the curve γ_3, the fifteen lines (ik) and the ten lines of intersection of the planes (ikl), (mnp); these lines (ikl), (mnp) are defined by the symbol (ikl). The four points of intersection of a line l and its image c_7 lie on S_4, hence a general line l contains no pair of associated points (ξ'), (ξ''). But it was seen that the lines joining (ξ') to (ξ'') are bisecants of γ_3, hence each of these bisecants contains an entire involution of associated points. The points on γ_3 constitute a pair and the remaining intersections with S_4 are the double points of the involution. The Weddle surface therefore cuts harmonically the bisecants of the cubic curve γ_3 which lies upon it.

7. In the involution T_3 of conjugate points as to γ_3, S_4 is invariant. Moreover, I can now be defined independently of (1) if the points $1, 2, \cdots, 6$ are given, since S_4 and γ_3 are fixed. Given a point P' in space. Let the bisecant of γ_3 from P' cut S_4 in M_1, M_2. The harmonic conjugate of P' as to M_1, M_2 is P''. From the theorem of STEPHANOS * it follows that the transformations T_3 and I are commutative.

* Mathematische Annalen, vol. 22 (1883), pp. 299–336. See page 313.

8. A plane π cuts its image F_7 in a curve of order seven and S_4 in a quartic which lies on F_7. The residual intersection consists of the three bisecants of γ_3 in π. A line cutting γ_3 goes into c_7 consisting of γ_3 and a rational quartic passing simply through all basis points and cutting the given line at the three residual intersections with S_4.

§ 3. *Transformations of S_4 into Itself.*

9. We may now enumerate the involutorial birational transformations of S_4 into itself:

The six nodal projections (N_1);

The transformation T_3, defined by conjugate points as to γ_3;

The fifteen transformations defined by a secant of γ_3 and one of its bisecants (ik). Given any point P on S_4. The cubic cone projecting γ_3 from P will be cut by the plane $P,(ik)$ in the two lines P_i and P_k and one other line. If the latter cuts S_4 in Q, between P and Q exists a birational correspondence. It will be designated by $T_{\gamma ik}$;

The interchange of points on every secant of two skew lines $T_{ik,\,lm}$ or $T_{ik,\,lmn}$;

The six transformations defined by the cubic curves passing through a point P and five basis points. The residual point of intersection of the curve with S_4 is fixed when P is known. We shall call these transformations C_i, i being the symbol of the basis point through which the curve does not pass;

The transformations similar to the last preceding ones defined by the cubic curves passing through four basis points and having the line joining the remaining pair for bisecant. These transformations will be designated by C_{ik};

An infinite number of further involutions can be defined as follows. Through any point on γ_3, a point of any fundamental line (ik) and a point D on S_4 not on any fundamental line pass a quadric surface. It will belong to (1), and cut S_4 in $\gamma_3,(ik)$ and a space quartic passing through the remaining nodes. To get the point associated with any point P on S_4, pass a plane through P and any two nodes l, m other than i, k. The plane cuts the space quartic in l, m and two other points. Pass a conic through these four points on the quartic and through P. It is fixed, and will cut S_4 in one remaining point Q. The conjugate of this conic in I is a sextic curve having double points at the basis points through which the conic does not pass, and passing simply through the six points in which the conic cuts S_4. The operation of interchanging P, Q will be denoted by X_D. The same

series of curves is invariant under x as under (N_l, N_m), but the former can be expressed in terms of the latter only when the line joining the points on the quartic is tangent to the curve elsewhere. But with the exception of the points i, k this is impossible, since one point lies on γ_3 and the other on (ik). The group generated by X_D, $X_{D'}$, is of infinite order.

10. The points of γ_3 are defined in terms of the homogeneous parameters σ, τ by the equations

$$\rho x_1 = A\sigma\tau(\sigma - \tau),$$

$$\rho x_2 = B\sigma(\sigma - \tau)(\sigma_1\tau - \sigma\tau_1),$$

$$\rho x_3 = C\sigma\tau(\sigma_1\tau - \sigma\tau_1),$$

$$\rho x_4 = D\tau(\sigma - \tau)(\sigma_1\tau - \sigma\tau_1),$$

wherein

$$A = a_1(a_2 - a_3)(a_2 - a_4)(a_4 - a_3), \qquad B = a_2(a_4 - a_3), \qquad C = a_3(a_2 - a_4),$$

$$D = a_4(a_2 - a_3), \qquad \sigma_1 = (a_2 - a_3)(a_1 - a_4), \qquad \tau_1 = (a_1 - a_2)(a_4 - a_3).$$

The values of the parameters at the basis points are:

at 1,	$\sigma = (a_2 - a_3)(a_1 - a_4)$,	$\tau = (a_1 - a_2)(a_4 - a_3)$,
at 2,	$\sigma = a_2 - a_3$,	$\tau = a_4 - a_3$,
at 3,	$\sigma = 1$,	$\tau = 0$,
at 4,	$\sigma = a_4(a_2 - a_3)$,	$\tau = a_2(a_4 - a_3)$,
at 5,	$\sigma = 1$,	$\tau = 1$,
at 6,	$\sigma = 0$,	$\tau = 1$.

These equations may be used in C_2 by replacing (a) by the coördinates of P. The other C_i can be obtained from it by linear transformations.

The equations of the Cremona transformation T_3 can be obtained from the preceding equations alone.

§ 4. *Special Positions of Basis Points.*

11. The quadric surface determined by the lines (12), (35), (46) has the equation

$$x_4[x_1(a_2 - a_3) + a_3x_2 - a_2x_3] + x_1[a_4x_3 - a_4x_2] = 0.$$

It cuts γ_3 in the six basis points; if upon the coefficients the condition is

imposed that one further point of γ_3 lies on the quadric, then γ_3 lies entirely upon it. This condition is

$$a_2(a_1 - a_3) = a_1(a_4 - a_3).$$

In this case the four nets $\{\gamma_3\}$, $\{12\}$, $\{35\}$, $\{46\}$ have a common quadric.

The intersection of a quadric with S_4 consists of a curve of order eight cutting each generator in both reguli in four points. On one regulus the points of intersection are on the three lines (12), (35), (46) and on γ_3; on the other, two of them are on γ_3, hence the residual intersection consists of a curve of order two cutting the generators of the second regulus in two points and not cutting those of the first, that is, it is composed of two straight lines u, v.

The equations of these lines are

$$lx_1 - mx_4 = 0, \qquad x_1(a_2 l - a_3 l) + x_2(a_3 l - a_4 m) + x_3(a_4 m - a_2 l) = 0,$$

the parameters $l : m$ being defined by the quadratic equation

$$a_1 l^2 - 2a_1 l_m + a_4 m^2 = 0.$$

If the points $1, 2, \cdots, 6$ are projected from any point of γ_3 upon a plane, the generator of the same regulus as u, v which passes through the center of projection will pierce the plane of projection in a point through which pass the projections of (12), (34), (56). The basis points thus project into six points in involution. Conversely, when from any point of γ_3 these points are projected into points in involution, they and γ_3 have a common transversal and the preceding condition applies. We shall speak of the six points $1, 2, \cdots, 6$ themselves as being in involution when their projections are in involution.

When the six basis points $1, 2, \cdots, 6$ *are in involution the surface* S_4 *contains two new lines* u, v. The generators of the regulus cutting u, v are bisecants of γ_3, cutting S_4 in the residual points on u, v. These four points of intersection are harmonic. In the axial involution determined by u, v the six points $1, 2, \cdots, 6$ are interchanged in pairs and γ_3 goes into itself, hence S_4 must remain invariant. *In this case the axial involution* I_1 *must be added to the preceding transformations.*

12. In the same manner it is found that (13), (24), (56) are in involution if the a_i satisfy the condition $a_1 a_2 - a_3 a_4 = 0$;

(16), (34), (25) are in involution if $a_1(a_3 - a_4) = a_3(a_2 - a_4)$;

(12), (36), (45) " " " " $a_1(a_3 - a_4) = a_2(a_1 - a_4)$;

(14), (23), (56) " " " " $a_2(a_4 - a_1) = a_3(a_4 - a_2)$;

(15), (26), (34) " " " " $a_1(a_4 - a_3) = a_4(a_2 - a_3)$.

These six conditions are all satisfied when the a_i have the values

$$a_1 = 1, \qquad a_2 = 2, \qquad a_3 = 1 + i, \qquad a_4 = 1 - i.$$

The surface S_4 now has six pairs of lines u_i'', v_i and is invariant under six axial involutions I_i. No two of these involutions can be commutative. In order that two axial involutions may be commutative it is necessary that their axes shall either intersect or belong to the same regulus. Let (u, v), (u', v') be the axes and suppose u, u' intersect in M. The plane of u, u' cuts v in K, v' in L. Any line through K cuts S_4 in a fourth point which is the harmonic conjugate of its intersection with u' as to its intersection with u and K. The locus of this point is a line through M. The residual intersection of the plane consists of another line, this latter passing through K. But this is impossible, as S_4 cannot contain so many coplanar lines.

If (u, v), (u', v') belong to the same regulus, they must be harmonic, since any transversal cuts S_4 only on these lines. The quadric determined by them would cut S_4 in four lines belonging to the other regulus, of which one at most can be a line (ik). Of the six lines (ilm) skew to (ik), three can be chosen that are mutually skew and do not have another fundamental line for common transversal. Since u, v, u', v' are all invariant under I, a generator of the same regulus goes into a sextic cutting every generator of each system in three points. But this curve has double points at the fundamental points except i and k, hence all the fundamental points lie on the quadric, which therefore belongs to the system (1). Since (lm) cuts u, v, it must therefore lie on the quadric, but this is impossible, as the entire intersection with S_4 has been accounted for. Hence in this case u, v, u', v' cannot belong to the same regulus. Similarly for the case when no line (ik) cuts both pairs of axes. This completes the proof that no two operations I_i, I_k are commutative.

§ 5. *Double* λ-*Space.*

In (1) let the λ_i be considered as homogeneous plane coördinates. The plane will define a quadric of the system (1). A linear equation will define a point in (λ). This equation and (1) will determine a net of quadrics, passing through $1, 2, \cdots, 6$ and two associated points (ξ'), (ξ''). If the coördinates of (ξ') are substituted in (1), the resulting equation will fix the same point in (λ). Thus, the equation (1) defines a (1, 2) correspondence between the points (λ) of a space Σ' and the points (ξ'), (ξ'') of another space Σ. Two linear equations in (λ) define a straight line in Σ'. From these equations and (1) a pencil of quadrics is fixed, whose basis curve passes through the basis points $1, 2, \cdots, 6$. Hence when (ξ') describe a plane in Σ, (λ) will describe a quartic surface. The complete image in Σ of this quartic surface in Σ' is the plane and its image F_7

in I. To a straight line in Σ corresponds a conic in Σ', whose complete image in Σ is the line and its image c_∞.

If in the first derivatives of (1) we eliminate x_i, the resulting equation of order four in λ_i will define those values for which $(\xi') \equiv (\xi'')$. They define a surface of class four, $\Delta(\lambda)$. If Σ' is regarded as a double space, one sheet corresponding to (ξ'), the other to (ξ''), the points of $\Delta(\lambda)$ are branch points. *The points of* $\Delta(\lambda)$ *and the points of* S_4 *are in* (1, 1) *correspondence.* Since an arbitrary line in Σ is not its own image in I, the image conics cannot intersect $\Delta(\lambda)$, hence:

The images of the ∞^4 *lines of* Σ *are* ∞^4 *conics which touch four tangent planes of* Δ.

The image of a general plane π touches Δ along a curve of order eight. To a plane section of Δ corresponds a curve of order eight, in which the image quadric cuts S_4. A tangent plane of Δ corresponds to a cone of Σ, belonging to (1). To the point of contact corresponds its vertex. To the four cones in a pencil of quadrics belonging to (1) correspond the four tangent planes through the image line of the basis quartic.

14. To an arbitrary point $\Sigma a_i \lambda_i$ corresponds a pair of points (ξ'), (ξ''). The ∞^2 planes of Σ' through this point correspond to the ∞^2 quadrics through $1, 2, \cdots, 6, (\xi'), (\xi'')$. In particular, to the tangent planes to Δ passing through the point correspond ∞^1 cones in the net of quadrics. The locus of the vertices of these cones is a curve of order eight on S_4, having double points at all eight basis points. If in Σ we project the ∞^2 basis curves of order four and this octic from one of the basis points, and take the dual of the configuration in Σ' as to a quadric surface, we have exactly the Geiser transformation which was discussed in a preceding paper.*

Let P'_{00} in Σ' correspond to the net of quadrics $\{\gamma_3\}$. It is a principal point of Σ'. Similarly, let P'_{ik} correspond to $\{ik\}$. It is also a principal point. Any plane π in Σ cuts γ_3 in three points and each of the fifteen lines (ik) once. Its image F' in Σ' passes three times through P'_{00} and once through each of the fifteen points P'_{ik}. Since γ_3 and (ik) all lie on S_4, P'_{00}, P'_{ik} all lie on Δ. The c'_8 of contact of Δ and F'_4 passes three times through P'_{00} and simply through P'_{ik}.

Every bisecant of γ_3 contains a whole involution of points (ξ'), (ξ''). Its image in Σ' is a straight line through P'_{00}, counted twice. *The bundle of lines through* P'_{00} *in* Σ' *and the* (1, 3) *congruence of bisecants of* γ_3 *are in* (1, 1) *correspondence.*

Since π cuts γ_3 in three points and contains three bisecants, F'_4 contains three double lines passing through P'_{00}. Thus F'_4 is a Steiner surface. A line l in π corresponds to a conic c'_2 on F'_4. The plane ω' of c'_2 corresponds to a

* Snyder: *Conjugate line congruences contained in a bundle of quadric surfaces*, these Transactions, vol. 11 (1910), pp. 371–387.

quadric H containing l and another line m lying in π; the image of m is a second conic k_2' in ω' on F_4'. These two conics intersect on the three double lines and in one other point, the image of lm. H touches π in lm; ω' touches F_4' in c_2', k_2'. Both of these conics are contact curves of Δ.

15. The points $1, 2, \cdots, 6$ are principal points of the transformation; the image of each is a plane π_i'. All the basis points lie on γ_3, hence the six planes π_i' pass through P_{00}'; (ik) passes through two, hence π_i', π_k' pass through P_{ik}'. The fifteen lines $P_{00}'P_{ik}'$ are therefore the intersections of the six planes π_i', π_k'. Since a plane in Σ' is the image of a quadric in Σ, the planes π_i' may be regarded as images of the cones in (1), having i for vertex and passing through the other basis points. A bisecant of γ_3 cuts S_4 in two points not on γ_3; its image cuts Δ in two points not at P_{00}'.

A basis c_4 in Σ cuts S_4 in four points besides the basis points, hence its image line cuts Δ in four points. *Δ is of order four. The surface of branch points in Σ' is of order and class four and has sixteen double points P_{00}', P_{ik}', hence it is a Kummer surface.* To a straight l' through P_{ik}' corresponds a quartic curve in Σ consisting of (ik) and a cubic curve cutting (ik) twice and passing through the other four basis points. The curve cuts S_4 in two points not at basis points nor on (ik), hence its image line cuts Δ in two points not at P_{ik}', or P_{ik}' is also a double point.

16. Any line l' in π_i' goes into a c_4 on $K_2^{(i)}$, but as this fundamental cone is the tangent cone to S_4 at i, c_4 has a double point at its vertex, and all the variable points of intersection with c_4, S_4 are at i, two on each nodal tangent. Thus l' is a double tangent to Δ, or π_i' contains a conic of contact. It passes through P_{00}' and P_{ik}', $(k = 1, 2, \cdots, 6, i \neq k)$.

The plane (123) goes into an F_4' consisting of π_1', π_2', π_3' and another plane π_{123}', whose complete image in Σ consists of (123) and its image in I, viz., (456). The line (123), (456) goes into the conic of contact of π_{123}'. The principal points lying upon it are P_{12}', P_{13}', P_{23}', P_{45}', P_{46}', P_{56}'. The distribution of singular points and singular planes can now be readily determined by observing the corresponding elements in Σ.

A line l through any basis point i in Σ corresponds to a line in Σ', for, if any quadric H of (1) corresponds to the plane p', l cuts H in as many variable points and the image cuts p'. But H passes through i, hence l corresponds to a straight line l'. Since l does not correspond to itself in I, l' is a bitangent of Δ. Thus the bitangents of Δ are arranged in six distinct systems corresponding to the six basis points $1, 2, \cdots, 6$, in Σ.

§ 6. *Image Congruences of the Bundles* $1, 2, \cdots, 6$.

17. Through any point P' in Σ' pass two lines of the congruence $[i]$, since P' has two image points (ξ'), (ξ'') in Σ, each of which defines a line of i. A

quadric H of (1) contains two lines of i, hence its image plane contains two lines of $[i]$. Any line l of i cuts S_4 in two residual points. These go into the points of contact of l' with Δ. The complete image in Σ of l' consists of l and its image in I, that is, a space cubic curve passing through the remaining basis points and cutting l at its residual intersection with S_4.

The arrangement of the lines of these congruences can now be ascertained. The bundle i can be considered as made up of a single infinity of plane pencils each in a plane through (ik). The image of each plane is a Steiner surface F'_4 consisting of π'_i and π'_k and a quadric surface. The generators of one regulus are the images of the lines of the pencil i, and those of the other regulus are images of the lines through k. These quadrics all pass through eight basis points and form a singly infinite quadratic system. Their envelope is the surface Δ.*

§ 7. *Depiction of S_4 on Δ.*

18. We can now study the transformations which leave Δ invariant by depicting S_4 upon Δ and considering the transformations into which those enumerated in No. 9 are transformed. From the preceding section it follows immediately that the nodal projections (N_i) become those of interchanging the points of contact on the lines of congruences $[i]$.†

The six operations $[i]$ generate an abelian group of order 32. The product of any two of them is an axial homology. By No. 14 the operation T_3 of conjugate points as to γ_3 becomes the nodal projection (P'_{00}) of Δ into itself from P'_{00}. The involution I is transformed into a Jonquières transformation leaving every point of Δ invariant. To find the image of any point A', connect it with P'_{00}. The line $A'P'_{00}$ cuts Δ in M'_1, M'_2. The harmonic conjugate of A' as to $M'_1 M'_2$ is the required image.

19. In Σ', the nodal projection (P'_{ik}) can be obtained by transforming (P'_{00}) through $[i][k]$,

$$(P'_{ik}) = [i][k](P'_{00})[k][i].$$

In Σ this operation requires that a point be projected from i, its image from k, this image found as to γ_3, then the first two operations repeated in reverse order Pass two quadrics H_1, H_2 of the system (1) through the line (ik). Their

* The equations of these quadrics, the configuration of the associated basis points, and all the properties of the congruences of order and class two are at once derived by projecting the cubic variety having 10 nodes in four way space from a point upon it, as is outlined in my paper: *Surfaces derived from the cubic variety having nine double points in four dimensional space*, these Transactions, vol. 10 (1909), pp. 71-78. In the notation of that paper, $\lambda = 1$ will define as contour a general Kummer surface.

† This result was first found by H. F. Baker by another method. A different proof is given in my paper: *Infinite discontinuous groups of birational transformations which leave certain surfaces invariant*, these Transactions, vol. 11 (1910), pp. 15-24. See page 17 for further references.

residual intersection consists of a cubic passing through the other basis points, and cutting (ik) twice. The cubic is fixed by any point P on S_4. The images in Σ' of H_1, H_2 are two planes each passing through P'_{ik}. Their intersection consists of a straight line which is fixed by P', the image of P.

The fifteen operations C_{ik} are transformed into the fifteen nodal projections (P'_{ik}).

$$C_{ik} = (N_i)(N_k)T_3(N_k)(N_i).$$

20. Any two of these operations generate a group of infinite order. Moreover, the index of this group as to that generated by two pairs will also be infinite. This can be most easily proved in Σ'. Let $(P'_{12})(P'_{34}) = s$ and $(P'_{15})(P'_{36}) = t$.

Pass a plane through $P'_{12}P'_{34}$ and any point P' on Δ. This plane cuts Δ in a general quartic curve having two nodes. When P' is operated upon by s, its image will always lie on the same quartic curve. Pass a plane through P', P'_{15}, P'_{36}. It will cut Δ in a quartic curve which intersects the preceding one in three points A'_1, A'_2, A'_3 besides P'. The images of P' when operated upon by t will lie in the second plane. If any power of s were equal to any power of t, the corresponding position of P' would be at A'_i, hence these respective operations repeated four times would cause the point to return to P'. If the period is finite for one point it has been proved to be finite for every point, but s and t are each of infinite order. Hence no relation of the form $s^m = t^n$ can be found.

Now multiply the series $1, s, s^2, s^3 \cdots$ by $t, t^2, \cdots$ on the right. If $s^k t^l = s^p t^q$, then $s^{k-p} = t^{q-l}$, which relation was proved to be impossible. It follows that no operation $s^k t^l$ appears twice in this set, or that the index of the group generated by s and t is of infinite order as to the group generated by s. In the same manner we can obtain a group generated by s, t, u, having an infinite index as to that generated by s, t, etc. All the operations of the group can be expressed in terms of (P'_{00}) and the operators of the linear group G_{16}.

21. Since the operations C_i and (N_i) are transformed into each other by I, it follows that C_i goes into $[i]$ in Σ'.

22. The straight lines defining T_{yik} go into conics in Σ' passing through P'_{00} and P'_{ik}. Since the lines in Σ do not go into themselves when operated upon by I, it follows that the image conics in Σ' touch Δ in the two points which are images of the residual points of intersection of the line and S_4. Through any point P of S_4 can be drawn but one line cutting γ and (ik); it will cut S_4 in Q. The basis curve composed of γ and (ik) will define a pencil of quadric surfaces, one of which passes through P. This quadric goes into the plane passing through P'_{00} and P'_{ik}, cutting Δ in a binodal quartic curve. Through P' can be passed four conics, all of which touch Δ at P', pass through P'_{00}, P'_{ik}, and touch Δ in another point. The generators of the other regulus on the quadric in Σ go into ∞^1 conics which are proper contact conics, that do not pass through

the nodes. Any binodal quartic can be generated in four ways as the envelope of a conic passing through the nodes; its pole as to the line joining them describes a conic. The four loci of poles are inscribed in the same quadrilateral, having a vertex at each node. The series of conics are rationally separated when one position of the pole is known. The second point of contact of one conic is Q'. When P' describes Δ, Q' can be distinguished from the other points of contact by means of the equation of the locus of the pole. Each of the other series of contact conics will also define birational transformations, but these are not images of straight lines in Σ. The conjugate operation in I is defined by the cubics passing through four basis points and cutting γ in one variable point.

In the same way, the operations $T_{ik,lm}$ are transformed into the operations defined by the series of contact conics passing through P'_{ik}, P'_{lm}. They are obtained from the preceding series by the linear transformations $[i]\,[k]$. Now consider the lines (123), (14) in Σ. The former goes into a conic through P'_{12}, P'_{13}, P'_{23}, P'_{45}, P'_{46}, P'_{56}, or the singular conic in the plane π'_{123}. The line (14) goes into the point P'_{14}. Any line in Σ cutting (123), (14) goes into a conic in Σ' passing through P'_{14}, touching the conic of contact in the singular plane π'_{123}, and touching Δ in the images of the residual intersections of the line and S_4. As before, three extraneous series of conics appear.

23. The conjugate of the lines cutting (ik), (lm) in I are quintic curves through i, k, l, m simply, through the remaining basis points twice, having one variable point on (ik) and on (il). They are uniquely fixed by a point P on S_4 and pass through Q, the residual point of intersection of the line through P cutting (ik), (lm). As in the case of the cubics C_i, these curves could be used to define the transformation. Whenever the corresponding transformation in Σ' is defined by rational contact curves, two distinct systems of rational curves in Σ will define the same transformation of points on S_4. On the other hand a Cremona transformation in Σ' may not be expressible in rational form in Σ, except for points of S_4.

24. To obtain the operation into which X is transformed in Σ', it will be observed that the quadric passing through γ_3 and (ik) goes into a plane, and the section of the former with S_4 becomes a plane quartic curve, section of Δ, having nodes at P'_{00} and P'_{ik}. This section is fixed when one point upon Δ is given.

Through the conic passing through l, m can be passed a quadric belonging to (1), hence its image in Σ' is a plane. The plane of the conic is transformed into a Steiner surface having π'_l, π'_m as factors, hence a quadric. The generators of one regulus are the images of the pencil of lines through l in the plane of the conic, and those of the other are images of the lines through m. The image of the conic is therefore a conic. On Σ, the space quartic passes through all the basis points except i and k. A plane through any other two will cut the curve in but two other points. The image quadric will therefore have but two points in common

with the image plane curve, that is, the conic in the plane of section will touch the curve at the two points. Through any point in the section of the quadric with Δ can be drawn a conic touching the curve which its plane cuts from Δ at two fixed points, the chosen point P' and at one other point Q'. This is the transform of the operation X.

§ 8. *The Tetrahedroid.*

25. When the points (1, 2), (3, 5), (4, 6) are in involution, it was seen that the nets $\{\gamma_3\}$, $\{12\}$, $\{35\}$, $\{46\}$ have a quadric in common. This quadric becomes a plane in Σ' passing through the four points P'_{00}, P'_{12}, P'_{35}, P'_{46}. Since the section of Δ made by the plane is a quartic having four double points, the curve of section consists of two conics. By the sixteen linear operations which leave Δ invariant the plane of the four nodes can be transformed into three others, forming with the given one a tetrahedron. By the operations $[i]$, the dual configuration is obtained, such that four points exist through each of which pass four of the singular planes of Δ. These planes intersect in six lines, on each of which lie two nodes, hence the nodes on any singular conic are in involution.

If the six nodes of S_4 are in involution, the six nodes in every singular conic of Δ are in involution. S_4 now has an extra pair of lines and Δ has four pairs of conics.

26. The axial involution in Σ goes into the operation defined by interchanging the points of contact of a conic which touches Δ in the images of the axes of the involution and in two other points. The four points of contact are harmonic. Δ is now a tetrahedroid; since the lines joining two nodes define an involution in each singular plane, and four singular planes pass through the center of this involution, Δ is now invariant under the central involutions defined by the vertices and opposite faces of the tetrahedron formed by the planes containing pairs of conics on Δ. These operations cannot be defined in terms of the axial involutions in Σ. Let P' be the center and π' the invariant plane. The image of P' in Σ is a pair of points ξ', ξ'' not lying on S_4; the image of π' is the quadric containing u, v. A line through P' cuts π' in Π' and Δ in two pairs of points M'_i, each harmonic with $P'\Pi'$. The image of the line is a quartic curve passing through the eight associated points $1, 2, \cdots, 6, \xi', \xi''$. It cuts the quadric in two points η', η'' and S_4 in four points, images of M'_i.

27. When the points $1, \cdots, 6$ are in multiple involution, Δ is a multiple tetrahedroid. In the case of six fold involution the new central involutorial perspective transformations and the linear G_{16} generate a group of order $24 \cdot 16$.*

*SEGRE, *Sur un cas particulier de la surface de Kummer*, Leipziger Berichte, vol. 36 (1884), pp. 132-135.

ON SEMI-DISCRIMINANTS OF TERNARY FORMS*

BY

O. E. GLENN

§ 1. *Introduction.*

It is well known that the number of independent conditional relations which must exist among the coefficients of a ternary form of order m in order that it should be factorable into linear factors, distinct term for term, is $\frac{1}{2}m(m-1)$. Several writers,† among them BRILL, and GORDAN, have published methods for the determination of such sets of relations. Their results are, as a rule, expressed in the form of a covariant, the identical vanishing of which gives necessary and sufficient conditions for the factorability.

These methods are somewhat indirect, and from certain standpoints are unsatisfactory for the additional reason that the set of conditions given by the identical vanishing of such a covariant is always redundant.

Our aim in this paper has been to develop a direct method of attacking this problem. Our method leads to a set of conditional relations containing the exact minimum number $\frac{1}{2}m(m-1)$; that is, it leads to a set of $\frac{1}{2}m(m-1)$ independent seminvariants of the form, whose simultaneous vanishing gives necessary and sufficient conditions for the factorability. We shall call these seminvariants *semi-discriminants*‡ of the form. They are all of the same degree $2m-1$; and are readily formed for any order m as simultaneous invariants of a certain set of *binary* quantics related to the original ternary form.

If a polynomial, f_{3m}, of order m, and homogeneous in three variables (x_1, x_2, x_3) is factorable into linear factors, its terms in (x_1, x_2) must furnish the (x_1, x_2) terms of those factors. Call these terms collectively a^m_{0x}, and the terms linear in x_3 collectively $x_3 a^{m-1}_{1x}$. Then if the factors of the former were known, and were distinct, say

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† THAER, "*Ueber die Zerlegbarkeit einer ebenen Linie dritter Ordnung in gerade Linien,*" Mathematische Annalen, vol. 14 (1879); GORDAN, "*Das Zerfallen der Curven in gerade Linien,*" Mathematische Annalen, vol. 45 (1894); BRILL, "*Ueber die Zerfällung der Ternärformen in lineare Factoren,*" Mathematische Annalen, vol. 50 (1898).

‡ It seems desirable to introduce a new term for the members of such a minimum set of functions of the form's coefficients. Since the name discriminant is already in common use, and since discriminants are *invariants*, it seems natural to adopt here the term *semi-discriminant*.

$$a_{0x}^{m} \equiv a_{00} \prod_{i=1}^{m} (r_2^{(i)} x_1 - r_1^{(i)} x_2) \div \prod_{i=1}^{m} (r_2^{(i)}),$$

the second would give by rational means the terms in x_3 required to complete the several factors. For we could find rationally the numerators of the partial fractions in the decomposition of a_{1x}^{m-1}/a_{0x}^{m}, viz.:

$$\frac{a_{1x}^{m-1}}{a_{0x}^{m}} \equiv \frac{\prod_{i=1}^{m} r_2^{(i)}}{a_{00}} \sum_{i=1}^{m} \frac{\alpha_i}{r_2^{(i)} x_1 - r_1^{(i)} x_2},$$

and the factors of the complete form will be, of course,

$$r_2^{(i)} x_1 - r_1^{(i)} x_2 + \alpha_i x_3 \qquad (i = 1, 2, \cdots, m).$$

Further, the coefficients of all other terms in f_{3m} are rational integral functions of the $r^{(i)}$ on the one hand, and of the α_i on the other — symmetrical in the sets $(r_2^{(i)}, -r_1^{(i)}, \alpha_i)$. We shall show in general that all these coefficients in the case of any linearly factorable form are *rationally* expressible in terms of those occurring in a_{0x}^{m}, a_{1x}^{m-1}. Hence will follow the important theorem (§ 5):

Theorem 1. *If a ternary form f_{3m} is decomposable into linear factors, all its coefficients, after certain $2m$, are expressible rationally in terms of those $2m$ coefficients. That is, in the space whose coördinates are all the coefficients of ternary forms of order m, the forms composed of linear factors fill a rational spread of $2m$ dimensions.*

We shall thus obtain the explicit form of the general ternary quantic which is factorable into linear factors (§ 5). Moreover, in case f_{3m} is not factorable a similar development will give the theorem (§ 3),

Theorem 2. *Every ternary form f_{3m}, for which the discriminant D of a_{0x}^{m} does not vanish, can be expressed as the sum of the product of m distinct linear forms, plus the square of an arbitrarily chosen linear form, multiplied by a "satellite" form of order $m-2$ whose coefficients are, except for the factor D^{-1}, integral rational seminvariants of the original form f_{3m}.*

§ 2. *A class of ternary seminvariants.*

Let us write the general ternary quantic in homogeneous variables as follows:

$$f_{3m} = a_{0x}^{m} + a_{1x}^{m-1} x_3 + a_{2x}^{m-2} x_3^2 + \cdots + a_{m0} x_3^{m},$$

where

$$a_{ix}^{m-i} \equiv a_{i0} x_1^{m-i} + a_{i1} x_1^{m-i-1} x_2 + a_{i2} x_1^{m-i-2} x_2^2 + \cdots + a_{im-i} x_2^{m-i} \quad (i = 1, 2, \cdots, m).$$

Then write

$$\frac{a_{1x}^{m-1}}{a_{0x}^{m}} = \frac{a_{1x}^{m-1}}{\prod_{k=1}^{m} (r_2^{(k)} x_1 - r_1^{(k)} x_2)} = \sum_{k=1}^{m} \frac{\alpha_k}{r_2^{(k)} x_1 - r_1^{(k)} x_2} \quad (a_{00} = r_2^{(1)} r_2^{(2)} \cdots r_2^{(m)});$$

and we have in consequence, assuming that $D \neq 0$, and writing

$$a'^{m}_{0r^{(k)}} = \left[\frac{\partial a^{m}_{0x}}{\partial x_1}\right]_{x_1 = r_1^{(k)},\ x_2 = r_2^{(k)}}, \qquad a''^{m}_{0r^{(k)}} = \left[\frac{\partial a^{m}_{0x}}{\partial x_2}\right]_{x_1 = r_1^{(k)},\ x_2 = r_2^{(k)}},$$

the results

$$\alpha_k = r_2^{(k)} a^{m-1}_{1r^{(k)}} / a'^{m}_{0r^{(k)}} = -\, r_1^{(k)} a^{m-1}_{1r^{(k)}} / a''^{m}_{0r^{(k)}}. \tag{1}$$

Hence also

$$a''^{m}_{0r^{(k)}} = -\frac{r_1^{(k)}}{r_2^{(k)}} a'^{m}_{0r^{(k)}}. \tag{2}$$

The discriminant of a^{m}_{0x} can be expressed in the following form:

$$D = \prod_{j=1}^{m} a'^{m}_{0r^{(j)}} / a_{00}(-1)^{\frac{1}{2}m(m-1)}, \tag{3}$$

and therefore

$$\alpha_k = \frac{r_2^{(k)} a^{m-1}_{1r^{(k)}} a'^{m}_{0r^{(1)}} a'^{m}_{0r^{(2)}} \cdots a'^{m}_{0r^{(k-1)}} a'^{m}_{0r^{(k+1)}} \cdots a'^{m}_{0r^{(m)}}}{a_{00}(-1)^{\frac{1}{2}m(m-1)} D}, \tag{4}$$

and in like manner we get

$$\prod_{k=1}^{m} \alpha_k = a^{m-1}_{1r^{(1)}} a^{m-1}_{1r^{(2)}} \cdots a^{m-1}_{1r^{(m)}} / (-1)^{\frac{1}{2}m(m-1)} D. \tag{5}$$

The numerator of the right hand member of this last equality is evidently the resultant (say R_m) of a^{m}_{0x} and a^{m-1}_{1x}.

Consider next the two differential operators

$$\Delta_1 = m a_{00} \frac{\partial}{\partial a_{10}} + (m-1) a_{01} \frac{\partial}{\partial a_{11}} + \cdots + a_{0m-1} \frac{\partial}{\partial a_{1m-1}},$$

$$\Delta_2 = m a_{0m} \frac{\partial}{\partial a_{1m-1}} + (m-1) a_{0m-1} \frac{\partial}{\partial a_{1m-2}} + \cdots + a_{01} \frac{\partial}{\partial a_{10}};$$

and particularly their effect when applied to a^{m-1}_{1x}. We get [see (2)]

$$\Delta_1 a^{m-1}_{1r^{(k)}} = a'^{m}_{0r^{(k)}}, \qquad \Delta_2 a^{m-1}_{1r^{(k)}} = a''^{m}_{0r^{(k)}} = -\frac{r_1^{(k)}}{r_2^{(k)}} a'^{m}_{0r^{(k)}}, \tag{6}$$

and from these relations we deduce the following:

$$\Delta_1 \prod_{k=1}^{m} \alpha_k = \frac{\Delta_1 R_m}{(-1)^{\frac{1}{2}m(m-1)} \lfloor 1 D} = a_{00} \Sigma \frac{a^{m-1}_{1r^{(1)}} a^{m-1}_{1r^{(2)}} \cdots a^{m-1}_{1r^{(m-1)}}}{a'^{m}_{0r^{(1)}} a'^{m}_{0r^{(2)}} \cdots a'^{m}_{0r^{(m-1)}}}, \tag{7}$$

or, from (1)

$$\frac{\Delta_1 R_m}{(-1)^{\frac{1}{2}m(m-1)} \lfloor 1 D} = \Sigma \alpha_1 \alpha_2 \cdots \alpha_{m-1} r_2^{(m)}. \tag{8}$$

In (7) the symmetric function Σ is to be read with reference to the r's, the super-

scripts of the r's replacing the subscripts usual in a symmetric function. Let us now operate with Δ_2 on both members of (7). This gives

$$\frac{\Delta_1\Delta_2 R_m}{(-1)^{\frac{1}{2}m(m-1)}\underline{|1}\,D} = a_{00}\Sigma\frac{a^{m-1}_{1r^{(1)}}a^{m-1}_{1r^{(2)}}\cdots a^{m-1}_{1r^{(m-2)}}}{a'^{m}_{0r^{(1)}}a'^{m}_{0r^{(2)}}\cdots a'^{m}_{0r^{(m-2)}}}\left(-\frac{r_1^{(m-1)}}{r_2^{(m-1)}}-\frac{r_1^{(m)}}{r_2^{(m)}}\right).$$

Let Σ_h represent an elementary symmetric function of the two groups of homogeneous variables r_1, r_2 which involves h distinct letters of each group, viz: $r_i^{(m-j+1)}$ $(j=1, 2, \cdots, h)$. Then we have

$$\frac{\Delta_1\Delta_2 R_m}{(-1)^{\frac{1}{2}m(m-1)}\underline{|1}\,\underline{|1}\,D} = \Sigma[(-1)\alpha_1\alpha_2\cdots\alpha_{m-2}\Sigma_2 r_1^{(m-1)}r_2^{(m)}]. \tag{9}$$

We are now in position to prove by induction the following fundamental formula:

$$\frac{\Delta_1^{m-s-t}\Delta_2^{t}R_m}{(-1)^{\frac{1}{2}m(m-1)}\underline{|m-s-t}\,\underline{|t}\,D} = \Sigma[(-1)^t\alpha_1\alpha_2\cdots\alpha_s\Sigma_{m-s}r_1^{(s+1)}r_1^{(s+2)}\cdots r_1^{(s+t)}r_2^{(s+t+1)}\cdots r_2^{(m)}] \tag{10}$$

$$(s=0, 1, \cdots, m;\ t=0, 1, \cdots, m-s),$$

where the outer summation covers all subscripts from 1 to m, superscripts of the r's counting as subscripts in the symmetric function. Representing by $J_{m-s-t,t}$ the left hand member of this equality we have from (6)

$$\Delta_2 J_{m-s-t,t} = \Sigma\left((-1)^{t+1}\frac{a^{m-1}_{1r^{(1)}}a^{m-1}_{1r^{(2)}}\cdots a^{m-1}_{1r^{(s-1)}}}{a'^{m}_{0r^{(1)}}a'^{m}_{0r^{(2)}}\cdots a'^{m}_{0r^{(s-1)}}}\right.$$

$$\left.\times r_2^{(1)}r_2^{(2)}\cdots r_2^{(s)}\frac{r_1^{(s)}}{r_2^{(s)}}\Sigma_{m-s}r_1^{(s+1)}\cdots r_1^{(s+t)}r_2^{(s+t+1)}\cdots r_2^{(m)}\right).$$

This equals

$$\Sigma(-1)^{t+1}\alpha_1\alpha_2\cdots\alpha_{s-1}S,$$

where S is a symmetric function each term of which involves $t+1$ letters r_1 and $m-s-t$ letters r_2. The number of terms in an elementary symmetric function of any number of groups of homogeneous variables equals the number of permutations of the letters occurring in any one term when the subscripts (here superscripts) are removed. Hence the number of terms in Σ_{m-s} is

$$\frac{\underline{|m-s}}{\underline{|m-s-t}\,\underline{|t}},$$

and the number of terms in S is

$$(m-s+1)\underline{|m-s}/\underline{|t}\,\underline{|m-s-t}.$$

But the number of terms in $\Sigma_{m-s+1}(r_1^{(s)} r_1^{(s+1)} \cdots r_1^{(s+t)} r_2^{(s+t+1)} \cdots r_2^{(m)})$ is

$$\lfloor m-s+1 \rfloor / \lfloor m-s-t \rfloor t+1.$$

Hence

$$S = (t+1)\Sigma_{m-s+1},$$

and so

$$\frac{\Delta_2 J_{m-s-t,\,t}}{t+1} = \Sigma\left[(-1)^{t+1}\alpha_1\alpha_2\cdots\alpha_{s-1}\Sigma_{m-s+1}\right].$$

This result, with (9), completes the inductive proof of formula (10).

Now the functions $J_{m-s-t,\,t}$ are evidently simultaneous invariants of the binary forms a_{0x}^m, $a_{0x}'^m$, $a_{0x}''^m$, a_{1x}^{m-1}. We shall show in the next section that the expressions

$$I_{m-s-t,\,t} \equiv Da_{st} - DJ_{m-s-t,\,t} \quad (s=2,\,3,\,\cdots,\,m\,;\ t=0,\,1,\,\cdots,\,m-s)$$

are, in reality, seminvariants of the form f_{3m} as a whole.

§ 3. *Structure of a ternary form.*

The structure of the right hand member of the fundamental equality (10) shows at once that the general (factorable or non-factorable) quantic f_{3m} $(D \neq 0)$ can be reduced to the following form:

$$(11)\qquad f_{3m} = \prod_{k=1}^{m}(r_2^{(k)}x_1 - r_1^{(k)}x_2 + \alpha_k) + \sum_{s=2}^{m}\sum_{t=0}^{m-s}(a_{st} - J_{m-s-t,\,t})x_1^{m-s-t}x_2'^t.$$

This gives explicitly the general "satellite" form of f_{3m}, with coefficients expressed rationally in terms of the coefficients of f_{3m}. It may be written

$$(12)\qquad D\mu_{m-2} = \sum_{s=2}^{m}\sum_{t=0}^{m-s}\left(Da_{st} - \frac{\Delta_1^{m-s-t}\Delta_2^t R_m}{(-1)^{\frac{1}{2}m(m-1)}\lfloor m-s-t \rfloor t}\right)x_1^{m-s-t}x_2'^t = \sum_{s=2}^{m}\sum_{t=0}^{m-s} I_{m-s-t,\,t}x_1^{m-s-t}x_2'^t.$$

Now the coefficients $I_{m-s-t,\,t}$ are seminvariants of f_{3m}. To fix ideas let $m=3$ and write the usual set of ternary operators,

$$\Omega_{x_1x_2} = a_{01}\frac{\partial}{\partial a_{00}} + 2a_{02}\frac{\partial}{\partial a_{01}} + 3a_{03}\frac{\partial}{\partial a_{02}} + a_{11}\frac{\partial}{\partial a_{10}} + 2a_{12}\frac{\partial}{\partial a_{11}} + a_{21}\frac{\partial}{\partial a_{20}},$$

$$\Omega_{x_2x_1} = 3a_{00}\frac{\partial}{\partial a_{01}} + 2a_{01}\frac{\partial}{\partial a_{02}} + a_{02}\frac{\partial}{\partial a_{03}} + 2a_{10}\frac{\partial}{\partial a_{11}} + a_{11}\frac{\partial}{\partial a_{12}} + a_{20}\frac{\partial}{\partial a_{21}},$$

$$\Omega_{x_3x_1} = a_{20}\frac{\partial}{\partial a_{30}} + 2a_{10}\frac{\partial}{\partial a_{20}} + 3a_{00}\frac{\partial}{\partial a_{10}} + Da_{11}\frac{\partial}{\partial a_{21}} + 2a_{01}\frac{\partial}{\partial a_{11}} + Da_{02}\frac{\partial}{\partial a_{12}},$$

etc. . . . . .

Then I_{10} is annihilated by $\Omega_{x_2x_1}$ but not by $\Omega_{x_1x_2}$, I_{01} is annihilated by $\Omega_{x_1x_2}$ but not by $\Omega_{x_2x_1}$, and I_{00} is annihilated by $\Omega_{x_1x_2}$ but not by $\Omega_{x_3x_1}$. In general $I_{m-s-t,\,t}$ fails of annihilation when operated upon by a general operator $\Omega_{x_ix_j}$ which contains a partial derivative with respect to a_{st}. We have now proved theorem 2.

§ 4. *The semi-discriminants.*

A necessary and sufficient condition that f_{3m} should degenerate into the product of m distinct linear factors is that μ_{m-2} should vanish identically. Hence, since the number of coefficients in μ_{m-2} is $\frac{1}{2}m(m-1)$, these equated to zero give a minimum set of conditions in order that f_{3m} should be factorable in the manner stated. As previously indicated we refer to these seminvariants as a set of semi-discriminants of the form f_{3m}. They are

$$(13)\qquad I_{m-s-t,\,t} = Da_{st} - \frac{\Delta_1^{m-s-t}\Delta_2^{t}R_m}{(-1)^{\frac{1}{2}m(m-1)}\,\underline{t}\,\underline{m-s-t}} \qquad \begin{pmatrix} s=2,\,3,\,\cdots,\,m; \\ t=0,\,1,\,\cdots,\,m-s \end{pmatrix}.$$

They are obviously independent since each one contains a coefficient (a_{st}) not contained in any other. They are free from adventitious factors, and each one is of degree $2m-1$.

In the case where $m=2$ we have

$$I_{00} = -a_{20}\begin{vmatrix} 2a_{00} & a_{01} \\ a_{01} & 2a_{02} \end{vmatrix} + \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & 0 \\ 0 & a_{10} & a_{11} \end{vmatrix}.$$

This is also the *ordinary* discriminant of the ternary quadratic.

The three semi-discriminants of the ternary cubic have been computed by the author by another method.* Corresponding results for the case $m=4$ have not been published. They are the following:

$$I_{00} = \tfrac{1}{2}\tfrac{1}{7}a_{40}(4i_1^3 - J_1^2) - R_4,$$

where

$$i_1 = a_{02}^2 - 3a_{01}a_{03} + 12a_{00}a_{04},$$

$$J_1 = 27a_{01}^2a_{04} + 27a_{00}a_{03}^2 + 2a_{02}^3 - 72a_{00}a_{02}a_{04} - 9a_{01}a_{02}a_{03},$$

$$R_4 = \begin{vmatrix} a_{10} & a_{11} & a_{12} & a_{13} & 0 & 0 \\ 0 & a_{10} & a_{11} & a_{12} & a_{13} & 0 \\ 0 & 0 & a_{10} & a_{11} & a_{12} & a_{13} \\ a_{01}a_{10} - a_{00}a_{11} & a_{02}a_{10} - a_{00}a_{12} & a_{03}a_{10} - a_{00}a_{13} & a_{04}a_{10} & 0 & 0 \\ a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & 0 \\ 0 & a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \end{vmatrix},$$

*American Journal of Mathematics, vol. 32 (1910), p. 89.

the other members of the set being obtained by operating upon R_4 with powers of Δ_1, Δ_2:

$$\Delta_1 = 4a_{00}\frac{\partial}{\partial a_{10}} + 3a_{01}\frac{\partial}{\partial a_{11}} + 2a_{02}\frac{\partial}{\partial a_{12}} + a_{03}\frac{\partial}{\partial a_{13}},$$

$$\Delta_2 = 4a_{04}\frac{\partial}{\partial a_{13}} + 3a_{03}\frac{\partial}{\partial a_{12}} + 2a_{02}\frac{\partial}{\partial a_{11}} + a_{01}\frac{\partial}{\partial a_{10}},$$

according to the formula

$$I_{4-s-t,\,t} = a_{st}D - \frac{\Delta_1^{4-s-t}\Delta_2' R_4}{\lfloor 4-s-t \rfloor \lfloor t} \qquad (s=2,\,3,\,4\,;\ t=0,\,1,\,\cdots,\,4-s).$$

§ 5. *Proof of Theorem* 1.

The factors of a_{0x}^m being assumed distinct we can always solve $I_{m-s-t,\,t}=0$ for a_{st}, the result being obviously rational in the coefficients occurring in a_{0x}^m, a_{1x}^{m-1}. This proves theorem 1, as far as the case $D \neq 0$ is concerned. Moreover by carrying the resulting values of a_{st} ($s=2,\,3,\,\cdots,\,m\,;\ t=0,\,1,\,\cdots,\,m-s$) back into f_{3m} we get the general form of a ternary quantic which is factorable into linear forms. In the result a_{0x}^m, a_{1x}^{m-1} are perfectly general (the former, however, subject to the negative condition $D \neq 0$), whereas

$$(-1)^{\frac{1}{2}m(m-1)} D a_{jx}^{m-j} \equiv \frac{\Delta_1^{m-j} R_m}{\lfloor m-j} x_1^{m-j} + \frac{\Delta_1^{m-j-1}\Delta_2 R_m}{\lfloor m-j-1 \rfloor 1} x_1^{m-j-1}x_2 + \cdots + \frac{\Delta_2^{m-j} R_m}{\lfloor m-j} x_2^{m-j}$$

$$(j=2,\,3,\,\cdots,\,m).$$

Assume next that $D=0$. Then there are two cases to consider. First, a_{0x}^m has multiple factors but f_{3m} as a whole has no multiple linear ternary factors, and a mere interchange of subscripts of the variables (x_1, x_2, x_3) transforms f_{3m} into a new quantic whose binary a_{0x}^m has no multiple factors. For this new quantic, then, $D \neq 0$. Secondly, f_{3m} as a whole may have repeated ternary linear factors. Let there be one such D factor, of multiplicity two. Then $J_{m-s-t,\,t}$, which we now write in the briefer form

$$J_{m-s-t,\,t} = \frac{\Delta_{st} R_m(a_{00},\ a_{01},\ \cdots)}{D(a_{00},\ a_{01},\ \cdots)},$$

is indeterminate. In fact, in this case, two of the α_i in the right hand member of (10), say α_1, α_2, are to be replaced by one and the same quantity

$$\alpha_{12} \equiv \frac{r_2^{(1)}\dfrac{\partial}{\partial r_1^{(1)}} a_{1r^{(1)}}^{m-1}}{\dfrac{\partial}{\partial r_1^{(1)}} a_{0r^{(1)}}^{m}}.*$$

*See Bulletin of the American Mathematical Society, vol. 17 (1911), p. 451, theorem in § 2.

Then it is not difficult to show that the corresponding true value* of $J_{m-s-t,\,t}$, and hence of a_{st}, is

$$a_{st}' = \frac{\Delta_{st}\dfrac{\partial}{\partial a_{0i}} R_m(a_{00}, \cdots)}{\dfrac{\partial}{\partial a_{0i}} D(a_{00}, \cdots)} \quad (i = \text{any number of the set } 0, 1, \cdots, m).$$

Likewise, when f_{3m} contains a linear factor of multiplicity three, three of the α_i in (10), say α_1, α_2, α_3 are to be replaced by the same quantity, viz.:

$$\alpha_{123} \equiv \frac{r_2^{(1)} \dfrac{\partial^2}{\partial r_1^{(1)2}} a_{1r^{(1)}}^{m-1}}{\dfrac{\partial^2}{\partial r_1^{(1)2}} a_{0r\,1)}^{m}}.$$

and then we get

$$a_{st} = \frac{\Delta_{st}\dfrac{\partial^2}{\partial a_{0i}^2} R_m(a_{00}, \cdots)}{\dfrac{\partial^2}{\partial a_{0i}^2} D(a_{00}, \cdots)} \quad (i = \text{any number of the set } 0, 1, \cdots, m).$$

Similarly in the case of a factor of f_{3m} of multiplicity higher than three. Hence in these cases also a_{st} is rational in the coefficients of a_{0x}^m, a_{1x}^{m-1}.

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* Cf. SALMON, *Modern Higher Algebra* (fourth edition), p. 90.

ON THE LIMIT OF THE DEGREE OF PRIMITIVE GROUPS*

BY

W. A. MANNING

To the theory of multiply transitive groups BOCHERT has contributed formulæ for the upper limit of the degree of a group when the number of letters displaced by a substitution in it is known.† But for simply transitive primitive groups a corresponding theorem of such elegance does not exist.‡ However, in the present paper is offered a theorem with some claims to simplicity. It may be stated thus:

If a simply transitive primitive group contains a substitution of prime order p and of degree pq (q less than $2p+3$), its degree is not greater than the larger of the two numbers $qp+q^2-q$, $2q^2-p^2$.

Also when the given substitution of prime order has more than $2p+2$ and less than p^2 cycles or when p is 2 or 3, our results are capable of fairly concise expression; but when q exceeds p^2-1 and p is greater than 3 the corresponding upper limit is unfortunately a more complicated function of p and q.

The subject matter of this study is not restricted to the simply transitive primitive groups, but it is proposed to find an upper limit for the degree of some transitive subgroup of a primitive group known to contain a substitution of order p on qp letters. It seems unnecessary to state that a simply transitive primitive group has no transitive subgroup of a lower degree.

A property of transitive groups generated by substitutions of prime order.

1. Let $s_1, s_2, \cdots$ be certain substitutions of prime order that generate a transitive group G.§ For the sake of brevity of statement we may assume that all substitutions of G which are similar to $s_1, s_2, \cdots$ have been added to and

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† BOCHERT, Mathematische Annalen, vol. 40 (1892), pp. 176–193; vol. 49 (1897), pp. 133–144.

‡ JORDAN, Bulletin de la Société Mathématique de France, vol. 1 (1873), pp. 175–221; Crelle's Journal, vol. 79 (1874), 248–258. The latter should be read in connection with the present paper. See also MANNING, Bulletin of the American Mathematical Society, vol. 13 (1907), p. 373.

§ In the American Journal of Mathematics, vol. 28 (1906), p. 226, the author imposed the condition that G is invariant in a primitive group. In these Transactions, vol. 10 (1909), p. 247, that condition was removed, but it was assumed that no substitution $s_1, s_2, \cdots$ has more than $p-1$ cycles. In the present investigation both these restrictions are removed. Note that the substitutions $s_1, s_2, \cdots$ are not said to be similar or yet of the *same* prime order.

are included in the series $s_1, s_2, \cdots$. Of these substitutions let $s_1, s_2, \cdots, s_i$, say, generate an intransitive subgroup H_i of G. If an arbitrary choice is made of a transitive constituent of H_i, it will ordinarily be possible to find among the substitutions of the series $s_1, s_2, \cdots$ a substitution s_{i+1} which replaces one of the letters of the arbitrarily chosen constituent by a letter of some other set of H_i. We seek the conditions under which it is impossible to affirm the existence of such a substitution.

Let the selected set of letters of H_i be $a_1, a_2, \cdots, a_{n_i}$. By hypothesis there is no substitution in the series $s_1, s_2, \cdots$ which replaces a letter a by a letter b, where b represents any letter displaced by H_i but not belonging to the set a. However, there must be at least one substitution in the series $s_1, s_2, \cdots$ that replaces an a by a letter α new to H_i, since G is transitive, and otherwise one of these generators would replace an a by a b. It is conceivable that there are a number of substitutions s_{i+1} that replace an a by an α. Let us select from $s_1, s_2, \cdots$ one that adds the least possible number of new letters to the set a. We may impose a second independent condition that s_{i+1} shall displace, in the cycles which do not involve letters a, as small a number of new letters β as possible. If no substitution s_{i+2} unites the extended set a of H_{i+1} to some other set of H_{i+1}, we select s_{i+2} in the same way. We may continue thus until we can find a substitution s_{k+2} which replaces a letter of the extended set $a_1, a_2, \cdots$ of H_{k+1} by a letter of some other transitive constituent of H_{k+1}. Let $a_1, a_2, \cdots, a_{n_k}$ be the letters of the set $a_1, \cdots$ of H_k, which we shall call the set a.

2. If p, the order of s_{k+1}, is an odd prime, and if s_{k+1} may be so chosen that it has not more than one letter new to H_k in any cycle, then we can show as follows that there is a substitution s in the series $s_1, s_2, \cdots$ which replaces a letter of the set a of H_k by a letter of another set of H_k, contrary to the hypothesis on which the group H_{k+1} was set up. Thus of course we may not have $s_{k+2} = (a_1 b_1 \cdots)\cdots$, where b_1 is one of the letters (b) of H_k not included in the set a. If $s_{k+2} = (\alpha_1 \beta_1 \cdots)\cdots$, where α_1 is a part of the set $a_1, a_2, \cdots$ of H_{k+1}, and β_1 is a letter of H_{k+1} but not in the set $a_1, a_2, \cdots$ nor yet in H_k, then $s_{k+1}^{-1} s_{k+2} s_{k+1}$ replaces an a of H_k by a b of H_k. If $s_{k+2} = (a_1 \beta_1 \cdots)\cdots$ or $(\alpha_1 b_1 \cdots)\cdots$, the same is true of $s_{k+1}^{-1} s_{k+2} s_{k+1}$ or else of $s_{k+1} s_{k+2} s_{k+1}^{-1}$. If $p = 2$ and s_{k+1} replaces an a by an a, the same result follows. For let $s_{k+2} = (a_1' \beta \cdots$ and $s_{k+1} = (a' a'')(b' \beta)\cdots$, where perhaps $a' = a''$. Some substitution H_k transforms s_{k+2} into $(a'' \beta \cdots$ and $s_{k+1}^{-1}(a'' \beta \cdots) s_{k+1} = (a' b' \cdots)$.

3. Now consider the substitution s_{k+1}. It bears the same relation to H_k as s_{i+1} does to H_i. We first note that every power of s_{k+1} replaces an a by an α. If there are two letters α in a cycle of s_{k+1}, in a certain power s_{k+1}^r these two letters α_1 and α_2 are adjacent. Now one of the generators of $s_{k+1}^{-r} H_k s_{k+1}^r$ will replace a letter a by an α unless s_{k+1}^r replaces each one of the n_k letters $a_1, a_2, \cdots, a_{n_k}$ by a letter α, new to H_k; if every a is followed by an α and if

there are two letters a in the same cycle, $s_{k+1}^{r} = (a_1\alpha' \cdots a_2\alpha'' \cdots)\cdots$ and $s_{k+1}^{xy} = (a_1 a_2 \cdots \alpha'\alpha'' \cdots)\cdots$. *Hence* s_{k+1} *either has at most one* α *in any cycle, and also at most one* β *in any one of the other cycles, or* s_{k+1} *is of the type*

$$(A)\quad (a_1\alpha_1'\alpha_1'' \cdots \alpha_1^{(p-1)})(a_2\alpha_2'\alpha_2' \cdots \alpha_2^{(p-1)}) \cdots (a_{n_k}\alpha_{n_k}'\alpha_{n_k}'' \cdots \alpha_{n_k}^{(p-1)}) \cdots;$$

and in this second case we should note that the minimum number of new letters displaced in cycles with $a_1, a_2, \cdots, a_{n_k}$ by any substitution of the series $s_1, s_2, \cdots$, which connects letters of the set a of H_k with other letters, is equal to the maximum number of such new letters.

4. It may be possible to find a substitution t that replaces an a by an a and an a by some letter ω not one of the letters $a_1, a_2, \cdots, a_{n_k}$, and such that *both* $t^{-1}H_k t$ *and* $tH_k t^{-1}$ *are subgroups of* G. Consider the group $t^{-1}H_k t$. One of the generators σ (a member of the series $s_1, s_2, \cdots$) of $t^{-1}H_k t$ replaces an a by a letter not belonging to the set a. This is clear since the set of letters in $t^{-1}H_k t$ which takes the place of the n_k letters a of H_k is now composed of two classes of letters, letters a and *not-a*. Then σ is of the type (A) with the n_k letters a involved in exactly n_k cycles, and the remaining places in these cycles are filled by letters new to H_k. Hence

$$t = (a\alpha' \cdots \alpha''\alpha \cdots b\alpha' \cdots),$$

that is, t replaces an a by an a, an a by a new letter α, and some letter b of H_k by a letter new to H_k, so that tHt^{-1} leaves fixed the letter α'', and one of its generators (belonging to $s_1, s_2, \cdots$) replaces an a by a *not-a*. This *not-a* cannot, by hypothesis, be a b from H_k, and is therefore an α. But since $tH_k t^{-1}$ leaves one a fixed, this generator is not of the type (A). *Hence, whenever a substitution* t *can be set up satisfying the conditions stated, the substitution* s_{k+1} *of prime order* p *has at most one new letter in any cycle.*

This is trivial when $p = 2$, for then the maximum number of new letters α in a cycle is one. But suppose that s_{k+1} is of type (A):

$$s_{k+1} = (a_1\alpha_1)(a_2\alpha_2) \cdots (a_{n_k}\alpha_{n_k}) \cdots,$$

and let us consider the group $t^{-1}H_k t$. From this it follows that t has the form displayed above. The transform of H_k by t^{-1} gives a substitution belonging to the series $s_1, s_2, \cdots$ which leaves fixed one of the letters a, and replaces at least one of them by a letter α. Thus when $p = 2$ the minimum number of new letters α is not in excess of $n_k - 1$. If there are two letters new to H_k in any cycle of s_{k+1} (of order 2), $s_{k+1}^{-1}H_k s_{k+1}$ leaves those two new letters fixed and one of its generators replaces an a by an a and an a by an α, since $s_{k+1}^{-1}H_k s_{k+1}$ has not more new letters in the set which takes the place of a than the number of new letters that s_{k+1} connects with the a's, that is, than $n_k - 1$.

5. It will be convenient to have a summary of the preceding results. We recall that $s_1, s_2, \cdots$ are substitutions of prime order that generate a transitive group G, and that $s_1, s_2, \cdots, s_i$ generate an intransitive subgroup H_i of G. The letters $a_1, a_2, \cdots, a_{n_i}$ compose a given set of intransitivity of H_i. If it is impossible to find a substitution in $s_1, s_2, \cdots$ that connects another set of H_i with the set a, we pass to the study of the group H_k. Now H_k coincides with or includes H_i and while the set $a_1, a_2, \cdots, a_{n_k}$ may contain more letters than the set $a_1, a_2, \cdots, a_{n_i}$ of H_i, it includes no letter of any other set of H_i. *But if there is no substitutiion among* $s_1, s_2, \cdots$ *possessing the required property (of connecting the set* $a_1, a_2, \cdots$ *of* H_k *with another set of* H_k*), all the substitutions of the series* $s_1, s_2, \cdots$ *that replace one of the letters* $a_1, a_2, \cdots, a_{n_k}$ *by a letter not a member of this set must be of the type*

$$(A) \quad (a_1, \alpha_1', \alpha_1'', \cdots, \alpha_1^{(p-1)})(a_2\alpha_2'\alpha_2'' \cdots \alpha_2^{(p-1)}) \cdots (a_{n_k}\alpha_{n_k}'\alpha_{n_k}'' \cdots \alpha_{n_k}^{(p-1)}) \cdots .$$

None of the letters α in (A) are displaced by H_k, or *a fortiori*, by H_i.

But we have another condition. Among $s_1, s_2, \cdots$ there certainly exists a substitution that replaces an a of H_k by a letter of another set of H_k, provided a substitution t can be set up that replaces an a by an a, and an a by a *not-a*, and having in addition the property that the two transformed groups $t^{-1}H_k t$ and $tH_k t^{-1}$ are both subgroups of G. In particular G itself contains such a substitution t whenever $a_1, a_2, \cdots, a_{n_k}$ do not form a system of imprimitivity of G.

This is only a partial solution of the problem proposed, but it is sufficient for our present purpose.

Other properties of the prime generators of a transitive group.

6. Suppose that there is in the series $s_1, s_2, \cdots$ a substitution s_{i+1} that replaces a letter of any given set of H_i by a letter of H_i not belonging to that set. From all the substitutions of the series $s_1, s_2, \cdots$ which replace an a by a b, let us select those which displace the minimum number of new letters. Now suppose that one of these substitutions thus selected is such that every power replaces an a by an a. *From this hypothesis we shall conclude that* s_{i+1} *has at most one new letter in any cycle.* Let new letters be denoted by α, whether they are transitively connected with $a_1, a_2, \cdots$ or not. If s_{i+1} has two letters adjacent in any cycle, $s_{i+1}^{-1}H_i s_{i+1}$ leaves fixed the second of them, so that by hypothesis $H' = \{H_i, s_{i+1}^{-1}H_i s_{i+1}\}$ can have no generator (belonging to $s_1, s_2, \cdots$) which replaces an a by a b. But H' has letters a and letters b in the same constituent, and all that we can affirm of this constituent is that it is a transitive group generated by certain substitutions of prime order. Bearing in mind the results of § 5, we know that there must be a set of letters $a_1, a_2, \cdots, a_{n_k}$ including the set $a_1, a_2, \cdots, a_{n_i} (n_i \leqq n_k)$, such that all the generators of H' which replace a letter of the set $a_1, a_2, \cdots, a_{n_k}$ by a letter

not in that set are of the type (A). Now s_{i+1} replaces an a by an a and an a by a b, that is by a letter not in the set $a_1, a_2, \cdots, a_{n_k}$. By hypothesis s_{i+1} has the three sequences $a_1 a_2,\ a_3 b_1,\ \alpha_1 \alpha_2$. Since $s_{i+1}^{-1} H_i s_{i+1}$ has at least one substitution of type (A), s_{i+1} must also have the two other sequences $a_4 \alpha_3$ and $b_2 a_5$. Now consider the group $H'' = \{ H_i,\ s_{i+1} H_i s_{i+1}^{-1} \}$. Since H'' does not displace α_1, no substitution (belonging to $s_1, s_2, \cdots$) of H'' replaces an a by a b. But $s_{i+1} H_i s_{i+1}^{-1}$ leaves fixed a_4 as well as α_1 and a generator certainly replaces an a by a *not-a*, no matter how far we have extended the original set $a_1, a_2, \cdots, a_{n_i}$ of H_i by new letters. That is to say, when we have formed the extended set $a_1, a_2, \cdots, a_{n_k}$ in the constituent $a_1, a_2, \cdots$ of H'', among the generators of $s_{i+1} H_i s_{i+1}^{-1}$ there is a substitution connecting other letters with the letters $a_1, a_2, \cdots, a_{n_k}$ and which is not of the type (A), inasmuch as it leaves fixed one of the letters a. Therefore, s_{i+1} has not two new letters adjacent in any cycle. If there are two new letters in one cycle of s_{i+1}, some power s_{i+1}^x brings them together. From what has just been said it follows that in s_{i+1}^x no a is followed by or preceded by a b, but $s_{i+1}^x = (a_2 \alpha_1 \cdots b_2 \alpha_2 \cdots) \cdots$, and $s_{i+1}^{xy} = (a_2 b_2 \cdots \alpha_1 \alpha_2 \cdots) \cdots$, which again is the case fully discussed. We have proved that *if one of the substitutions* $s_1, s_2, \cdots$, *which replaces an a by a b and displaces the minimum number of new letters, has the property that no power of it replaces every a by a not-a, then there is at most one new letter in any one of its cycles.*

7. If $b', b'', \cdots, b^{(\rho)}$ are sets b of H_i which have at least one letter in the same cycle of s_{i+1} with a letter a (a arbitrarily chosen), and unless s_{i+1} has a power which replaces every b' by a *not-b'* for at least one of the sets $b', b'', \cdots, b^{(\rho)}$, then s_{i+1} has at most one new letter to a cycle. This is an easy inference from the preceding section.

8. Let s_{i+1}, one of the substitutions $s_1, s_2, \cdots$, be chosen subject to the conditions:

(1) It unites at least two sets of H_i;

(2) it displaces as few new letters as any substitution that satisfies (1);

(3) it unites as few sets as any of the substitutions satisfying (1) and (2); then if s_{i+1} does not displace all the letters of all the sets united it has at most one new letter to a cycle. This follows from § 6.

9. We again select from $s_1, s_2, \cdots$ those substitutions which replace a letter of the set a of H_i by a letter of some other set, and which displace the minimum number of letters α, new to H_i (assuming that at least one such substitution exists). What follows if one of these substitutions (s_{i+1}) has a cycle composed entirely of new letters? As a matter of notation let $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{i+1}, \bar{H}_i$, etc., be the substitutions and groups remaining after the erasure of the letters of $\{ H_i, s_{i+1} \}$ that do not form a part of the set $a_1, a_2, \cdots$. The group K_{i+1} generated by the conjugates of H_i is invariant in H_{i+1}, and the constituent $\bar{K}_{i+1}$

is invariant in $\overline{H}_{i+1}$. If $\overline{K}_{i+1}$ is intransitive its sets of intransitivity are systems of imprimitivity of $\overline{H}_{i+1}$ and these systems are permuted by s_{i+1}. If $\overline{K}_{i+1}$ is transitive, we note that, since $\overline{K}_{i+1}$ leaves fixed certain letters α, no generator (belonging to $\overline{s}_1, \overline{s}_2, \cdots$) can replace an a of H_i by a b. Then $\overline{K}_{i+1}$ has a subgroup $\overline{H}_k$, the set $a_1, a_2, \cdots, a_{n_i}, \alpha_1, \cdots, \alpha_{n_k - n_i}$ of which is a system of imprimitivity of $\overline{H}_{i+1}$. Since $\overline{s}_{i+1}$ replaces an a by a b, $\overline{s}_{i+1}$ replaces the system $a_1, \cdots, \alpha_1, \cdots$ by another system. Hence the result:

If s_{i+1} connects transitively a given set a of H_i and another set, and displaces the minimum number of new letters, it cannot contain a cycle composed entirely of new letters unless it actually displaces the n_i letters a and has not more than one letter a in any cycle.

Applications to primitive groups.

10. Now let $s_1, s_2, \cdots$ be a complete set of conjugate substitutions of prime order p and of degree pq in a primitive group. The subgroup $\{s_1, s_2, \cdots\}$, because invariant in a primitive group, is transitive. As before let all the substitutions of $\{s_1, s_2, \cdots\}$ that are similar to s_1 be associated with the original set of generators $s_1, s_2, \cdots$. Since we have to do with a primitive group in which $\{s_1, s_2, \cdots\}$ is invariant, there exists a substitution t satisfying the conditions of § 5, and in consequence there is in the series $s_1, s_2, \cdots$ a substitution s_{i+1} that replaces a letter of any given set of H_i by a letter of H_i not belonging to that set (an a by a b if we continue the former notation). Let N_i be the degree of H_i, c_i the number of its transitive constituents, and as before let n_i be the degree of the constituent a. We shall make use of a number θ which represents $2p/(p-1)$ when p is odd, but is equal to 2 when p is 2. Again suppose that we have before us all the substitutions of the series $s_1, s_2, \cdots$ that (1) connect the given system a with other letters of H_i, and (2) displace the minimum number of new letters. If all these substitutions have a power that replaces every a by a *not-a*, if each one of them displaces all the letters of one of the sets $b', b'', \cdots$ whose letters are found in cycles with a, and if more than one new letter is in some cycle, we have two cases to consider. Each of these substitutions may have at least one cycle entirely new to H_i, in which case the following inequalities hold true:

$$(B) \quad N_{i+1} \leqq N_i + pq - n_i - p, \qquad c_{i+1} \leqq c_i - 1 + q - n_i, \qquad n_{i+1} \geqq p n_i.$$

On the other hand if even one substitution has no cycle of new letters only, the three conditions just stated being satisfied, we conclude that

$$(C) \qquad N_{i+1} \leqq N_i + pq - n_i - p, \qquad c_{i+1} \leqq c_i - 1, \qquad n_{i+1} \geqq \theta n_i.$$

The inequality $n_{i+1} \geqq \theta n_i$ requires explanation. If $n_{i+1} < \theta n_i$, every power of s_{i+1} replaces one letter a at least by an a. Then the number of new letters is

q at most, with not more than one in any cycle. In fact, if there is one of the substitutions that satisfy (1) and (2) for which any one of the three conditions under which (B) and (C) were formed fails, that substitution s_{i+1} has not more than one new letter in any cycle. Hence

$$(D) \qquad N_{i+1} \leqq N_i + q, \qquad c_{i+1} \leqq c_i - 1, \qquad n_{i+1} \geqq n_i + p.$$

11. In the formation of the groups $H_2, H_3, \cdots$ there is another point of view which is of value. We no longer base our reasoning upon an arbitrarily chosen set a of H_i, but impose upon s_{i+1} the conditions of § 8, which we need not repeat, and in addition the condition (4) that no substitution of $s_1, s_2, \cdots$ which satisfies (1), (2) and (3) has fewer cycles in the letters of the extended set formed by the union of sets of H_i by s_{i+1}. Let $\bar{H}_i, \bar{H}_{i+1}, \bar{K}_{i+1}$, have the same meaning as before, that is, $\bar{H}_{i+1}$ is transitive in the letters of the sets $a, b, \cdots$ of H_i and certain new letters which s_{i+1} joins to these sets. If $\bar{s}_{i+1}$ is of higher degree than any of the substitutions $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_i$, then by condition (4), no substitution of $\bar{K}_{i+1}$, similar to one of the substitutions $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_i$, can connect in its cycles two of the sets of $a, b, \cdots$. Then if two of the sets $a, b, \cdots$ are in one transitive constituent of $\bar{K}_{i+1}$, there are substitutions in $\bar{K}_{i+1}$ that displace all the letters of one of the sets, as a, and such that all powers of these substitutions replace every a by a new letter:

$$r = (a_1 \alpha_1' \cdots \alpha_1^{(p-1)}) \cdots (a_{n_i} \alpha_{n_i}' \cdots \alpha_{n_i}^{(p-1)}) \cdots .$$

The transforms of r by substitutions of H_i generate a group with a transitive constituent of degree pn_i in the letters $a_1, a_2, \cdots, \alpha_1', \cdots, \alpha_{n_i}^{(p-1)}$. If $\bar{K}_{i+1}$ does not unite transitively two sets of H_i, the sets of intransitivity of $\bar{K}_{i+1}$ are systems of imprimitivity of $\bar{H}_{i+1}$ which $\bar{s}_{i+1}$ must permute. Then the degree of $\bar{H}_{i+1}$ is pn_i or more. If one system is made up entirely of new letters, the transforms of $\bar{s}_{i+1}$ by substitutions of $\bar{H}_i$ generate a transitive group. If no system is made up entirely of new letters, exactly p distinct sets of $a, b, \cdots$ of H_i are united by $\bar{s}_{i+1}$. Consider in particular a substitution $\bar{s} = (a_1 a_2 \cdots a_p) \cdots$ of $\bar{H}_i$. The group $\{ r, s^{-1} r s \}$ has a constituent of degree p^2, and for it

$$(E) \qquad N_2 \leqq 2pq - p^2, \qquad c_2 \leqq 1 + 2(q - p), \qquad n_2 \geqq p^2.$$

Likewise, when $\bar{K}_{i+1}$ has p sets, if a power of $\bar{s}_{i+1}$ replaces each of the letters $a_1, a_2, \cdots a_p$ by a new letter α, the same inequalities are satisfied by the group $\{ s_{i+1}, s^{-1} s_{i+1} s \}$. If no power of s_{i+1} replaces each of the letters $a_1, a_2, \cdots, a_p$ by letters new to H_i, s_{i+1} involves all the letters of p cycles of s, so that the inequalities hold for $\{ s, s_{i+1} \}$.

Now we assume that $\bar{s}_{i+1}$ is not of higher degree than the substitutions of highest degree among $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_i$, and that s_{i+1} has two new letters in some one of its cycles. We know (§ 8) that $\bar{s}_{i+1}$ displaces all the letters of $\bar{H}_i$ so that

$\bar{s}_{i+1}$ displaces no new letter. From this it follows that every power of s_{i+1} replaces a letter of one set of $\bar{H}_i$ by a letter of another set of $\bar{H}_i$. Let s_{i+1}^x have two new letters adjacent. Because of condition (2) no generator of $\{\bar{H}_i, \bar{s}_{i+1}^{-x}\bar{H}_i\bar{s}_{i+1}^x\}$ has letters of two sets of H_i in one of its cycles. But since there are no new letters α in this group, $\bar{s}_{i+1}$ permutes cyclically p sets $a, b, \cdots$ of $\bar{H}_i$. Again if $\bar{s}$ is a generator of $\bar{H}_i$, the group $\{s, s_{i+1}\}$ satisfies inequalities (E) provided $\bar{s}$ displaces letters of all the p sets $a, b, \cdots$; but if $\bar{s}$ leaves fixed all the letters of one of the sets $a, b, \cdots$, the group $\{s, s^{-1}s_{i+1}s\}$ satisfies inequalities (E). The result which we are going to use is this:

If we have before us a group H_i we may be able to find in $s_{i+1}, s_{i+2}, \cdots$ a substitution which unites two or more sets of H_i and has at most one new letter in any cycle, thus satisfying the inequalities:

$$(F) \qquad N_{i+1} \leqq N_i + q, \qquad c_{i+1} \leqq c_i - 1.$$

A third inequality $n_{i+1} \leqq n_i$ is irrelevant since the largest set of H_i is not taken into account and may not be enlarged by s_{i+1}. But if $s_{i+1}, s_{i+2}, \cdots$ do not include such a substitution, we can say that there are two substitutions in the series $s_1, s_2, \cdots$ which we are at liberty to use for s_1 and s_2 which generate a group satisfying (E). The constituent of degree p^2 or more is imprimitive.

12. Should it be possible to find for all the groups $H_i (i = 1, 2, \cdots)$ up to the transitive group H_λ a substitution s_{i+1} with at most one new letter to a cycle, then the degree of H_λ is not greater than $pq + (q-1)q$. This is certainly true if q is less than or equal to p. But if at any point in the formation of this chain of groups, there is no substitution s_{i+1} with at most one new letter in a cycle, we begin our series $H_1, H_2, \cdots$ with the H_2 satisfying (E). Let us first impose upon q the condition $p < q < 2p + 3$, and suppose p odd. If then the upper limit of the degree of H_λ is not given by $pq + q^2 - q$, we begin with inequalities (E). In the next step we cannot have inequalities (D) since then the p^2 (or more) letters of a set a of H_2 would have to be present in at least $\theta p = 2p + 2 + 2/(p-1)$ cycles of s_3. Then the degree of H_λ is not greater than $N_2 + (c_2 - 1)q = 2q^2 - p^2$. Note that if $q = p + 1$, $pq + q^2 - q > 2q^2 - p^2$, and if $q = 2p$, $pq + q^2 - q < 2q^2 - p^2$; it is sufficiently precise to state that *a primitive group that contains a substitution of order p and degree pq (p an odd prime, $p < q < 2p + 3$), contains a transitive subgroup the degree of which is not greater than the larger of the two numbers $pq + q^2 - q$ and $2q^2 - p^2$.*

13. Let us now suppose that q is subject to the inequality $2p + 3 \leqq q < p^2$, and that p is odd. Let H_2 satisfy (E). If we have before us a group H_i, the question arises: What comparison exists between the degree of H_λ when H_{i+1} satisfies (C) and H_{i+2} satisfies (E) and the degree when H_{i+1} is subject to (E) while H_{i+2} obeys (C)? A simple calculation shows that there is no difference in the degrees. But to have H_{i+1} satisfy (C) and H_{i+2} (D) is more unfavor-

able than the reverse. Then let H_3, H_4, $\cdots$, H_x obey (C), so that

$$N_3 \leqq N_2 + pq - n_2 - p, \qquad c_3 \leqq c_2 - 1, \qquad n_3 \leqq \theta p^2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$N_x \leqq N_{x-1} + pq - n_{x-1} - p, \qquad c_x \leqq c_{x-1} - 1, \qquad n_x \leqq \theta^{x-2} p^2.$$

The remaining groups H_{x+1}, $\cdots$, $H_{2(q-p+1)}$ satisfy (D):

$$N_{x+1} \leqq N_x + q, \qquad c_{x+1} \leqq c_x - 1, \qquad n_{x+1} \leqq \theta^{x-2} p^2 + p,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$N_{2(q-p+1)} \leqq N_{2(q-p)+1} + q, \qquad c_{2(q-p+1)} \leqq c_{2(q-p)+1},$$

whence

$$N_{2(q-p+1)} \leqq xpq - p^2 - (x-2)p - (p^2 + p^2\theta + \cdots p^2\theta^{x-3}) + (2q - 2p + 2 - x)q,$$

or

$$N_{2(q-p+1)} \leqq 2q^2 - p^2 + (pq - q - p)(x-2) - p^2 \frac{\theta^{x-2} - 1}{\theta - 1}.$$

A condition which n_{x-1} must satisfy is

$$n_{x-1} \leqq q \frac{p-1}{2},$$

that is

$$\theta^{x-3} p^2 \leqq q \frac{p-1}{2}.$$

If H_{x-1} instead of H_x were the last group subject to inequality (C), then

$$N_{2(q-p+1)} \leqq 2q^2 - p^2 + (pq - q - p)(x-3) - p^2 \frac{\theta^{x-3} - 1}{\theta - 1}.$$

This expression is always numerically less than the other. For the difference

$$(pq - q - p) - p^2\theta^{x-3} \geqq pq - q - p - q\frac{p-1}{2} \geqq \frac{p-1}{2}(q - \theta),$$

is positive. Hence the limiting value of the degree of H_λ is obtained by replacing x in the first formula by the largest integer which satisfies $\theta^{x-3} p^2 \leqq q(p-1)/2$, and this is the largest integer in $2 + \log_\theta (q/p)$.

If a primitive group contains a substitution of odd prime order p on more than $2p+2$ and less than p^2 cycles, it contains a transitive subgroup of degree not greater than

$$2q^2 - p^2 + (pq - q - p)\mu - p^2 \frac{\theta^\mu - 1}{\theta - 1},$$

where μ is the integral part of $\log_\theta (q/p)$ and $\theta = 2p/(p-1)$.

14. We finally take q so large $(q \geqq p^2)$ that the inequality (B) is possible. The case $p = 2$ is not excluded. As before it is evident that it is more unfavor-

able to have H_{i+1} subject to (B) and H_{i+2} to (D) than for H_{i+1} to satisfy (D) and H_{i+2} to satisfy (B). Now compare (B) and (C) in relation to H_{i+1} and H_{i+2}. If the inequalities are applied in the order $(B), (C)$, we have

$$N_{i+2} \leqq N_i + 2p(q-1) - n_i - pn_i, \qquad c_{i+2} \leqq c_i - 2 + q - n_i, \qquad n_{i+2} \geqq \theta p n_i;$$

and if in the order $(C), (B)$:

$$N'_{i+2} \leqq N_i + 2p(q-1) - n_i - \theta n_i, \qquad c'_{i+2} \leqq c_i - 2 + q - \theta n_i, \qquad n'_{i+2} \geqq p\theta n_i.$$

If p is 2 or 3, $\theta = p$, and then obviously the first arrangement is the more unfavorable. When p is greater than 3 the matter is not so simple. However we can show that the addition of one new set of intransitivity by s_{i+1} is equivalent in the end to the introduction of q or more new letters. For one more set means one more group H in the chain $H_1, H_2, \cdots$ before arrival at a transitive group H_λ. This is evidently true if the extra group gives rise to a repetition of the inequalities (D). There only remains the question whether $pq - n_i - p$ can be less than q. Now (B) and (C) are not necessary unless $n_i \leqq (p-1)/2q$. From

$$pq - n_i - p < q,$$

we have

$$(p-1)q - p < n_i \leqq \frac{p-1}{2} q,$$

whence

$$q < \theta,$$

which is false under the present assumptions. Hence, while N'_{i+2} exceeds N_{i+2} by $(p-\theta)n_i$, if we multiply $(\theta - 1)n_i$, the difference between c_{i+2} and c'_{i+2}, by q, it is apparent that the first order is the more unfavorable.

We now form the successive inequalities in their most unfavorable order:

$$N_2 \leqq 2pq - p^2, \qquad c_2 \leqq 1 + 2(q-p), \qquad n_2 \geqq p^2,$$

$$N_3 \leqq N_2 + pq - n_2 - p, \qquad c_3 \leqq c_2 + q - n_2 - 1, \qquad n_3 \geqq p^3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$N_x \leqq N_{x-1} + pq - n_{x-1} - p, \qquad c_x \leqq c_{x-1} + q - n_{x-1} - 1, \qquad n_x \geqq p^x,$$

$$N_{x+1} \leqq N_x + pq - n_x - p, \qquad c_{x+1} \leqq c_x - 1, \qquad n_{x+1} \geqq p^x \theta,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$N_{x+y} \leqq N_{x+y-1} + pq - n_{x+y-1} - p, \qquad c_{x+y} \leqq c_{x+y-1} - 1, \qquad n_{x+y} \geqq p^x \theta^y,$$

$$N_{x+y+1} \leqq N_{x+y} + q, \qquad c_{x+y+1} \leqq c_{x+y} - 1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$N_\lambda \leqq N_{x+y} + (c_{x+y} - 1)q.$$

From these inequalities we derive the following expression for the upper limit

of the degree of H_λ, in which x and y remain to be determined:

$$N_\lambda \leqq \frac{p^2(q-p+2)}{p-1} + q^2x + (pq-q-p)(x+y-2) - \left(\frac{q+1}{p-1} - \frac{1}{\theta-1}\right)p^x - \frac{p^x\theta^y}{\theta-1}.$$

To cut short unprofitable refinements we now regard this expression, following Jordan's lead, as a continuous function of two independent variables x and y and equate to zero the partial derivatives

$$q^2 + pq - q - p - \left(\frac{q+1}{p-1} - \frac{1}{\theta-1}\right)p^x \log p - \frac{p^x\theta^y}{\theta-1}\log p = 0,$$

$$pq - q - p - \frac{p^x\theta^y}{\theta-1}\log\theta = 0.$$

We solve, substitute for N_λ, reduce, and have for the limit of the degree of the transitive subgroup H_λ

$$\frac{p^2(q-p+2)}{p-1} + (q-1)(q+p)\log_p\frac{ac}{e} + a\log_\theta\frac{c\theta^2\log\theta}{\theta-1},$$

in which

$$a = pq - q - p, \qquad b = \frac{q+1}{p-1} - \frac{1}{\theta-1}, \qquad abc = \frac{q^2+a}{\log p} - \frac{a}{\log\theta}.$$

15. When p is 2 or 3, the formula can be greatly simplified, for then $\theta = p$, and $y = 0$. For $p = 3$ it becomes

$$\tfrac{1}{2}(q+3) + (q+3)(q-1)\mu - \tfrac{1}{2}(q+1)3^\mu,$$

where μ is the integral part of $\log_3 3q$.

16. When $p = 2$ inequality (F') may be replaced by

$$(F') \qquad N_{i+1} \leqq N_i + q - 1,$$

whence

$$N_\lambda \leqq N_x + (c_x - 1)(q-1) \leqq 2(q+1) + (q^2-1)x - q\cdot 2^x.$$

If 2^{x-1} is less than q, the difference of N_λ with respect to x is

$$(q^2-1) - q2^{x-1}$$

and is positive. If 2^{x-1} is equal to q, this difference is negative, but the difference with respect to x of

$$2(q+1) + (q^2-1)(x-1) - q2^{x-1}$$

is positive. Hence when $p = 2$ $(q \geqq 4)$ the limit of the degree of a transitive subgroup H_λ of G is

$$2(q+1) + (q^2-1)\mu - q\cdot 2^\mu,$$

where μ is the largest whole number less than $\log_2 2q$.

17. A theorem was stated in the introduction which was proved in § 12 for p an odd prime. But if p is 2, we know from recent researches * on the class of primitive groups that the theorem is true except perhaps when the given substitution of order 2 has 6 cycles and the group in question is of class 12. But the limit of the preceding paragraph, obtained without any restriction on the class of the group, is 71, while the limit required for the theorem is 68. The inequalities of § 14 allow $H_3(p=2,\ q=6)$ ten transitive constituents. A moment's consideration shows that, if the entire group is of class 12, H_3 cannot have so many constituents and that the limit of the degree of H_λ is 66.

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*C. JORDAN, Liouville's Journal, ser. 2, vol. 17 (1872), p. 363; NETTO, *Theory of Substitutions*, COLE'S translation (1892), p. 133; MILLER, American Mathematical Monthly, vol. 9 (1902), p. 63; MANNING, American Journal of Mathematics, vol. 32 (1910), pp. 235–257, and vol. 28 (1906), p. 226.

ISOMORPHISMS OF A GROUP WHOSE ORDER IS A POWER OF A PRIME*

BY

G. A. MILLER

§ 1. *Groups of isomorphisms involving operators of order* p^{m-1}.

Let G be any group of order p^m, p being a prime number, and let I represent its group of isomorphisms. A necessary and sufficient condition that I involves operators of order p is that $m > 1$, since G can always be made isomorphic with one of its invariant subgroups of order p which corresponds to the identity in this isomorphism when $m > 1$. Hence the order of I is divisible by p whenever $m > 1$ and only then. Although the order of I may be divisible by a much higher power of p than p^m, it is very easy to prove that the order of every operator of I is a divisor of $p^{m-1}k$, k being prime to p. That is, I cannot involve any operator whose order is divisible by p^m.

To prove this theorem we may assume that t is an operator of I which has been so selected that its order is the highest possible power of p. As t and G generate a group whose order is a power of p it is possible to select a series of invariant subgroups of G

$$H_0,\ H_1,\ H_2,\ \cdots,\ H_m = G,$$

whose orders are respectively $1,\ p,\ p^2,\ \cdots,\ p^m$ and such that each one except the last is contained in the one which follows it and that t transforms each operator of each of these subgroups except the first into itself multiplied by an operator in the one which precedes it in the given series. Hence we have the following equations, s_α being an operator of H_α,

$$t^{-1}s_\alpha t = s_{\alpha-1}s_\alpha, \qquad t^{-p}s_\alpha t^p = s_{\alpha-2}s_\alpha \qquad (\alpha \geqq 2),$$

since $s_{\alpha-1}^p = s_{\alpha-2}$ and $s_{\alpha-1}^{-1}s_{\alpha-2}s_{\alpha-1} = s'_{\alpha-2}$, $s'_{\alpha-2}$ being an operator of $H_{\alpha-2}$.

If we replace t^p by t_1 it results from similar considerations that

$$t_1^{-p}s_\alpha t_1^p = s_{\alpha-3}s_\alpha \qquad (\alpha \geqq 3).$$

In general,

$$t^{-p^\beta}s_\alpha t^{p^\beta} = s_{\alpha-\beta-1}s_\alpha \qquad (\alpha \geqq \beta+1).$$

From the last equation it follows directly that the order of t is a divisor of p^{m-1},

* Presented to the Society April 28, 1911.

and hence the following theorem has been proved: *The group of isomorphisms of a group of order p^m does not involve any operator whose order is divisible by p^m.*

It is well known that the group of isomorphisms of the cyclic group of order p^m, $p > 2$, contains operators of order p^{m-1}. We proceed to prove that no other group of order p^m with the exception of the non-cyclic group of order p^2 can have operators of order p^{m-1} in its group of isomorphisms. To prove this statement it is only necessary to observe that every non-cyclic group of order p^m contains an invariant non-cyclic subgroup of order p^2. Hence it results that

$$t^{-p^{m-3}} s_\alpha t^{p^{m-3}} = s_{\alpha-m+2} s_\alpha \qquad (\alpha \geqq m-2).$$

If we let $\alpha = m$ and observe that the orders of all the operators of H_2 divide p, it results that the $t^{p^{m-2}}$ is commutative with every operator of G. That is, *the group of isomorphisms of a non-cyclic group of order p^m cannot contain any operator whose order is divisible by p^{m-1} when $p > 2$ and $m > 2$.*

When $p = 2$ it is known that the group of isomorphisms of the cyclic group of order p^m is the direct product of the cyclic group of order 2^{m-2} and the group of order 2. Hence there is no operator of order 2^{m-1} in the I of the cyclic group of order 2^m when $m > 2$. Moreover it results from the preceding paragraph that the I of any group of order 2^m, $m > 3$, cannot involve any operator of order 2^{m-1} whenever this group of order 2^m involves the non-cyclic group of order 4 invariantly. A group of order 2^m contains an invariant non-cyclic group of order 4 when the number of its cyclic groups of this order is even. Hence it results that if the I of a group of order 2^m, $m > 3$, contains an operator of order 2^{m-1} this group must be one of the three non-cyclic groups of order 2^m which involve an odd number of cyclic subgroups of order 4.*

Two of these non-cyclic groups are the dihedral and the dicyclic groups, and it is easy to prove that the I of each of these groups involves operators of order 2^{m-1}. In fact, the I of such a group of order 2^m must involve an operator which is commutative with each of its operators in the invariant cyclic group of order 2^{m-1} but transforms the remaining operators into themselves multiplied by an operator of order 2^{m-1}. The order of such an operator in I is evidently 2^{m-1}, and hence it results that the groups of isomorphisms of the dihedral group and of the dicyclic group of order 2^m involve operators of order 2^{m-1}. The remaining one of the three non-cyclic groups of order 2^m which involve an odd number of cyclic subgroups of order 4 involves exactly 2^{m-2} non-invariant operators of order 2 when $m > 3$, and its group of isomorphisms may be represented as an intransitive substitution group having two transitive constituents of degree 2^{m-2}. Hence this I cannot involve any operator of order 2^{m-1} so that we have arrived at the theorem: *A necessary and sufficient condition that a group of order 2^m,*

*These Transactions, vol. 6 (1905), p. 58.

$m > 3$, *have an operator of order* 2^{m-1} *in its group of isomorphisms is that the group be either dihedral or dicyclic.* When $m \leqq 3$ every group of order 2^m, with the exception of the cyclic group of order 8, has an operator of order 2^{m-1} in its group of isomorphisms, as one may readily verify.

§ 2. *Groups of isomorphisms involving operators of order* p^{m-2}, $p > 2$.

Having considered all the possible groups of order p^m which involve an operator of order p^{m-1} in their groups of isomorphisms we proceed to consider those whose groups of isomorphisms involve operators of order p^{m-2} but none of order p^{m-1}. We shall again represent the group under consideration by G and its group of isomorphisms by I. Suppose that G contains an invariant subgroup of order p^4 which does not involve any operator of order p^3 and that the series of invariant subgroups

$$H_0, H_1, H_2, \cdots, H_m = G$$

has the same properties as in the preceding section.

From the equation

$$t^{-p^\beta} s_\alpha t^{p^\beta} = s_{\alpha-\beta-1} s_\alpha \qquad (\alpha \geqq \beta + 1),$$

where s_α is any operator of H_α, it results that

$$t^{-p^{m-5}} s_m t^{p^{m-5}} = s_4 s_m \qquad (m > 4).$$

Let $t_1 = t^{p^{m-5}}$. We proceed to prove that $t_1^{p^2} = t^{p^{m-3}} = 1$. In fact,

$$t_1^{-p} s_{m\alpha} t_1^p = s_2 s_3^{kp} s_4^p s_m = s_3 s_m \qquad (p > 2).$$

Since s_3 is commutative with s_2 and as G involves an invariant non-cyclic subgroup of order p^2, it results that the order of s_3 cannot exceed p. This proves that $t^{p^{m-3}} = 1$ whenever $m > 4$ and $p > 2$, since t_1^p must then be commutative with s_2. We have now proved that G cannot have an operator of order p^{m-2}, $m > 4$, in its group of isomorphisms when G involves an invariant subgroup of order p^4 which does not involve any operator of order p^3. It is evident that this proves also that G could not involve such an operator if it contained an invariant subgroup of order p^3 involving only operators of order p besides the identity. In seeking all the groups whose groups of isomorphisms involve operators of order p^{m-2} but none of order p^{m-1}, we may therefore confine our attention to such groups as give rise to a cyclic quotient group with respect to the invariant non-cyclic group of order p^2 and in which operators of order p^2 correspond to the operators of order p in this quotient group. This clearly implies that the groups in question contain operators of order p^{m-1}.

From the preceding paragraph it follows that if the I of G involves operators of order p^{m-2} but none of order p^{m-1}, G must be one of the two non-cyclic groups

of order p^m which involve operators of order p^{m-1}. As the cyclic group of order p^{m-1} involves operators of order p^{m-2} in its group of isomorphisms it is evident that the I's of both of the non-cyclic groups in question involve operators of order p^{m-2}. Hence the theorem: *A necessary and sufficient condition that the group of isomorphisms of a group of order p^m, $p > 2$ and $m > 4$, involves operators of order p^{m-2} is that this group of order p^m involves operators of order p^{m-1}.* When $m = 3$ it is evident that the groups of isomorphisms of all the possible non-cyclic groups involve operators of order p^{m-2} but none of order p^{m-1}. When $m = 4$ a group of order p^m which does not involve operators of order p^{m-1} may also have operators of order p^{m-2} in its group of isomorphisms, as may readily be verified.

Incidentally the developments of this section have led to an important general theorem which may be stated as follows: *A necessary and sufficient condition that a group of order p^m, $p > 2$ and $m > 4$, contain an invariant subgroup of order p^4 which does not involve any operator of order p^3 is that the group of order p^m does not involve any operator of order p^{m-1}.* In fact, we may easily derive the more general theorem that the number of the subgroups of order p^4 which do not contain any operator of order p^3 is always of the form $1 + kp$ when G does not involve any operator of order p^{m-1}. When G involves such an operator this number is evidently zero, so that we may say that every group of order p^m, $p > 2$ and $m > 4$, contains either no subgroup of order p^4 which involves only operators whose orders divide p^2, or the number of its subgroups which have this property is of the form $1 + kp$. When it involves no such subgroup it must be one of the three groups of order p^m which involve operators of order p^{m-1}.

To prove this theorem we may confine our attention to the invariant subgroups of order p^4 which do not involve any operators of order p^3. From the theorem stated at the beginning of the preceding paragraph it results that G contains at least one such invariant subgroup if it does not involve any operator of order p^{m-1}. In what follows we shall assume that this condition is satisfied and let

$$K_1, K_2, \cdots, K_\lambda$$

be the totality of the invariant subgroups of order p^4 in G which do not involve operators of order p^3. We proceed to prove that the group generated by these λ subgroups does not involve any operator of order p^3. This fact will evidently establish the theorem in question.

To prove this fact we let t be any operator of the group generated by K_1, $K_2, \cdots, K_\lambda$ such that $t^{p^2} = 1$, while s_α represents an operator of K_α, $1 \leqq \alpha \leqq \lambda$, and s_3, s_2 represent operators belonging to invariant subgroups of orders p^3 and p^2 respectively in K_α, these subgroups being invariant under both K_α and t. We may evidently assume that s_2 is of order p and that it is invariant under

both of the operators t^p and s_3^p. Hence the following equations

$$(s_\alpha t)^p = s_\alpha t s_\alpha t s_\alpha t \cdots = s_\alpha t s_\alpha t^{-1} t^2 s_\alpha t^{-2} t^3 \cdots t^p$$

$$= s_\alpha s_3 s_\alpha s_2 s_3^2 s_\alpha \cdots s_2' s_3^{p-1} s_\alpha t^p = s_2'' s_3^p s_\alpha^p t^p,$$

where s_2, s_2', s_2'' are operators of the same invariant subgroup of order p^2 in K_α. As all of the operators s_2'', s_3^p, s_α^p, t^p are commutative with s_2'' and s_3^p, and the order of each is a divisor of p it results that the order of $s_\alpha t$ cannot exceed p^2. This proves that the group generated by K_1, K_2, $\cdots$, K_λ involves no operator of order p^3 and hence we have established the theorem: *If a group of order* p^m, $p > 2$ *and* $m > 4$, *involves no operator of order* p^{m-1} *the number of its subgroups of order* p^4 *which do not involve any operator of order* p^3 *is of the form* $1 + kp$. In other words, it has been proved that if G is any non-cyclic group of order p^m, $p > 2$ and $m > 4$, which does not involve any operator of order p^{m-1}, the number of its cyclic subgroups of order p^4 is congruent mod p to the number of its non-cyclic subgroups of order p^4 which involve an operator of order p^3, each of these numbers being divisible by p.

§ 3. *Groups of isomorphisms involving operators of order* 2^{m-2}.

In the present section we shall assume that the order of G is 2^m and that its group of isomorphisms I involves operators of order 2^{m-2} but none of order 2^{m-1}. It has already been observed that the I of the cyclic group of order 2^m contains operators of order 2^{m-2} but none of order 2^{m-1}, so that G may be cyclic. In what follows we shall consider the possible cases when G is non-cyclic. We begin, as in the preceding section, by finding an upper limit for m when G involves an invariant subgroup of order p^4 which does not contain any operator of order p^3. The equation

$$t^{-p^{m-5}} s_m t^{p^{m-5}} = s_4 s_m \qquad (m > 4),$$

which was obtained in the preceding section is evidently true also when $p = 2$, and if we assume that $m > 5$ it follows that

$$t_1^{-p} s_m t_1^p = s_2 s_4^2 s_m = s_3 s_m,$$

t_1, s_2, s_3, s_4, s_m having the same meaning as in the preceding section. As the order of s_3 divides 2 and as t_1^p is commutative with s_3, it results that the order of t must divide 2^{m-3}. That is, *if a group of order* 2^m *contains an invariant subgroup of order* 16 *which does not include any operator of order* 8, *the group of isomorphisms of this group of order* 2^m *does not involve any operator of order* 2^{m-2} *when* $m > 5$.

We shall now consider some of the properties of the groups of order 2^m, $m > 5$, which do not include an invariant subgroup of order 16 involving no

operator of order 8. It is evident that all the groups of order 2^m which involve operators of order 2^{m-1} are included among these groups. If G is any other group of order 2^m which belongs to this category it involves a non-cyclic invariant subgroup of order 4 and the corresponding quotient group contains no invariant non-cyclic group of order 4. Hence this quotient group contains operators of order 2^{m-3}. To the invariant operator of order 2 in this quotient group there must correspond operators of order 4 in G and hence G involves operators of order 2^{m-2}. That is, every group of order 2^m, $m > 5$, whose group of isomorphisms involves operators of order 2^{m-2} must itself involve operators of this order. In other words, *if a group of order* 2^m *does not contain any operator of order* 2^{m-2} *its group of isomorphisms cannot contain any operator of this order when* $m > 5$.

As all the groups of order 2^m which involve operators of order 2^{m-2}, as well as all those whose orders divide 32, are known the preceding theorem reduces our problem to the study of known groups. Two of the six groups of order 2^m, $m > 3$, which involve operators of order 2^{m-1} have operators of order 2^{m-1} in their groups of isomorphisms in accord with the results of section 1. It is easy to see that two others have operators of order 2^{m-2} in their groups of isomorphisms while the largest operators in the I's of the remaining two groups are of order 2^{m-3}. That the group of order 2^m which is generated by an operator of order 2^{m-1} and an operator of order 2 which transforms this into its $(2^{m-2}-1)$th power involves an operator of order 2^{m-2} in its I, results immediately from the fact that the I of this group contains an operator which is commutative with each operator of the cyclic subgroup of order 2^{m-1} and which transforms the remaining operators into themselves multiplied by an operator of order 2^{m-2}. In fact, there is such an operator in its group of cogredient isomorphisms. As the cyclic group of order 2^m has also operators of order 2^{m-2} in its I, we have found the two groups of order 2^m whose I's contain operators of order 2^{m-2} but none of order 2^{m-1}.

Each of the two remaining groups of order 2^m which involve operators of order 2^{m-1} contains two cyclic subgroups of order 2^{m-1}. The I's of these G's may evidently be represented as substitution groups on two sets of 2^{m-2} letters corresponding to the operators of order 2^{m-1} in the G's. When I is represented in this way it contains an intransitive subgroup of half its own order composed of all its substitutions which do not interchange the two cyclic subgroups of order 2^{m-1} in G. These transitive constituents clearly involve operators of order 2^{m-3} but none of order 2^{m-2} and there is a $(2, 2)$ isomorphism between them, since the cyclic subgroups of order 2^{m-1} in G have their operators whose orders divide 2^{m-2} in common. One of the remaining operators of I is of order 2 and is commutative with all the operators of G whose orders divide 2^{m-2}. It must therefore be commutative with all the operators of I which transform each

operator of order 2 in G into itself. From this fact it follows directly that I does not contain any operator of order 2^{m-2}, and hence the theorem: *The six groups of order* 2^m, $m > 3$, *which involve operators of order* 2^{m-1} *may be divided into three sets of two each such that the groups of isomorphisms of the first set involve operators of order* 2^{m-1}, *those of the second set involve operators of order* 2^{m-2} *but none of order* 2^{m-1}, *while the largest order of the operators in the groups of isomorphisms of the two groups of the third set is* 2^{m-3}. *All these groups of isomorphisms involve only operators whose orders are powers of* 2.

It remains to consider the groups of order 2^m, $m > 5$, which do not involve operators of order 2^{m-1} but contain operators of order 2^{m-2} in their groups of isomorphisms. It has been observed that the quotient group with respect to an invariant non-cyclic group of order 4 in such a group G cannot include a non-cyclic invariant subgroup of order 4, and that the operators of G which correspond to the invariant operator of order 2 in this quotient group must be of order 4. As this quotient group cannot be cyclic and does not involve an invariant non-cyclic subgroup of order 4 it must be one of three groups, viz., the dihedral, the dicyclic, or the group which involves both a dihedral and a dicyclic subgroup of half its own order. In the last two cases G involves operators of order 8 which correspond to operators of order 4 in some non-invariant subgroup of order 4 of the quotient group.

From the preceding paragraph it results that the only groups which remain to be considered are those which have an invariant subgroup of order 2^{m-1} involving two cyclic subgroups of order 2^{m-2}. This invariant subgroup must be one of two groups and the rest of the operators of G transform each operator of this subgroup into its inverse multiplied by one of the operators of the invariant non-cyclic subgroup of order 4 contained in G. Since t transforms each operator of the cyclic subgroup of order 2^{m-3} in the quotient group of G, with respect to its non-cyclic subgroup of order 4, into itself multiplied by an operator of lower order, it results that t must transform the operators of G which correspond to this cyclic subgroup in this quotient group according to an operator whose order cannot exceed 2^{m-3}. Hence t cannot be of order 2^{m-2} unless it transforms the remaining operators of G according to an operator of this order. It must therefore give rise to a commutator of order 2^{m-2} whenever its order is 2^{m-2}.

It is now easy to prove that the subgroup of G which involves two cyclic subgroups of order 2^{m-2} must be abelian.* If it were non-abelian, t could not transform the operators corresponding to non-invariant subgroups of order 2 or 4 in the given quotient group into themselves multiplied by operators of order 2^{m-2}, since the latter are not commutative with all the operators of order 2 in

* This is a special case of the general theorem that a commutator arising from either cogredient or contragredient isomorphisms of a group must be commutative with all the operators of every characteristic complete set of conjugates which involves only two operators or subgroups.

the non-cyclic invariant subgroup of order 4 in G. We shall therefore assume in what follows that G contains an abelian subgroup K of order 2^{m-1} involving two cyclic subgroups of order 2^{m-2}, and we shall consider the various possible groups when the quotient group as regards the non-cyclic invariant subgroup of G is one of the three possible subgroups.

If this quotient group is dihedral and all the operators of G which are not also in K are of order 2, it is evident that the I of G involves operators of order 2^{m-2}, since t transforms each operator of K into its inverse. When t transforms all the operators of K into their inverses and is of order 4, its square may be equal to the square of an operator of order 4 in K, or it may be one of the other two operators of order 2 in K. In the latter case, the possible group when t^2 is one of these operators of order 2 is evidently conjugate with the one when t^2 is the other operator of order 2. Hence there are three groups of order 2^m which involve K and in which all the operators of K are transformed into their inverses by t. In each of these three groups the group of isomorphisms clearly involves operators of order 2^{m-2}.

When t transforms half the operators of K into their inverses and the rest into their inverses multiplied by an operator of order 2, it is clearly only necessary to consider two of the three operators of order 2 in K. Moreover the two cyclic subgroups of order 2^{m-2} in K are evidently conjugate under I, and hence we need to consider only one of them. That is, it is necessary to consider only the four cases when the operators of G which are transformed into their inverses by t constitute either the cyclic subgroup of order 2^{m-2} or the non-cyclic subgroup of this order, and the remaining operators of K are transformed into their inverses multipled by one of two operators of order 2. One of these cases does not give rise to any group since the t which transforms all the operators of a cyclic subgroup of order 2^{m-2} into their inverses cannot transform the remaining operators into their inverses multiplied by an operator of order 2 that is included among these operators. It is therefore only necessary to consider three cases.

There are just two groups when t transforms all the operators of a cyclic subgroup of order 2^{m-2} into their inverses and the remaining operators of G into their inverses multiplied by the square of an operator of order 4 in G. In one of these G's half the operators which are not also in K are of order 2, while the other half are of order 4; in the other G all of these operators are of order 8. The I's of these groups must involve operators of order 2^{m-2} since operators of this order are transformed into their inverses by the operators of G which are not also in K, and hence one of the latter operators may be made to correspond to itself multiplied by an operator of order 2^{m-2} in K.

When t transforms all the operators of the non-cyclic subgroup of order 2^{m-2} in K into their inverses there are four possible groups, but only one of these

four has operators of order 2^{m-2} in its group of isomorphisms. It is very evident that the two of these groups in which half the operators which are not in K are of order 2 cannot have operators of order 2^{m-2} in their groups of isomorphisms since the product of any one of these operators of order 2 and an operator of order 2^{m-2} in K is of order 4. In the other two possible groups all the operators of G which are not also in K are of order 4 and these operators have two distinct squares. When one of these squares is the square of an operator of order 4 in K the corresponding I involves operators of order 2^{m-3} but none of order 2^{m-2}, since this square is a characteristic operator of G. On the other hand, the G in which the squares of these operators of order 4 are the other two operators of order 2 in K has operators of order 2^{m-2} in its I. This completes the consideration of the possible cases when all the operators of K are transformed under G into their inverses multiplied by operators of its non-cyclic invariant subgroup of order 4.

It remains to consider the case when the quotient group of G with respect to this non-cyclic subgroup of order 4 contains both the dihedral group and the dicyclic group of half its own order. In this case half the operators of G which are not also in K are of order 8 while the orders of the other half of these operators divide 4. It is known that there is one and only one such group,* and that it involves $2^{m-3}+3$ operators of order 2. Hence the transitive constituents of its group of isomorphisms, which correspond to the operators of order 2 in G, cannot involve any operator of order 2^{m-2}. The group of isomorphisms of G may evidently be represented as a substitution group in which a transitive constituent corresponds to operators of highest order in G while the other constituents correspond to operators of order 2. As neither of these constituents could involve substitutions of order 2^{m-2}, it has been proved that the I of the group under consideration cannot involve any operator of order 2^{m-2}. We have therefore established the following theorem: *There are exactly six groups of order* 2^m, $m>5$, *which do not involve operators of order* 2^{m-1} *but contain operators of order* 2^{m-2} *in their groups of isomorphisms. Each of these six groups includes an abelian subgroup of order* 2^{m-1} *and of type* $(m-2, 1)$. Hence the total number of groups of order 2^m, $m>5$, which have operators of order 2^{m-2} in their groups of isomorphisms is 10 and the number of those which have operators of order 2^{m-2} but none of order 2^{m-1} in their groups of isomorphisms is 8.

§ 4. *Sylow subgroups of the groups of isomorphisms.*

Since every group G of order p^m contains a series of invariant subgroups of orders $1, p, p^2, \cdots, p^m$ which are also invariant under a Sylow subgroup of order p^α in the group of isomorphisms of G, it follows from the preceding

* These Transactions, vol. 2 (1901), p. 271.

theorems that $\alpha \leqq m-1+m-2+\cdots+2+1$. Hence it results that if p^α is the order of a Sylow subgroup in the group of isomorphisms of G then $\alpha \leqq \frac{1}{2}m(m-1)$. When G is cyclic the value of α is $m-1$, hence α cannot have this maximal value for every group of order p^m when $m>2$. On the other hand, there is always at least one group of order p^m for which $\alpha=\frac{1}{2}m(m-1)$, viz., the abelian group of type $(1, 1, \cdots)$. In the present section we shall determine all the possible groups of order p^m whose groups of isomorphisms involve Sylow subgroups whose orders are $p^{\frac{1}{2}m(m-1)}$. In what follows we shall represent such a group by G and its group of isomorphisms by I. A series of invariant subgroups of G which are also invariant under a Sylow subgroup of I and have the orders $1, p, p^2, \cdots, p^m$ (each including those which precede it) will be represented by

$$H_0, H_1, H_2, \cdots, H_m=G.$$

The maximum value of α evidently implies that the isomorphism of each operator in one of the $p-1$ divisions of $H_\beta - H_{\beta-1}$ with respect to $H_{\beta-1}$, $\beta \leqq m$, is independent of those of $H_{\beta-1}$ and hence G cannot involve any operator whose order exceeds p^2. For the same reason the order of the commutator subgroup of G cannot be divisible by p^2, and all the operators of order p^2 in G must generate the same subgroup of order p. This common subgroup must be the commutator subgroup when G is both non-abelian and also involves operators of order p^2. All the operators of $H_\beta - H_{\beta-1}$ are of the same order and H_{m-1} must be abelian, otherwise an operator of one of the $p-1$ divisions of $H_m - H_{m-1}$ could not be made to correspond to every other operator of such a division when the identical operators of H_{m-1} would correspond.

When G is abelian there are only two groups of order p^m, $m>1$, which satisfy the conditions imposed in the preceding paragraph. One of these is of type $(1, 1, \cdots)$ and the other is of type $(2, 1, 1, \cdots)$. It is evident that the I of each of these two groups involves a Sylow subgroup of order $p^{\frac{1}{2}m(m-1)}$. Hence the theorem: *There are two and only two abelian groups of order* p^m, $m>1$, *whose groups of isomorphisms have orders which are divisible by* p^α, *where* $\alpha=\frac{1}{2}m(m-1)$. *One of these is of type* $(1, 1, 1, \cdots)$, *while the other is of type* $(2, 1, 1, \cdots)$.

It remains to determine the possible groups when G is non-abelian. When $p=2$ and all the operators of $H_m - H_{m-1}$ are of order 2, the abelian subgroup H_{m-1} must be of type $(2, 1, 1, 1, \cdots)$, and hence G is the direct product of a group of type $(1, 1, 1, \cdots)$ and the octic group. When all the operators of $H_m - H_{m-1}$ are of order 4, H_{m-1} must again be of type $(2, 1, 1, \cdots)$, since the product of two such operators of order 4 is of order 2 when they are commutative and of order 4 when they are non-commutative. Hence G is the Hamiltonian group of order 2^m in this case. That is, it is the direct product of

the quaternion group and the group of type $(1, 1, \cdots)$. Hence we have proved the theorem: *If the order of the group of isomorphisms of a non-abelian group of order 2^m is divisible by $2^{m(m-1)/2}$ it is either the Hamiltonian group or it is the direct product of the octic group and a group of type* $(1, 1, 1, \cdots)$.

When G is non-abelian and $p > 2$ all the operators of $G_m - G_{m-1}$ may be of order p. As one of the subgroups of order p^{m-2} is composed of invariant operators it must be characteristic and hence may be used for H_{m-2}. Hence there is one and only one such group in which all the operators have orders which divide p. Since the product of two operators of order p in G is of order p it remains to consider the case when all the operators of $G_m - G_{m-1}$ are of order p^2. If t and s are two such operators $(ts)^p = t^p s^p$ and hence the orders of all the operators of G_{m-1} must again divide p. This abelian group is therefore again completely determined and hence we have only one such group. This proves the theorem: *Every non-abelian group of order p^m, $p > 2$, whose group of isomorphisms has an order which is divisible by $p^{m(m-1)/2}$ is the direct product of a non-abelian group of order p^3 and an abelian group of type* $(1, 1, 1, \cdots)$, *and every such direct product has a group of isomorphisms whose order is divisible by $p^{m(m-1)/2}$.*

If we combine this theorem with what precedes it results that there are just four groups of order p^m, $m > 2$, which have groups of isomorphisms whose orders are divisible by p^α, where $\alpha = m(m-1)/2$. Two of these groups are abelian and two are non-abelian. If p^β is the order of an operator in such a group of isomorphisms $\beta \leqq m/2$ when m is even and $\beta \leqq (m+1)/2$ when m is odd. When $p = 2$, no two of the four given groups are conformal but when $p > 2$, each of the two abelian groups is conformal with one of the two non-abelian groups.

Since the series of subgroups $H_0, H_1, H_2, \cdots, H_m$ remains fixed under all the isomorphisms of a Sylow subgroup of order p^α in I, it results directly that *a necessary and sufficient condition that the group of isomorphisms of a group of order p^m involves only one Sylow subgroup of order p^α is that this group of order p^m involves a characteristic subgroup of order p^γ, for every value of γ from 1 to $m-1$.* In particular, every group of order p^m, $m > 3$, which involves operators of order p^{m-1} has a group of isomorphisms involving only one Sylow subgroup of order p^α. When the I of G involves only one Sylow subgroup of order p^α all of its operators whose orders are prime to p must have orders whose prime factors divide $p-1$. Hence the theorem: *A necessary and sufficient condition that the order of the group of isomorphisms of a group of order 2^m is of the form 2^α is that this group of order 2^m has a characteristic subgroup of order 2^γ, $\gamma = 1, 2, \cdots, m-1$.*

When two operators of the group of isomorphisms of any group give rise to

commutators which are invariant under these operators then these operators have evidently the same orders and the same relative properties as their respective commutators. In particular, if a group G contains a subgroup H composed of operators which arise as commutators under operators of the group of isomorphisms of G which are commutative with each operator of H, then this group of isomorphisms involves a subgroup which is simply isomorphic with H. A number of other general theorems relating to the groups of isomorphisms are given in the article entitled: "The groups of isomorphisms of the groups whose degree is less than eight," *Philosophical Magazine*, vol. 231 (1908), p. 223. The present article has close contact with the one just mentioned.

§ 5. *Isomorphisms in which a large number of operators correspond to their inverses.*

If more than three fourths of the operators of a group correspond to their inverses in some one of its possible automorphisms the group must be abelian,* and all the operators of any abelian group evidently correspond to their inverses in one of its automorphisms. The totality of the operators which correspond to their inverses in an automorphism of any abelian group constitutes a subgroup whenever this totality does not include all the operators of the group, but the totality of the operators which correspond to their inverses in an automorphism of a non-abelian group does not necessarily constitute a subgroup, as one may directly see by means of the known theorem: A necessary and sufficient condition that exactly three fourths of the operators of a group correspond to their inverses in one of the possible automorphisms of the group is that its group of cogredient isomorphism is the four-group.† Hence the octic group and the quaternion group are the two groups of smallest order which admit automorphisms in which exactly three fourths of the operators correspond to their inverses.

In the present section we shall consider the non-abelian groups of order p^m in which the largest possible number of operators correspond to their inverses in some one automorphism. In particular, we shall prove the theorem: *A necessary and sufficient condition that a group of order p^m, $p > 2$, be abelian is that more than p^{m-1} of its operators correspond to their inverses in one of its possible automorphisms.* That is, at most p^{m-1} of the operators of any non-abelian group G of order p^m correspond to their inverses in a possible automorphism of G whenever $p > 2$. In what follows it will be assumed that $p > 2$ and that G is non-abelian, unless the contrary is stated. If exactly p^{m-1} of the operators of G correspond to their inverses in an automorphism of G these p^{m-1} operators do not necessarily constitute a subgroup as may be seen by

* Annals of Mathematics, ser. 2, vol. 7 (1906), p. 55.

† MANNING, these Transactions, vol. 7 (1906), p. 233.

considering the automorphisms of the non-abelian group of order p^3 which involves no operator of order p^2.

Suppose that more than p^{m-1} of the operators of G should correspond to their inverses in some automorphism of G. Every invariant operator s would correspond to its inverse in this automorphism. If this were not the case we could consider the quotient group of G with respect to the invariant subgroup generated by s. As not more than $1/p$ of the operators of G which would correspond to an operator of this quotient group could correspond to their inverses, it results that no more than p^{m-1} of the operators of G would correspond to their inverses. This proves that whenever more than p^{m-1} of the operators of a group of order p^m correspond to their inverses in an automorphism of the group then every invariant operator under the group must correspond to its inverse. This theorem applies also to the excluded case where $p = 2$.

To simplify the proof of the theorem that no more than p^{m-1} of the operators of G correspond to their inverses in a possible automorphism, we may consider the quotient group of G with respect to an invariant subgroup of order p, and then the quotient group of this first quotient group with respect to one of its invariant subgroups of order p, etc. By this process we must arrive after a finite number of steps at an abelian quotient group. As more than $1/p$ of the operators of G are supposed to correspond to their inverses in the automorphism under consideration it results that all the operators of this abelian quotient group must correspond to their inverses, while the quotient group which immediately precedes this abelian quotient group has a commutator subgroup of order p. Hence G cannot have an automorphism in which more than p^{m-1} of its operators correspond to their inverses unless a group of order p^β whose commutator subgroup is of order p has an automorphism in which more than $p^{\beta-1}$ of its operators correspond to their inverses.

Suppose that K is such a group of order p^β. Some non-invariant operator s of K must correspond to its inverse in the automorphism under consideration. Let K_1 be the subgroup of order $p^{\beta-1}$ composed of all the operators of K which are commutative with s. As the commutators of K are invariant under K they must correspond to their inverses in the automorphism in question, and hence all the operators of the invariant subgroup generated by s and by these commutators must have the same property. Consider the quotient group of K with respect to this invariant subgroup and observe that not more than $1/p$ of a set of operators of K which correspond to an operator of this quotient group can correspond to their inverses whenever this operator in the quotient group corresponds to operators of K which are not also in K_1.

If K_1 were non-abelian we would repeat the argument of the preceding paragraph as regards one of its non-invariant operators. Hence it remains only to consider the case when K_1 is abelian. As more than $1/p$ of its operators corre-

spond to their inverses in the automorphism under consideration it results that all of these operators must correspond to their inverses. We proceed to prove that none of the operators of K which are not also in K_1 could correspond to its inverse. The group of cogredient isomorphisms of K is assumed to be of order p^2 so that K_1 involves $p^{\beta-2}$ operators which are invariant under K, and K_1 is generated by s and these invariant operators. Let s_0, t represent respectively a commutator of order p and an operator of K which is not also in K_1, and assume that

$$t^{-1}st = s_0 s, \quad \text{or that} \quad tst^{-1} = s_0^{-1} s.$$

Since s_0, s_0^{-1} and s, s^{-1} are two pairs of corresponding operators in the automorphism under consideration, as it was assumed that all the operators of K_1 correspond to their inverses in this automorphism, it results that t cannot correspond to t^{-1}. We have thus arrived at an absurdity by assuming that more than p^{m-1} of the operators of a group of order p^m can correspond to their inverses in some automorphism of the group and hence we have arrived at the theorem announced in the second paragraph of the present section, which may also be stated as follows: *A non-abelian group of order p^m, $p > 2$, has no automorphism in which more than p^{m-1} of its operators correspond to their inverses.* When $p = 2$ there are non-abelian groups of order p^m, for every value of $m > 2$, in which $\frac{3}{2}p^{m-1}$ of the operators correspond to their inverses, as was noted above.

The p^{m-1} operators of a non-abelian group of order p^m which correspond to their inverses in a possible automorphism of the group do not necessarily constitute a subgroup. If they constitute a subgroup this must be abelian but it does not follow that the operators of every abelian subgroup of order p^{m-1} in a non-abelian group of order p^m can correspond to their inverses in a possible automorphism of the group. In fact, it is very easy to see that all the operators of any one of the p cyclic subgroups of order p^{m-1} in the group of order p^m, $p > 2$, which involves operators of order p^{m-1} may correspond to their inverses in automorphisms of this group but the operators of the non-cyclic group of order p^{m-1} cannot all correspond to their inverses in a possible automorphism of this group. That is, *the group of isomorphisms of the non-abelian group of order p^m, $p > 2$, which involves operators of order p^{m-1} contains exactly p^2 operators of order 2 which transform p^{m-1} of the operators of the group into their inverses.*

This special theorem may serve to illustrate the more general developments of the following paragraphs. In the first place, it should be observed that every automorphism of a non-abelian group in which exactly p^{m-1} of the operators correspond to their inverses is affected by an operator of order 2 in the group of isomorphisms of this non-abelian group. This is evident in case the operators which correspond to their inverses generate the entire group. If they do not

generate the entire group they constitute an abelian subgroup and each of the remaining operators is transformed into itself multiplied by an operator of this subgroup. As the commutators must all correspond to their inverses, the theorem has been proved. It may be observed that this proof does not necessarily hold when the group of order p^m is abelian, since in this case the commutators need not correspond to their inverses.

Suppose that s_1, s_2 are two non-commutative operators of any group, which correspond to their inverses in some automorphism of the group. From the equations $s_0 = s_2^{-1} s_1^{-1} s_2 s_1$, $s_0 s_1 s_2 = s_1 s_2 s_0$, we deduce the following:

$$s_2^{-1} s_1 s_2 = s_1 s_0^{-1}, \qquad s_0^{-1} = s_1 s_2 s_1^{-1} s_2^{-1}, \qquad s_1^{-1} s_0^{-1} = s_2 s_1^{-1} s_2^{-1}.$$

Hence it results that s must correspond to itself in this automorphism. In other words, if the product of two operators, which correspond to their inverses in an automorphism of a group, is commutative with their commutator then this commutator must correspond to itself in this automorphism. Since no operator is transformed into its inverse under the group of cogredient isomorphisms of a group of order p^m, $p > 2$, it results that *if two non-commutative operators of a group of order p^m, $p > 2$, correspond to their inverses in an automorphism of the group their commutator cannot correspond to its inverse in this automorphism.*

This theorem is a special case of the theorem, if two operators and their commutator correspond to their inverses in an automorphism of a group this commutator is transformed into its inverse by the product of these two operators. The proof of this more general theorem results immediately from the following considerations. Since s_1, s_2, $s_1^{-1} s_2^{-1} s_1 s_2$ correspond to their inverses in some automorphisms of the group involving s_1, s_2 we have that $s_1^{-1} s_2^{-1} s_1 s_2$ must correspond to both of the operators $s_2^{-1} s_1^{-1} s_2 s_1$ and $s_1 s_2 s_1^{-1} s_2^{-1}$ in the automorphism in question. That is,

$$(s_1 s_2)^{-1} s_2 s_1 = s_1 s_2 (s_2 s_1)^{-1}, \qquad \text{or} \qquad (s_1 s_2)^2 = (s_2 s_1)^2.$$

From the last equations and the fact that each of two operators having a common square transforms into its inverse the product of one of these operators and the inverse of the other * the theorem in question results immediately.

Suppose that G is a non-abelian group of order p^m, $p > 2$, in which p^{m-1} operators correspond to their inverses in an automorphism and these p^{m-1} operators constitute a subgroup H. It has been observed that H is abelian. We proceed to prove that G involves operators of order p which are not contained in H. Let s be any operator of G that is not also in H and let s_1 be the corresponding operator in the given automorphism. From the fact that the commutators of G correspond to their inverses it results that $s_1 = s' s$ where s' is an operator of H.

*Archiv der Mathematik und Physik (3), vol. 9 (1905), p. 6.

Since $s_a s$ must correspond to $s_a^{-1} s' s$, s_a being any operator of H, it results that some operators of G which are not also in H must correspond to themselves in the given automorphism. These operators must be of order p since their pth powers correspond to their inverses. Hence it results that *a necessary and sufficient condition that a non-abelian group of order* p^m, $p > 2$, *has an automorphism in which* p^{m-1} *operators forming a subgroup correspond to their inverses is that this non-abelian group involves an abelian subgroup of order* p^{m-1} *which does not include all the operators of order* p *in the group.*

When the p^{m-1} operators of G which correspond to their inverses do not constitute a subgroup we may consider the smallest non-abelian quotient group of G. As the smallest non-abelian quotient group of any group of order p^m has a commutator subgroup of order p and a group of cogredient isomorphisms of order p^2, it results that the smallest non-abelian quotient group of G is of order p^3 since its generators correspond to their inverses in the given automorphism. As the commutators of order p of this quotient group correspond to themselves, the subgroup of G which corresponds to this commutator subgroup must involve p^{m-3} operators which correspond to their inverses and form an abelian invariant subgroup of G.

This invariant subgroup of order p^{m-3} must be composed of invariant operators under G since the commutator of any one of its operators and any other operator of G which corresponds to its inverse in the given automorphism is the identity, as this commutator corresponds to its inverse in this automorphism. Hence it results that G involves an invariant abelian subgroup of order p^{m-2} which corresponds to the commutator subgroup of its smallest non-abelian quotient group, and that the group of cogredient isomorphisms of G is either of order p^2 or of order p^3. This proves the theorem: *If a group of order* p^m, $p > 2$, *admits an automorphism in which* p^{m-1} *operators, which do not constitute a subgroup, correspond to their inverses, the order of its group of cogredient isomorphisms is a divisor of* p^3.

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ON MINIMAL LINES AND CONGRUENCES IN FOUR-DIMENSIONAL SPACE*

BY

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A study of the ∞^5 minimal lines in four-dimensional space and the configurations obtained by considering certain ensembles of these will yield interesting results, if we employ a method similar to that used by LIE † in the case of minimal lines of ordinary space.

§ 1. *The minimal cone.*

Consider a minimal element-cone in space S_4; its equation may be written

$$dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2 = 0. \tag{1}$$

A flat space

$$AX_1 + BX_2 + CX_3 + DX_4 + E = 0 \tag{2}$$

is said to be tangent to (1) whenever the following relation between the coefficients holds:

$$A^2 + B^2 + C^2 + D^2 = 0. \tag{3}$$

We shall call (2) a minimal space, when (3) is satisfied. There exist therefore ∞^3 minimal spaces.

We shall now express A, B, C, D as functions of two parameters so that (3) is identically satisfied, by writing

$$A = s + t, \qquad B = i(t - s), \qquad C = st - 1, \qquad D = i(st + 1),$$

so that the equation of the minimal space takes the form

$$(s + t)X_1 + i(t - s)X_2 + (st - 1)X_3 + i(st + 1)X_4 + E = 0. \tag{4}$$

If now we wish to obtain a minimal M_3, we put $E = F(s, t)$ in (4) and differentiate with respect to s and t. We have then the following system

$$\begin{aligned} (s + t)X_1 + i(t - s)X_2 + (st - 1)X_3 + i(st + 1)X_4 + F &= 0, \\ X_1 - iX_2 + tX_3 + itX_4 + F'_s &= 0, \\ X_1 + iX_2 + sX_3 + isX_4 + F'_t &= 0, \end{aligned} \tag{5}$$

* Presented to the Society, September 6, 1910.

† LIE-SCHEFFERS, *Geometrie der Berührungstransformationen*, vol. 1, chap. 10, pp. 411–480, and chap. 12.

from which by eliminating s and t we get the required minimal manifoldness M_3 expressed in cartesian coördinates. If s and t are considered fixed, we have the equations of a minimal line, which may be written, solving (5) for X_1, X_2, and X_3,

$$X_1 = \frac{-i(s+t)}{1+st}X_4 - \frac{(s+t)F + (1-s^2)F'_s + (1-t^2)F'_t}{2(1+st)},$$

$$X_2 = \frac{t-s}{1+st}X_4 + i\frac{(s-t)F - (1+s^2)F'_s + (1+t^2)F'_t}{2(1+st)}, \tag{6}$$

$$X_3 = \frac{-i(st-1)}{1+st}X_4 + \frac{F - sF'_s - tF'_t}{1+st}.$$

If we consider s, t, F, F'_s, F'_t as the coördinates of the minimal line (6), we see at once that *in four-dimensional space there exist* ∞^5 *minimal lines, and through any fixed point pass* ∞^2 *such lines.*

If

$$F = ast + bs + ct + d,$$

we obtain the following system of lines

$$X_1 = \frac{-i(s+t)}{1+st}X_4 - \frac{(b+c)(1+st) + (a+d)(s+t)}{2(1+st)},$$

$$X_2 = \frac{t-s}{1+st}X_4 + i\frac{(c-b)(1+st) + (a+d)(s-t)}{2(1+st)}, \tag{7}$$

$$X_3 = \frac{i(1-st)}{1+st}X_4 + \frac{d-ast}{1+st},$$

which may be written in the form

$$X_1 + \frac{b+c}{2} = -\frac{i(s+t)}{1+st}\left[X_4 - \frac{i(a+d)}{2}\right],$$

$$X_2 - \frac{i(c-b)}{2} = \frac{t-s}{1+st}\left[X_4 - \frac{i(a+d)}{2}\right], \tag{8}$$

$$X_3 - \frac{d-a}{2} = \frac{i(1-st)}{1+st}\left[X_4 - \frac{i(a+d)}{2}\right].$$

Eliminating s and t, we obtain the minimal cone

$$\left(X_1 + \frac{b-c}{2}\right)^2 + \left(X_2 - \frac{i(c-b)}{2}\right)^2 + \left(X_3 - \frac{d-a}{2}\right)^2 + \left(X_4 - \frac{i(a+d)}{2}\right)^2 = 0,$$

which may be considered as the ensemble of all the minimal lines passing through the point

$$-\frac{b+c}{2}, \quad \frac{i(c-b)}{2}, \quad \frac{d-a}{2}, \quad \frac{i(d+a)}{2}.$$

If now we consider F, s and t as cartesian coördinates of a three-dimensional space S_3, we may state the result obtained thus:

To all the ∞^4 paraboloids $F = ast + bs + ct + d$ in the space S_3 there correspond ∞^4 minimal cones in S_4.

To all the ∞^2 points of the paraboloid correspond the ∞^2 minimal lines which pass through the point

$$(9)\quad \alpha = -\frac{c+b}{2}, \qquad \beta = \frac{i(c-b)}{2}, \qquad \gamma = \frac{d-a}{2}, \qquad \delta = \frac{i(d+a)}{2}.$$

There is a one-to-one correspondence between the paraboloids

$$F = ast + bs + ct + d$$

of the space S_3 and the points of a four-dimensional space. The addition of

$$ast + bs + ct + d$$

to the function F in (6) *has the effect of translating the minimal M_3 to the point $(\alpha, \beta, \gamma, \delta)$. Finally, by virtue of equations* (5) *there is established a one-to-one correspondence between the ∞^5 minimal lines of S_4 and the ∞^5 surface-elements F, s, t, F'_s, F'_t of S_3.* This correspondence will be discussed in Sections 7 and 8 of this paper.

The system (6) defines an isotropic congruence of lines in four-dimensional space. If we interpret the quantities dX_1/dX_4, dX_2/dX_4, dX_3/dX_4 as coördinates in space at infinity, we see at once that any one of these lines meets the imaginary sphere at infinity in a point; hence we may consider the quantities

$$(10)\qquad \frac{-i(s+t)}{1+st}, \qquad \frac{t-s}{1+st}, \qquad \frac{-i(st-1)}{1+st}$$

as coördinates of points on this sphere. If we have an isotropic congruence, the quantities F, F'_s and F'_t are fixed whenever s and t are; hence, *to every point on the sphere at infinity corresponds a minimal line of the congruence.*

§ 2. *The focal surface.*

We may obtain a focal surface of the congruence (6) in the following manner. We choose a generator D of the congruence and pass through it a developable surface of two dimensions, an M_2, whose generators belong to the congruence. If s_0 and t_0 are the parameters belonging to the generator D, it is clear that we must find a curve determined by a relation $t = \phi(s)$ such that the generators are tangent to it and moreover such that $t_0 = \phi(s_0)$. The curve will then be the edge of regression of the minimal developable M_2. Writing the condition that two consecutive generators intersect, we have in addition to the three equations (5) the following three:

$$(X_1 + iX_2)dt + dF - sdF'_s - F'_s ds = 0,$$

$$(X_3 + iX_4)dt + dF'_s = 0, \tag{11}$$

$$(X_3 + iX_4)ds + dF'_t = 0.$$

If from the six equations (5) and (11) we eliminate X_1, X_2, X_3, X_4, we obtain the two relations

$$dF - F'_s ds - F'_t dt = 0,$$

$$F''_{s^2} ds^2 - F''_{t^2} dt^2 = 0.$$

The first relation is obviously satisfied for any isotropic congruence provided $F(s, t)$ is finite and continuous and possesses finite and continuous derivatives. The second condition shows that the edge of regression is determined as either of the two solutions of the differential equation

$$F''_{s^2} ds^2 - F''_{t^2} dt^2 = 0 \tag{12}$$

which for $s = s_0$ gives $t_0 = \phi(s_0)$. But such a solution always exists, hence *there exist two minimal developables passing through an arbitrary generator D of the congruence.* In order to obtain the focal surface of the ∞^1 minimal developables we substitute the value of dt from (12) in the last of equations (11), so that we have

$$X_3 + iX_4 + F''_{st} + \sqrt{F''_{s^2} F''_{t^2}} = 0, \tag{13}$$

or, using the second value of ds,

$$X_3 + iX_4 + F''_{st} - \sqrt{F''_{s^2} F''_{t^2}} = 0. \tag{13'}$$

The equations of the focal surface may now be obtained by solving (5) and (13), or (5) and (13'), for X_1, X_2, X_3, and X_4. We find

$$\begin{aligned}
X_1 &= \tfrac{1}{2}[(s+t)(F''_{st} \pm \sqrt{F''_{s^2} F''_{t^2}}) - F'_t - F'_s], \\
X_2 &= -\frac{i}{2}[(s-t)(F''_{st} \mp \sqrt{F''_{s^2} F''_{t^2}}) - F'_t + F'_s], \\
X_3 &= \tfrac{1}{2}[(st-1)(F''_{st} \pm \sqrt{F''_{s^2} F''_{t^2}}) - tF'_t - sF'_s + F], \\
X_4 &= \frac{i}{2}[(st+1)(F''_{st} \pm \sqrt{F''_{s^2} F''_{t^2}}) - tF'_t - sF'_s + F].
\end{aligned} \tag{14}$$

We have thus proved the

THEOREM. *An isotropic congruence of four-dimensional space has a focal surface consisting of two sheets. The edges of regression on the focal surface are minimal curves satisfying the differential equation*

$$F''_{s^2} ds^2 - F''_{t^2} dt^2 = 0.$$

Through each generator of the congruence pass two minimal developables M_2 which are tangent to the surface.

The linear element $d\sigma$ of the surface (14) may be obtained without difficulty. We find

$$d\sigma^2 = -\sqrt{F''_{s^2}F''_{t^2}}\,(\sqrt{F''_{s^2}}\,ds - \sqrt{F''_{t^2}}\,dt)^2 \tag{15}$$

for the focal sheet S_1 and

$$d\sigma^2 = \sqrt{F''_{s^2}F''_{t^2}}\,(\sqrt{F''_{s^2}}\,ds + \sqrt{F''_{t^2}}\,dt)^2 \tag{15'}$$

for the second sheet S_2. Introducing the minimal curves on S_1 as one set of coördinate curves, we may reduce the element (15) or (15′) to the form

$$d\sigma^2 = \lambda^2 d\beta^2,$$

where λ is a function of β and a second coördinate α which for (15) is such that the expression

$$\frac{i\sqrt[4]{F''_{s^2}F''_{t^2}}\,(\sqrt{F''_{s^2}}\,ds - \sqrt{F''_{t^2}}\,dt)}{\lambda} \tag{16}$$

is an exact differential. Similarly for (15′). The element is thus brought into the same form as that of a minimal developable in ordinary space referred to its minimal curves as one set of coördinate lines. *The focal surface* (14) *is, however, not developable, for the curves $\beta = const.$ are not minimal straight lines, but curves.* To show this it is only necessary to take a single example. We put $F = s^2t^2$ and form the corresponding equations of the focal sheets:

$$(17)\qquad S_1:\quad \begin{aligned} X_1 &= 2st(s+t),\\ X_2 &= -2ist(s-t),\\ X_3 &= \tfrac{3}{2}st(st-2),\\ X_4 &= \tfrac{3}{2}ist(st+2). \end{aligned} \qquad S_2:\quad \begin{aligned} X_1 &= 0,\\ X_2 &= 0,\\ X_3 &= -\tfrac{1}{2}st(st+2),\\ X_4 &= -\frac{i}{2}st(st-2). \end{aligned}$$

The linear elements of these surfaces are respectively

$$d\sigma_1^2 = -\,[\,2\sqrt{st}\,t\,ds - 2\sqrt{st}\,s\,dt\,]^2,$$

$$d\sigma_2^2 = [\,2\sqrt{st}\,t\,ds + 2\sqrt{st}\,s\,dt\,]^2.$$

If we consider the focal sheet S_1, we have

$$\lambda = 2i\sqrt{st}\,st,$$

so that we get

$$\frac{ds}{s} - \frac{dt}{t} = d\beta.$$

Integrating and putting e^s equal to a new β, we have

$$\frac{s}{t} = \beta.$$

Choosing the lines t for the second set of coördinates, the equations for S_1 become

$$X_1 = 2t^3\beta(1+\beta), \qquad X_2 = -2it^3\beta(\beta-1),$$
$$X_3 = \tfrac{3}{2}t^2\beta(t^2\beta-2), \qquad X_4 = \tfrac{3}{2}it^2\beta(t^2\beta+2).$$

The curves $\beta = \text{const.}$ are minimal curves, since

$$\Sigma\left(\frac{\partial X_i}{\partial t}\right)^2 = 0;$$

it is also readily verified that

$$\Sigma\frac{\partial X_i}{\partial t}\cdot\frac{\partial X_i}{\partial \beta} = 0.$$

The second sheet S_2 reduces to a curve in the plane $X_1=0$, $X_2=0$. We have

$$\frac{ds}{s} + \frac{dt}{t} = d\beta,$$

hence, putting $st = \beta$ the equations become

$$X_1 = 0, \qquad X_2 = 0, \qquad X_3 = -\tfrac{1}{2}\beta(\beta+2), \qquad X_4 = \frac{-i}{2}\beta(\beta-2).$$

§ 3. *The linear element.*

In ordinary space a ruled surface whose generators are minimal lines is necessarily a minimal developable and its linear element is a perfect square. It is different in four-dimensional space. A ruled surface with minimal lines as generators is not in general a minimal developable. Let there be given a ruled surface

$$X_1 = f_1\alpha + \phi_1, \qquad X_2 = f_2\alpha + \phi_2,$$
$$X_3 = f_3\alpha + \phi_3, \qquad X_4 = f_4\alpha + \phi_4,$$

and let us suppose that $\beta = \text{const.}$ are minimal lines, in which case we must have

$$f_1^2 + f_2^2 + f_3^2 + f_4^2 = 0. \tag{18}$$

If this surface is a minimal developable, two consecutive lines on the surface must intersect, hence we must have

$$X_1 = (f_1 + df_1)\alpha + \phi_1 + d\phi_1 = f_1\alpha + \phi_1,$$
$$X_2 = (f_2 + df_2)\alpha + \phi_2 + d\phi_2 = f_2\alpha + \phi_2,$$
$$X_3 = (f_3 + df_3)\alpha + \phi_3 + d\phi_3 = f_3\alpha + \phi_3,$$
$$X_4 = (f_4 + df_4)\alpha + \phi_4 + d\phi_4 = f_4\alpha + \phi_4,$$

from which we get

$$\alpha f''_1 + \phi'_1 = 0, \qquad \alpha f''_2 + \phi'_2 = 0, \qquad \alpha f''_3 + \phi'_3 = 0, \qquad \alpha f''_4 + \phi'_4 = 0,$$

or,

$$\frac{\phi'_1}{f''_1} = \frac{\phi'_2}{f''_2} = \frac{\phi'_3}{f''_3} = \frac{\phi'_4}{f''_4}, \tag{19}$$

which is the condition that the ruled surface shall be developable.

The linear element is now a perfect square. In fact, taking account of the relation (19) and the identities

$$\sum_1^4 f_i^2 = 0, \qquad \sum_1^4 f_i f''_i = 0, \qquad \sum_1^4 f_i \phi'_i = 0,$$

we have

$$ds^2 = \sum_1^4 (\alpha f''_i + \phi'_i)^2 d\beta^2 = \left(\alpha + \frac{\phi'_4}{f''_4}\right)^2 \Sigma f''^2_i \cdot d\beta^2.$$

The edge of regression is

$$X_1 = -f_1 \frac{\phi'_1}{f''_1} + \phi_1, \qquad X_2 = -f_2 \frac{\phi'_2}{f''_2} + \phi_2,$$

$$X_3 = -f_3 \frac{\phi'_3}{f''_3} + \phi_3, \qquad X_4 = -f_4 \frac{\phi'_4}{f''_4} + \phi_4,$$

which is seen to be a minimal curve, if account is taken of the conditions (18) and (19). We have thus proved the

THEOREM. *If a ruled surface in four-dimensional space whose generators are minimal lines is a developable surface, its linear element is a perfect square.*

We have already seen that the converse of this theorem is not true, since the focal surface of an isotropic congruence is not developable, although its element is a perfect square.

We shall now prove the

THEOREM. *If a surface in four-dimensional space has its linear element a perfect square, it is either a minimal developable, or the focal surface of* ∞^1 *such developables.*

Let the linear element be written

$$ds^2 = (\sqrt{E}\,du + \sqrt{G}\,dv)^2.$$

If we put

$$\frac{\sqrt{E}\,du + \sqrt{G}\,dv}{\lambda} = d\beta,$$

where λ is a function of u and v which will make the left side of the equation an exact differential, the linear element takes the form

$$ds^2 = \lambda^2 d\beta^2. \tag{20}$$

There exist therefore on the surface ∞^1 minimal curves $\beta = \text{const.}$ (a single family it should be noticed) which may serve as one set of lines of reference, an arbitrary system $\alpha = \text{const.}$ being chosen as the second set. A minimal curve $\beta = \text{const.}$ determines a minimal developable whose edge of regression is the minimal curve; *this minimal developable will touch the surface along a curve whose direction is conjugate* to the curve* $\beta = const.$ In fact, we have from (20),

$$\Sigma\left(\frac{\partial X_i}{\partial \alpha}\right)^2 = 0, \qquad \Sigma \frac{\partial X_i}{\partial \alpha}\frac{\partial X_i}{\partial \beta} = 0. \tag{21}$$

Differentiating the first with respect to α and β, we have

$$\Sigma \frac{\partial X_i}{\partial \alpha}\frac{\partial^2 X_i}{\partial \alpha^2} = 0, \qquad \Sigma \frac{\partial X_i}{\partial \alpha}\frac{\partial^2 X_i}{\partial \alpha \partial \beta} = 0. \tag{22}$$

From (21) and (22) it follows that the determinant

$$e = \left|\frac{\partial X_1}{\partial \alpha}, \frac{\partial X_2}{\partial \beta}, \frac{\partial^2 X_3}{\partial \alpha^2}, \frac{\partial^2 X_4}{\partial \alpha \partial \beta}\right|$$

vanishes. But this is precisely the condition that two consecutive tangent planes at (α, β) and $(\alpha + d\alpha, \beta)$ shall intersect in a line. The lines β and the lines whose direction is the direction of the line of intersection of the two tangent planes are therefore conjugate lines. Since the system $\alpha = \text{const.}$ was arbitrary, we may introduce this conjugate system as our lines of reference. The new lines α' will then be the solution of the differential equation

$$2f d\alpha + g d\beta = 0,$$

where f and g are the determinants

$$\left|\frac{\partial X_1}{\partial \alpha}, \frac{\partial X_2}{\partial \beta}, \frac{\partial^2 X_3}{\partial \alpha^2}, \frac{\partial^2 X_4}{\partial \beta^2}\right|, \qquad \left|\frac{\partial X_1}{\partial \alpha}, \frac{\partial X_2}{\partial \beta}, \frac{\partial^2 X_3}{\partial \alpha \partial \beta}, \frac{\partial^2 X_4}{\partial \beta^2}\right|.$$

* There exists on a surface M_2 in S_4 a double family of curves which have been called asymptotic lines by K. KOMMERELL. They form a unique system of conjugate lines, that is to say, if we pass along an asymptotic line of the first set with a tangent plane to the surface, any two consecutive planes will intersect in a line whose direction is the direction of an asymptotic line of the second set. (It should be remembered that in S_4 planes intersect in general in a point.) In ordinary space there exists at any point on a surface an involution of such lines passing through the point; in S_4 there are only two conjugate directions at any point on an M_2. See KOMMERELL, Thesis, pp. 30 and 31. Also a paper by the same author, *Riemannsche Flächen in ebenem Raum von vier Dimensionen*, Mathematische Annalen, vol. 60 (1905), pp. 548–596. KOMMERELL's thesis is entitled: *Die Krümmung der zweidimensionalen Gebilde in ebenem Raum von vier Dimensionen*, Thesis, Tübingen, 1897.

The differential equation of these asymptotic-conjugate lines is:

$$e d\alpha^2 + 2f d\alpha d\beta + g d\beta^2 = 0,$$

where e, f and g are the determinants mentioned above.

The surface being thus referred to its unique system of conjugate-asymptotic lines, it follows from the property of such a system that if we pass along a curve $\alpha = \text{const.}$, the consecutive tangent planes will intersect in minimal tangent lines whose directions are those of the minimal curves $\beta = \text{const.}$ The surface is therefore the focal surface of the ∞^2 minimal lines determined by the curves $\beta = \text{const.}$ If the curves $\beta = \text{const.}$ are straight lines, the surface is a minimal developable.

Consider now the focal surface (14) and a generator D_1 tangent to the minimal curves C_1 and C_2 on the two sheets S_1 and S_2 respectively. Let T_1 and T_2 be the curves conjugate to C_1 and C_2 respectively and passing through the focal points F_1 and F_2. The tangent planes at consecutive points on an edge of regression intersect in a line which must be tangent to the conjugate curve T_2 at F_2. Likewise, tangent planes at F_2 and at a consecutive point on T_2 intersect in a line which is D_1. Hence D_1 and the tangent to T_2 are conjugate and asymptotic directions on S_2 at F_2. The same will be true on S_1 for the curves C_1 and T_1.

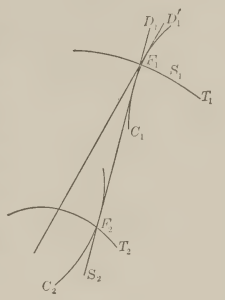

FIG. 1.

If the curves C_2 satisfy the differential equation

$$\sqrt{F''_{s^2}}\,ds - \sqrt{F''_{t^2}}\,dt = 0, \tag{23}$$

the curves T_2 will satisfy the equation

$$\sqrt{F''_{s^2}}\,ds + \sqrt{F''_{t^2}}\,dt = 0. \tag{24}$$

Again, the equation (24) will be satisfied by the curves C_1 on S_1, that is, by the edges of regression, while the curves T_1 will satisfy the equation (23). We have thus proved the

THEOREM. *On each sheet of the focal surface the family of lines whose differential equation is*

$$F''_{s^2}ds^2 - F''_{t^2}dt^2 = 0$$

constitutes a double set of conjugate and asymptotic lines.

§ 4. *The middle surface.*

The generator D which determines a pair of minimal developables is a common tangent to the two edges of regression on S_1 and S_2 respectively. Suppose now we take the mid-points of all the segments F_1F_2, F_1 and F_2 being the focal points. We obtain a new locus which we shall call the *middle surface* of the congruence. From equations (14) we find that the equations of this surface are

$$X_1 = \tfrac{1}{2}[(s+t)F''_{st} - F'_t - F'_s],$$

$$
\begin{aligned}
X_2 &= -\frac{i}{2}[(s-t)F''_{st} - F'_t + F'_s],\\
X_3 &= \tfrac{1}{2}[(st-1)F''_{st} - tF'_t - sF'_s + F],\\
X_4 &= \frac{i}{2}[(st+1)F''_{st} - tF'_t - sF'_s + F].
\end{aligned} \tag{25}
$$

The lines s and t on the surface are minimal lines. In fact, if we calculate the linear element dS we find that

$$\Sigma\left(\frac{\partial X_i}{\partial s}\right)^2 \quad \text{and} \quad \Sigma\left(\frac{\partial X_i}{\partial t}\right)^2$$

vanish and the following simple form results

$$dS^2 = F''_{t^2}F''_{s^2}ds\,dt. \tag{26}$$

We shall next prove the following

THEOREM. *The surface* (25) *is the envelope of the* ∞^2 *minimal spaces,*

$$(s+t)X_1 + i(t-s)X_2 + (st-1)X_3 + i(st+1)X_4 + F = 0. \tag{27}$$

In fact, this space passes through the four vertices $(s,\ t)$; $(s+ds,\ t)$; $(s,\ t+dt)$; $(s+ds,\ t+dt)$ of the elementary quadrilateral formed by the coördinate lines s and t. From (25) we have

$$
\begin{aligned}
&(s+t)\frac{\partial X_1}{\partial s} + i(t-s)\frac{\partial X_2}{\partial s} + (st-1)\frac{\partial X_3}{\partial s} + i(st+1)\frac{\partial X_4}{\partial s} = 0,\\
&(s+t)\frac{\partial X_1}{\partial t} + i(t-s)\frac{\partial X_2}{\partial t} + (st-1)\frac{\partial X_3}{\partial t} + i(st+1)\frac{\partial X_4}{\partial t} = 0,\\
&(s+t)\frac{\partial^2 X_1}{\partial s\,\partial t} + i(t-s)\frac{\partial^2 X_2}{\partial s\,\partial t} + (st-1)\frac{\partial^2 X_3}{\partial s\,\partial t} + i(st+1)\frac{\partial^2 X_4}{\partial s\,\partial t} = 0,
\end{aligned} \tag{28}
$$

which proves the statement. The minimal lines of an M_3 being given by the equations

$$
\begin{aligned}
(s+t)X_1 + i(t-s)X_2 + (st-1)X_3 + i(st+1)X_4 + F &= 0,\\
X_1 - iX_2 + t(X_3 + iX_4) + F'_s &= 0,\\
X_1 + iX_2 + s(X_3 + iX_4) + F'_t &= 0,
\end{aligned} \tag{29}
$$

the middle surface may be obtained by adding to this set the minimal space

$$X_3 + iX_4 + F'''_{st} = 0 \tag{30}$$

and solving for X_1, X_2, X_3 and X_4. The minimal space (30) is obtained by differentiating either the second of (29) with respect to s, or the third with respect to t.

If we choose F as a solution of the differential equation

$$X_4 = \frac{i}{2}\left[(st+1)F''_{st} - tF'_t - sF'_s + F\right] = 0, \tag{31}$$

we shall obtain a surface situated in ordinary space (X_1, X_2, X_3). The general solution of (31) is

$$F = 2tS + 2sT - (1+st)(S' + T'), \tag{32}$$

where S and T are functions of s and t respectively. Introducing this value of F in equations (25) we find the following equations

$$\begin{aligned} X_1 &= \tfrac{1}{2}(1-s^2)S'' + sS' - S + \tfrac{1}{2}(1-t^2)T'' + tT' - T, \\ X_2 &= \frac{i}{2}(1+s^2)S'' - isS' + iS - \frac{i}{2}(1+t^2)T'' + itT' - iT, \\ X_3 &= sS - S' + tT'' - T', \\ X_4 &= 0, \end{aligned} \tag{33}$$

which we recognize at once as the most general equations of a minimal surface in the so-called Weierstrassian form. The corresponding focal surface is obtained from (14) by substituting for F the value obtained above. We have then, calling the coördinates of the focal surface $\bar{X}_1$, $\bar{X}_2$, $\bar{X}_3$, $\bar{X}_4$ and those of the middle surface X_1, X_2, X_3, X_4, $(X_4 = 0)$,

$$\begin{aligned} \bar{X}_1 &= X_1 \pm \tfrac{1}{2}(s+t)(1+st)\sqrt{S'''T'''}, \\ \bar{X}_2 &= X_2 \mp \frac{i}{2}(s-t)(1+st)\sqrt{S'''T'''}, \\ \bar{X}_3 &= X_3 \pm \tfrac{1}{2}(st-1)(1+st)\sqrt{S'''T'''}, \\ \bar{X}_4 &= \pm \tfrac{1}{2}(st+1)^2\sqrt{S'''T'''}, \end{aligned} \tag{34}$$

where the upper sign goes with the focal sheet S_1 and the lower with S_2. From these equations it appears that the focal surface may be obtained in the following manner when the minimal surface is given: We construct in ordinary space, $X_4 = 0$, the point (X_1, X_2, X_3) on the minimal surface; from this point we lay off a distance iR, where R is one of the principal radii of curvature of the minimal surface at the point (X_1, X_2, X_3), parallel to the X_4-axis; the point thus found is a point on the focal sheet S_1, if the upper sign is used. If the lower sign is used, we obtain a point on S_2 which is symmetrical to the first, the center of symmetry being the point s, t on the minimal surface. That $\bar{X}_4 = \pm iR$ may be verified in the usual manner. (See Darboux, *Théorie générale des surfaces*, vol. 1, p. 302, equations (15).) We may therefore say:

A minimal surface in ordinary space is the middle surface of a certain minimal congruence of four-dimensional space, viz., the congruence for which F is defined by equations (32).

This new definition of an ordinary minimal surface suggests a definition for a minimal surface in four-dimensional space. The usual definition fails when extended to higher space, since two surfaces meet usually in discrete points.*

Definition. A minimal surface in four-dimensional space is the middle surface of an isotropic congruence.

The most general equations of such a surface referred to its minimal curves are given by (25). They include ordinary minimal surfaces in three-dimensional space, namely when F is a solution of any one of the four differential equations

$$\begin{aligned}
(s+t)F''_{st} - F'_t - F'_s &= 0,\\
(s-t)F''_{st} - F'_t + F'_s &= 0,\\
(st-1)F''_{st} - tF'_t - sF'_s + F &= 0,\\
(st+1)F''_{st} - tF'_t - sF'_s + F &= 0.
\end{aligned} \tag{35}$$

Either one of the four values of F introduced in (25) will give the equations of an ordinary minimal surface in the most general form. The solution of the last equation will furnish, as we have seen, the Weierstrassian form; the other forms may be deduced from this by the usual substitutions employed in the theory of minimal surfaces.

§ 5. *Orthogonal projections.*

We shall put the equations of the focal surface (14) in the following form:

$$\begin{aligned}
\bar{X}_1 &= X_1 \pm \tfrac{1}{2}(s+t)\sqrt{F''_{s^2}F''_{t^2}},\\
\bar{X}_2 &= X_2 \mp \frac{i}{2}(s-t)\sqrt{F''_{s^2}F''_{t^2}},\\
\bar{X}_3 &= X_3 \pm \tfrac{1}{2}(st-1)\sqrt{F''_{s^2}F''_{t^2}},\\
\bar{X}_4 &= X_4 \pm \frac{i}{2}(st+1)\sqrt{F''_{s^2}F''_{t^2}},
\end{aligned} \tag{36}$$

where X_i are the coördinates of the corresponding middle or minimal surface. The following relation between the coördinates $\bar{X}_i$ and X_i is seen to hold:

$$(\bar{X}_1 - X_1)^2 + (\bar{X}_2 - X_2)^2 + (\bar{X}_3 - X_3)^2 + (\bar{X}_4 - X_4)^2 = 0. \tag{37}$$

We shall now make use of a transformation which has been employed by Lie in the case of $n = 3$.† Let (X_1, X_2, X_3, X_4) be the vertex of a minimal

*See KOMMERELL, loc. cit., p. 588.

†See LIE-SCHEFFERS, loc. cit., pp. 428–429.

cone in S_4 and (x_1, x_2, x_3, x_4) the running coördinates. The equation of the cone is then

$$(38) \qquad (x_1 - X_1)^2 + (x_2 - X_2)^2 + (x_3 - X_3)^2 + (x_4 - X_4)^2 = 0.$$

If we cut this cone by the space $x_4 = 0$, we obtain the sphere in three-dimensional space S_3

$$(39) \qquad (x_1 - X_1)^2 + (x_2 - X_2)^2 + (x_3 - X_4)^2 + X_4^2 = 0.$$

We now establish a correspondence between the points of the space (X_1, X_2, X_3, X_4) and the spheres in S_3 by means of the transformation

$$(40) \qquad x_1 = X_1, \qquad x_2 = X_2, \qquad x_3 = X_3, \qquad R = iX_4,$$

such that the vertex of the minimal cone is projected on $X_4 = 0$ into the center of the sphere, the radius being equal to $R = iX_4$. To every point (X_1, X_2, X_3, X_4) corresponds a sphere of radius R and center $(x_1, x_2, x_3, 0)$, and conversely, to every sphere in the space $x_4 = 0$ with center (x_1, x_2, x_3) and radius R there correspond the two points

$$(41) \qquad X_1 = x_1, \qquad X_2 = x_2, \qquad X_3 = x_3, \qquad X_4 = \pm iR$$

in S_4. If two points (X_1, X_2, X_3, X_4) and (X_1', X_2', X_3', X_4') lie in a minimal line, the corresponding minimal cones are tangent to each other, and the corresponding spheres in S_3 will also be tangent to each other, since

$$(X_1 - X_1')^2 + (X_2 - X_2')^2 + (X_3 - X_3')^2 = (R - R')^2.$$

To all the points of a minimal line correspond ∞^1 spheres which touch at the point of intersection of the minimal line with the space $X_4 = 0$. This means that the spheres have a surface-element in common at that point.

Consider now a flat space E; it intersects the space at infinity in a plane. The pole of this plane with respect to the (imaginary) sphere at infinity is the point at infinity which determines the direction of the ∞^3 normals to E. If this pole lies on the sphere, ∞^2 of these normals will lie in E and E is a minimal space.

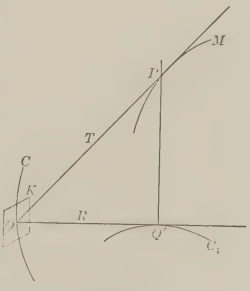

FIG. 2.

The tangents to a minimal curve form a minimal developable. Let T be any one of these tangents and P the point of contact; the intersection of the space $X_4 = 0$ with this developable will be a twisted curve C of that space. We shall now project the space S_4 orthogonally on the space $X_4 = 0$. Let Q be the point of intersection of the minimal line T with $X_4 = 0$, and let K, the intersection of $X_4 = 0$ with the minimal space E determined by T, be the

plane of the surface element at Q. Since this plane is contained in E, the line T, being normal to E, must be normal to every line and plane in E; hence T is normal to K. The projection of T on X_4, or $\overline{QQ'}$, must also be perpendicular to K. Hence, when the point P moves along the minimal curve, the projected point Q' moves along the curve C' which is an evolute of C. The line C is therefore a line of curvature on the surface obtained by intersecting the isotropic congruence with the space $X_4 = 0$. Hence the

THEOREM. *The orthogonal projection of a minimal curve M on the space $X_4 = 0$ is an evolute of the curve of intersection of the developable of M with that space.*

We have seen that a congruence of minimal lines determines a double family of minimal developables and that it has a focal surface which in general consists of two sheets on which are situated the ∞^1 edges of regression of the developables. If now this entire congruence be projected orthogonally, we obtain curves and surfaces which we shall study in detail. It will be convenient to use the analytical method, although a purely geometric method might also be applied.

If we cut the minimal congruence (6), § 1, by the space $X_4 = 0$, we obtain a surface in three-dimensional space whose coördinates we shall denote by x_1, x_2, and x_3. We have then

$$
\begin{aligned}
x_1 &= -\frac{(s+t)F + (1-s^2)F'_s + (1-t^2)F'_t}{2(1+st)}, \\
x_2 &= i\frac{(s-t)F - (1+s^2)F'_s + (1+t^2)F'_t}{2(1+st)}, \qquad (42) \\
x_3 &= \frac{F - sF'_s - tF'_t}{1+st}.
\end{aligned}
$$

We shall call this surface the *surface of reference* of the congruence. We see at once that the equations (42) are those of any surface in S_3 referred to the curves of contact of tangent cones which have their vertices on the imaginary circle at infinity; these curves (s and t) have for spherical images the rectilinear generators of a sphere. (Compare DARBOUX, l. c., vol. 1, p. 243, and equations (33), p. 246.)

To the minimal curves on the focal surface of the congruence correspond lines of curvature on the surface of reference. In fact, the differential equation of the minimal curves is

$$F''_{s^2}ds^2 - F''_{t^2}dt^2 = 0, \qquad (12)$$

which is also the differential equation of the lines of curvature on the surface (42). The directions of these lines at any point s, t are the bisectors of the angles between the coördinate curves.*

*See DARBOUX, l. c., vol. 1, p. 243.

If now we project the focal sheets orthogonally on the space $X_4 = 0$ by means of the transformation (40), we obtain the equations

$$
\begin{aligned}
\bar{x}_1 &= \tfrac{1}{2}[(s+t)(F''_{st} \pm \sqrt{F''_{s^2}F''_{t^2}}) - F'_t - F'_s],\\
\bar{x}_2 &= -\frac{i}{2}[(s-t)(F''_{st} \pm \sqrt{F''_{s^2}F''_{t^2}}) - F'_t + F_s],\\
\bar{x}_3 &= \tfrac{1}{2}[(st-1)(F''_{st} \pm \sqrt{F''_{s^2}F''_{t^2}}) - tF'_t - sF'_s + F],\\
R &= -\tfrac{1}{2}[(st+1)(F''_{st} \pm \sqrt{F''_{s^2}F''_{t^2}}) - tF'_t - sF'_s + F].
\end{aligned}
\tag{43}
$$

The first three equations are seen to be the equations of the surface of centers of the surface of reference, while the last equation gives the values of the principal radii of curvature of the surface. *The minimal curves on the focal sheets in S_4 become geodesics on the surface of centers* (by the theorem on p. 416). The isotropic congruence becomes the normal congruence with respect to the lines of curvature of the surface of reference. The middle surface of the minimal congruence becomes the middle surface of the normal congruence.

If, in particular, F is a solution of the equation

$$(st+1)F''_{st} - tF'_t - sF'_s + F = 0,$$

the system (43) reduces to (34), while the value of R is

$$R = \pm \tfrac{1}{2}(1+st)^2\sqrt{S'''T'''},$$

which is the expression for the principal radii of curvature. We notice that they are numerically equal with opposite signs; hence, the middle surface is an ordinary minimal surface, as was proved before on p. 413, and it is identical with the surface of reference of the minimal congruence. Hence the

THEOREM. *If the middle surface of a minimal congruence in four-dimensional space is a minimal surface in ordinary space, it is identical with the surface of reference.*

The two conjugate-asymptotic directions on the focal surface in S_4 become conjugate directions on the surface of centers, the geodesics being one family (corresponding to the minimal curves) and the second, the curves of contact of the developable surfaces determined by the second family of lines of curvature on the surface of reference. In this particular case, then, conjugate directions are preserved when the surface is projected orthogonally on $X_4 = 0$. As an example we shall take the case $F = s^2t^2$ used before. The focal surface (14) becomes, when projected orthogonally on to the space (x_1, x_2, x_3),

$$
\begin{aligned}
\bar{x}_1 &= 2\alpha\sqrt{\frac{\alpha}{\beta}}(1+\beta), \qquad \bar{x}_3 = -2i\alpha\sqrt{\frac{\alpha}{\beta}}(\beta-1),\\
\bar{x}_3 &= \tfrac{3}{2}a(a-2), \qquad iR = \tfrac{3}{2}ia(a+2).
\end{aligned}
$$

Since the lines $\beta = \text{const.}$ are minimal curves on the focal surface (14), they are geodesics on the surface of centers. The lines $\alpha = \text{const.}$ are conjugate to these, since the expression

$$HD' = \begin{vmatrix} \frac{\partial \bar{x}_2}{\partial \alpha} & \frac{\partial \bar{x}_3}{\partial \alpha} \\ \frac{\partial x_2}{\partial \beta} & \frac{\partial \bar{x}_3}{\partial \beta} \end{vmatrix} \frac{\partial^2 x_1}{\partial \alpha \partial \beta} + \begin{vmatrix} \frac{\partial \bar{x}_3}{\partial \alpha} & \frac{\partial \bar{x}_1}{\partial \alpha} \\ \frac{\partial \bar{x}_3}{\partial \beta} & \frac{\partial \bar{x}_1}{\partial \beta} \end{vmatrix} \frac{\partial^2 \bar{x}_2}{\partial \alpha \partial \beta} + \begin{vmatrix} \frac{\partial \bar{x}_1}{\partial \alpha} & \frac{\partial_2 \bar{x}}{\partial \alpha} \\ \frac{\partial \bar{x}_1}{\partial \beta} & \frac{\partial \bar{x}_2}{\partial \beta} \end{vmatrix} \frac{\partial^2 \bar{x}_3}{\partial \alpha \partial \beta}$$

vanishes identically, as a simple calculation will show.* The surface of reference is

$$x_1 = \frac{(\beta+1)(\alpha-2)\sqrt{\alpha}\alpha}{2(1+\alpha)\sqrt{\beta}}, \qquad x_2 = \frac{i(1-\beta)(\alpha-2)\sqrt{\alpha}\alpha}{2(1+\alpha)\sqrt{\alpha}}, \qquad x_3 = \frac{-3\alpha^2}{1+\alpha},$$

where α and β are lines of curvature. This surface, though written in imaginary form, is real; in fact, writing the surface in the form

$$x_1 = -\frac{st(s+t)(2-st)}{2(1+st)}, \qquad x_2 = \frac{ist(t-s)(st-2)}{2(1+st)}, \qquad x_3 = \frac{3s^2t^2}{1+st},$$

we may obtain a real form by making s and t conjugate imaginaries; thus, suppose we put

$$s = p + iq, \qquad t = p - iq,$$

then

$$x_1 = -\frac{2(p^2+q^2)(2-p^2-q^2)p}{2(1+p^2+q^2)}, \qquad x_2 = \frac{2(p^2+q^2)(p^2+q^2-2)q}{2(1+p^2+q^2)},$$

$$x_3 = \frac{-3(p^2+q^2)^2}{1+p^2+q^2}.$$

All the focal surfaces (14) in S_4 are imaginary, as are also the minimal developables belonging to a focal surface. The surface of reference, as well as the surface of centers and middle surface, will in general have real sheets which are obtained by making s and t conjugate variables. The middle surface of a minimal congruence will have real sheets only if it lies in a three-dimensional space, in which case it is an ordinary minimal surface.

§ 6. *Translation surfaces in S_4.*

We shall now consider the case where the middle surface, or "minimal" surface of the congruence is a translation-surface. The most general solution of the differential equation

* The second focal sheet being a focal line, the second sheet of the surface of centers is also a line. DARBOUX has discussed such surfaces, l. c., vol. 1, p. 226. The surface of reference in this case is the envelope of a sphere of variable radius whose center describes the curve $\bar{x}_1 = 0$, $\bar{x}_2 = 0$, $\bar{x}_3 = -\alpha/2(\alpha+2)$, the radius being $R = -\alpha/2(\alpha-2)$; the surface is a canal surface.

$$X_4 = \frac{i}{2}\left[(st+1)F''_{st} - tF'_t - sF'_s + F\right] = \frac{i}{2}(S+T), \tag{44}$$

where S and T are two arbitrary functions of s and t respectively, will give a value of F such that the corresponding surface is one of translation. If we put $F''_{st} = S' + T'$, the left side of (44) becomes

$$X_4 = \frac{i}{2}\left[S_1 + T_1 - tT'_1 - sS'_1 - S' - T'\right], \tag{45}$$

which is of the form $S + T$. Introducing the function F obtained by integrating the equation

$$F''_{st} = S' + T',$$

that is to say,

$$F = tS + sT + S_1 + T_1 \tag{46}$$

in the system (25), we obtain expressions similar to (45) for the remaining coördinates, viz:

$$\begin{aligned}
X_1 &= \tfrac{1}{2}\left[sS' - S - S'_1\right] + \tfrac{1}{2}\left[tT' - T - T'_1\right],\\
X_2 &= -\frac{i}{2}\left[sS' - S + S'_1\right] + \frac{i}{2}\left[tT' - T + T'_1\right],\\
X_3 &= -\tfrac{1}{2}\left[sS'_1 + S' - S_1\right] - \tfrac{1}{2}\left[tT'_1 + T' - T_1\right],\\
X_4 &= -\frac{i}{2}\left[sS'_1 - S' - S_1\right] - \frac{i}{2}\left[tT'_1 - T' - T_1\right],
\end{aligned} \tag{47}$$

which may be written in the following form

$$\begin{aligned}
X_1 &= \tfrac{1}{2}\int (sS'' - S''_1)\,ds + \tfrac{1}{2}\int (tT'' - T''_1)\,dt,\\
X_2 &= -\frac{i}{2}\int (sS'' + S''_1)\,ds + \frac{i}{2}\int (tT'' + T''_1)\,dt,\\
X_3 &= -\tfrac{1}{2}\int (sS''_1 + S'')\,ds - \tfrac{1}{2}\int (tT''_1 + T'')\,dt,\\
X_4 &= -\frac{i}{2}\int (sS''_1 - S'')\,ds - \frac{i}{2}\int (tT''_1 - T'')\,dt.
\end{aligned} \tag{47'}$$

The focal surface may now be obtained by substituting the value of F in equations (36). We find, denoting the coördinates by $\bar{X}_i$ and those of the middle surface by X_i,

$$\begin{aligned}
\bar{X}_1 &= X_1 \pm \tfrac{1}{2}(s+t)\sqrt{(tS'' + S''_1)(sT'' + T''_1)},\\
\bar{X}_2 &= X_2 \mp \frac{i}{2}(s-t)\sqrt{(tS'' + S''_1)(sT'' + T''_1)},\\
\bar{X}_3 &= X_3 \pm \tfrac{1}{2}(st-1)\sqrt{(tS'' + S''_1)(sT'' + T''_1)},
\end{aligned} \tag{48}$$

$$\bar{X}_4 = X_4 \pm \frac{i}{2}(st+1)\sqrt{(t\bar{S}'' + \bar{S}_1'')(s\bar{T}'' + \bar{T}_1'')},$$

which is not a translation-surface.

If, in particular, we put $\bar{S} = 2\bar{S} - s\bar{S}'$, $T = 2\bar{T} - t\bar{T}'$, $S_1 = -\bar{S}'$, $T_1 = -\bar{T}'$, we obtain the ordinary minimal surface (33). From the form of equations (47) we see that these translation-surfaces are imaginary. Since the equations

$$\begin{aligned} \bar{X}_1 &= sS' - S - S_1', & \bar{X}_2 &= -i(sS' - S + S_1'), \\ \bar{X}_3 &= -(sS_1' + S' - S_1), & \bar{X}_4 &= -i(sS_1' - S' - S_1) \end{aligned} \tag{49}$$

give the most general parametric representation of a minimal curve in S_4, it would seem that (47) represents the most general translation-surface whose translation curves are minimal, and that they are therefore all imaginary. This is, however, not so. In fact, a real translation-surface may be obtained in the following manner: Consider two curves like (49); the locus of the middle points of all the chords joining the two curves will be a translation surface (47), that is, we have

$$\begin{aligned} X_1 &= \tfrac{1}{2}(\bar{X}_1 + \bar{\bar{X}}_1), & X_2 &= \tfrac{1}{2}(\bar{X}_2 + \bar{\bar{X}}_2), \\ X_3 &= \tfrac{1}{2}(\bar{X}_3 + \bar{\bar{X}}_4), & X_4 &= \tfrac{1}{2}(\bar{X}_4 + \bar{\bar{X}}_4), \end{aligned} \tag{50}$$

$\bar{X}_i$ being the coördinates in (49) and $\bar{\bar{X}}_i$ the coördinates of the curve

$$\begin{aligned} \bar{\bar{X}}_1 &= tT' - T - T_1', & \bar{\bar{X}}_2 &= i(tT' - T + T_1'), \\ \bar{\bar{X}}_3 &= -(tT_1' + T' - T_1), & \bar{\bar{X}}_4 &= -i(tT_1' - T' - T_1). \end{aligned} \tag{51}$$

The surface (47) has no real sheet; if, however, we project the curve (49) by means of the transformation $\bar{\bar{X}}_1 = \bar{X}_1$, $\bar{\bar{X}}_2 = \bar{X}_2$, $\bar{\bar{X}}_3 = \bar{X}_3$, $\bar{\bar{X}}_4 = -\bar{X}_4$, i. e., if we reflect the curve with respect to the space $\bar{X}_4 = 0$, we get the curve

$$\begin{aligned} \bar{X}_1 &= sS' - S - S_1', \\ \bar{X}_2 &= -i(sS_1' - S + S_1'), \\ \bar{X}_3 &= -(sS_1' + S' - S_1), \\ \bar{X}_4 &= i(sS_1' - S' - S_1). \end{aligned} \tag{50'}$$

If now we use (50′) instead of (49), we get the surface

$$\begin{aligned} X_1 &= \tfrac{1}{2}[sS' - S - S_1'] + \tfrac{1}{2}[tT' - T - T_1'], \\ X_2 &= -\frac{i}{2}[sS' - S + S_1'] + \frac{i}{2}[tT' - T + T_1'], \\ X_3 &= -\tfrac{1}{2}[sS_1' + S - S_1] - \tfrac{1}{2}[tT_1' + T' - T_1], \\ X_4 &= \frac{i}{2}[sS_1' - S' - S_1] - \frac{i}{2}[tT_1' - T' - T_1], \end{aligned} \tag{52}$$

and this surface is real when s and t are chosen conjugate imaginaries, as an easy calculation will show. Hence we have the

THEOREM. *To the imaginary translation-surface in S_4 which is the middle surface of an isotropic congruence there corresponds always a real surface obtained by reflecting in one of the coördinate spaces.*

The real surface (52) is not a "minimal" surface, that is, the middle surface of an isotropic congruence. It may be constructed as follows: From the point S_0, the projection on the space $X_4 = 0$ of a point on the curve (50′), we draw a parallel S_0M to the chord ST which joins a point S of (50) to a point T on (51). We project P, which is a point on the surface (50), drawing the line PP' which meets the line S_0M at P'; this point is then a point on the surface (52). (See figure 3.) We notice that P' is the middle point of a chord joining S', the reflection of S, to T.

Two curves, (49) and (51), being given, there exist two real translation-surfaces whose generating curves are minimal curves. To a point Q on one real

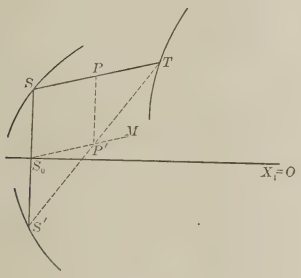

FIG. 3.

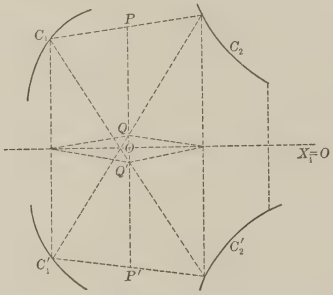

FIG. 4.

surface obtained by using the curves C_2 and C'_2 corresponds a point Q', the reflection of Q, on another real surface obtained by using the curves C_1, C_2 and C'_1 (C'_1 and C'_2 are reflections of C_1 and C_2 with respect to the space $X_4 = 0$).

The two real surfaces are therefore reflections of each other. In the same way, the imaginary translation-surfaces determined by C_1, C_2 and C'_1, C'_2 are reflections. If the translation-surface is contained in the space $X_4 = 0$, the points P, Q, P', Q' all coincide at O, and the surface becomes a minimal surface in ordinary space (Fig. 4). To these real surfaces belong the "R-Flächen" studied by K. Kommerell.* These so-called "minimal" surfaces are,

*K. KOMMERELL, l. c., pp. 543, 596. The equations of the R-Flächen are:

$$X_1 = \frac{s+t}{2}, \quad X_2 = \frac{i(s-t)}{2}, \quad X_3 = \tfrac{1}{2}(S+T), \quad X_4 = -\frac{i}{2}(S-T).$$

by our definition, not minimal, since they can only become middle surfaces when contained in the space $X_4 = 0$, in which case they become ordinary minimal surfaces.

§ 7. *Line and sphere geometry.*

We have found (p. 405) that to the ∞^4 hyperboloids

$$F = ast + Cs + ct + d$$

in the space (F, s, t) correspond ∞^4 minimal cones in S_4. If now we transform the space (F, s, t) by means of Euler's transformation

$$(1) \quad x = F'_s, \qquad y = t, \qquad z = F - sF'_s, \qquad p = -s, \qquad q = F'_t,$$

the hyperboloids will be transformed into the ∞^4 lines of ordinary space, viz.:

$$(2) \qquad x = ay + b, \qquad z = cy + d,$$

to which must be added the following:

$$(2') \qquad q = -ap + c,$$

which determines the relation between the coördinates p, q and -1 of the planes of the ∞^2 surface-elements of the line (2), so that to the ∞^2 surface-elements of a given hyperboloid there correspond the ∞^2 surface-elements of the line. We have thus a one-to-one correspondence between the ∞^4 minimal cones and the ∞^4 lines of a three-dimensional space S_3. To a line (a, b, c, d) in S_3 corresponds a minimal cone whose vertex is (p. 404)

$$(3) \quad \alpha = -\frac{b+c}{2}, \qquad \beta = \frac{i(c-b)}{2}, \qquad \gamma = \frac{d-a}{2}, \qquad \delta = \frac{i(a+d)}{2}.$$

To two lines that intersect there correspond two minimal cones that are tangent to each other; in fact, the condition that two lines

$$(4) \qquad \begin{aligned} x &= ay + b, & z &= cy + d, \\ x &= a'y + b, & z &= c'y + d, \end{aligned}$$

shall intersect is

$$(5) \qquad (a' - a)(d - d') - (c' - c)(b - b') = 0.$$

The coördinates of the vertices of the corresponding cones are

$$\alpha = -\frac{b+c}{2}, \qquad \beta = \frac{i(c-b)}{2}, \qquad \gamma = \frac{d-a}{2}, \qquad \delta = \frac{i(a+d)}{2},$$

$$\alpha' = -\frac{b'+c'}{2}, \qquad \beta' = \frac{i(c'-b')}{2}, \qquad \gamma' = \frac{d'-a'}{2}, \qquad \delta' = \frac{i(a'+d')}{2},$$

from which we obtain by solving

$$a = -(\gamma + i\delta), \qquad b = -(\alpha - i\beta), \qquad c = -(\alpha + i\beta), \qquad d = \gamma - i\delta,$$

$$a' = -(\gamma' + i\delta'), \qquad b' = -(\alpha' - i\beta'), \qquad c' = -(\alpha' + i\beta'), \qquad d' = \gamma' - i\delta'.$$

Substituting these values in (5), we have

$$(\alpha' - \alpha)^2 + (\beta' - \beta)^2 + (\gamma' - \gamma)^2 + (\delta' - \delta)^2 = 0, \tag{6}$$

which is the condition that two minimal cones shall be tangent to each other. In particular, if the two lines are consecutive, the relation (6) becomes

$$d\alpha^2 + d\beta^2 + d\gamma^2 + d\delta^2 = 0. \tag{7}$$

If the line (a, b, c, d) be fixed and the line (a', b', c', d') variable, the locus of the corresponding variable vertex is a minimal cone. Hence, to all the ∞^3 lines of a special null-system in S_3 correspond the ∞^3 minimal cones in S_4 whose vertices lie on another minimal cone.

The equations (2) and (2′) show that if the coördinates y and p of the surface-element are fixed, the remaining three, viz.: x, z and q, are determined. But fixing y and p also fixes s and t. Therefore to all the ∞^2 surface-elements of a line in S_3 correspond the ∞^2 minimal lines of a minimal cone in S_4.

Let there be given in S_3 a non-special null-system which may be written

$$Ad - Bb + C(bc - ad) + Da + Ec + G = 0;$$

to it will correspond in S_4 the sphere

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\frac{B-E}{C}\alpha-\frac{i(B+E)}{C}\beta+\frac{A-D}{C}\gamma-\frac{i(A+D)}{C}\delta+\frac{G}{C}=0. \tag{8}$$

which may be written

$$\left(\alpha-\frac{B-E}{2C}\right)^2+\left(\beta-\frac{i(B+E)}{2C}\right)^2+\left(\gamma-\frac{D-A}{2C}\right)^2+\left(\delta-\frac{i(A+D)}{2C}\right)^2 = -\frac{BE+AD+CG}{C^2}. \tag{9}$$

Hence, to all the ∞^3 lines of a non-special null-system in S_4 correspond the ∞^3 minimal cones in S_4 having their vertices on a fixed sphere. If we consider each point of the sphere (9) as the carrier of a minimal cone, we have a one-to-one correspondence between the ∞^5 null-systems of ordinary space and the ∞^5 spheres of four-dimensional space. If the null-system is special,

$$BE + AD + CG = 0,$$

and this means that the fixed sphere (19) is a minimal cone.

The geometry of a linear complex in S_3 is thus seen to be equivalent to the geometry of a complex of minimal cones in S_4. It should be noted, however, that the null-system must be conceived of as an aggregate of surface-elements, that is to say, the ∞^3 lines and their corresponding surface-elements of which each line has ∞^2. To each surface-element of a line belonging to the null-system corresponds a minimal line passing through a point on the fixed sphere, that is, to a surface-element corresponds a minimal element. We shall return to this interesting fact later and proceed now to explain briefly the distribution of the linear elements of the minimal cone.

We have shown, p. 405, that s and t may be regarded as curvilinear coördinates on the imaginary sphere at infinity, which we shall denote by S_∞, the curves $s = \text{const.}$ and $t = \text{const.}$ being the generators of the sphere. By Euler's transformation, $y = t$, $p = -s$; hence all the surface-elements of the line (2), p. 422, for which $y = \text{const.}$, give minimal lines through the vertex of the minimal cone in S_4 which meet a t-generator of S_∞. Similarly for the surface-elements $p = \text{const.}$ The minimal lines through the vertex of the minimal cone are thus arranged in two families of planes corresponding to the two families of generators t and s on S_∞. These planes are the minimal planes of the cone.

§ 8. *Lie's line-sphere geometry.*

We shall now return to the congruence of minimal lines (5), § 1, and write it as follows:

$$\begin{aligned} tX_1 + itX_2 - X_3 + iX_4 + F - sF'_s &= 0, \\ X_1 - iX_2 + tX_3 + itX_4 + F'_s &= 0, \\ X_1 + iX_2 + sX_3 + isX_4 + F'_t &= 0. \end{aligned} \tag{10}$$

Introducing Euler's transformation (1), this congruence becomes:

$$\begin{aligned} yX_1 + iyX_2 - X_3 + iX_4 + z &= 0, \\ X_1 - iX_2 + yX_3 + iyX_4 + x &= 0, \\ X_1 + iX_2 - pX_3 - ipX_4 + q &= 0, \end{aligned} \tag{11}$$

which equations solved for X_1, X_2 and X_3 may be written

$$X_1 = \frac{i(y-p)}{1-py} X_4 - \frac{x}{2} - \frac{y(z-yq)+q-pz}{2(1-py)},$$

$$X_2 = \frac{y+p}{1-py} X_4 - \frac{y}{2} - i\frac{y(z+yq)-q+pz}{2(1-py)}, \tag{12}$$

$$X_3 = \frac{i(1+py)}{1-py} + \frac{z-py}{1-py}.$$

The condition that two consecutive minimal lines shall intersect we obtain as before by differentiating the system (11) with respect to the parameters x, y, z, p, and q. We have

$$\begin{aligned} (X_1 + iX_2)\,dy + dz &= 0, \\ (X_3 + iX_4)\,dy + dx &= 0, \\ -(X_3 + iX_4)\,dp + dq &= 0. \end{aligned} \tag{13}$$

Eliminating the coördinates X_i from (11) and (13) we find the following conditions:

$$\begin{aligned} dz - p\,dx - q\,dy &= 0, \\ dp\,dx + dq\,dy &= 0. \end{aligned} \tag{14}$$

The first equation expresses the condition that two surface-elements corresponding to the two consecutive minimal lines shall be united. The second expresses the condition that the line of intersection of the planes of the surface-elements joins the two consecutive points x, y, z, and $x+dx$, $y+dy$, $z+dz$. If therefore we have in S_4 a minimal curve, there will correspond to it in S_3 an element-band the planes of whose surface-elements osculate the point-locus of the band. To a surface $z=f(x,y)$ in S_3 corresponds a minimal congruence $\phi(X_1, X_2, X_3, X_4)=0$. To the two minimal developables determined by a generator D of the congruence correspond two asymptotic lines or bands on the surface $z=f(x,y)$, one of each family; hence, *to the two families of ∞^1 minimal developables contained in the congruence $\phi=0$ correspond in S_3 the two families of asymptotic bands on the surface, and to a minimal curve of the developable (an edge of regression) corresponds an asymptotic curve.*

We shall now project the space S_4 into a space $\bar{S}_3$ by means of the transformation (40), p. 415, which transforms the minimal cone

$$(X_1-\alpha)^2+(X_2-\beta)^2+(X_3-\gamma)^2+(X_4-\delta)^2=0 \tag{15}$$

into a sphere in ordinary space (X_1, X_2, X_3), viz.:

$$(X_1-\alpha)^2+(X_2-\beta)^2+(X_3-\gamma)^2+\delta^2=0 \tag{16}$$

(α, β, γ) being the center and $\pm i\delta$ the radius. We notice that the correspondence is a one-to-two owing to the double sign of the radius $R=\pm i\delta$, that is to say, to the cone corresponds a sphere, while to the sphere correspond two cones, viz.:

$$(X_1-\alpha)^2+(X_2-\beta)^2+(X_3-\gamma)^2+(X_4-iR)^2=0 \tag{17}$$

and

$$(X_1-\alpha)^2+(X_2-\beta)^2+(X_3-\gamma)^2+(X_4+iR)^2=0. \tag{17'}$$

To a minimal line in S_4 corresponds a surface-element in S_3, while to a minimal cone $(\alpha, \beta, \gamma, \delta)$ corresponds a sphere of radius $i\delta$ considered as an ensemble of its ∞^2 surface-elements. Further, to a minimal congruence will correspond a surface and to the ∞^1 minimal developables the ∞^1 lines of curvature on the surface, these being considered as bands (curvature bands). Comparing now the two spaces S_3 and $\bar{S}_3$ with S_4, we see that the same configuration in S_4 has been transformed into two different configurations in S_3 and $\bar{S}_3$ of such a nature that lines in S_3 correspond to spheres in $\bar{S}_3$ and asymptotic lines to lines of curvature. Further, since by the transformation (40), p. 415, the relation (6), § 7, becomes

$$(\alpha-\alpha')^2+(\beta-\beta')^2+(\gamma-\gamma')^2=(R-R')^2,$$

it follows also that to lines that intersect in S_3 correspond spheres that touch. The relation between the surface-elements of S_3 and $\bar{S}_3$ must therefore be of such a nature that a contact-transformation transforms the elements of S_3 into those of $\bar{S}_3$. This contact-transformation is obtained at once by putting $X_4=0$ in equations (12), p. 424, viz.:

$$\begin{aligned} X_1 &= -\frac{x}{2}-\frac{y(z-yq)+q-pz}{2(1-py)}, \\ X_2 &= -\frac{iy}{2}-i\frac{y(z+yq)-q+pz}{2(1-py)}, \\ X_3 &= \frac{z-py}{1-py}. \end{aligned} \tag{18}$$

If to (18) we add the two equations giving the values of the coördinates P_1 and P_2 (obtained from (12)), we have

$$(19) \qquad P_1 = \frac{y-p}{1+yp}, \qquad P_2 = \frac{-i(y+p)}{1+yp}.$$

Equations (18) and (19) give the required contact-transformation, which we recognize as the well known line-sphere contact transformation of Lie. This method of deriving the transformation is, we believe, new and perhaps not without interest.

In conclusion we give the following table showing the relation between the elements of the three spaces S_3, $\bar{S}_3$, and S_4:

SPACE $S_3(X_1, X_2, X_3, P_1, P_2)$.	SPACE $S_4(X_1, X_2, X_3, X_4)$.	SPACE $\bar{S}_3(x, y, z, p, q)$.
I. A surface-element.	I. A minimal line.	I. A surface-element.
II. *a*) $dX_3 - P_1 dX_1 - P_2 dX_2 = 0$. *b*) $(dX_1 + P_1 dX_3)dP_1 - (dX_2 + P_2 dX_3)dP_2 = 0$.	II. Intersecting consecutive minimal lines.	II. *a*) $dz - pdx - qdy = 0$. *b*) $dpdx + dqdy = 0$.
III. The ∞^2 surface-elements of a sphere, center α, β, γ, radius $\pm i\delta$.	III. The ∞^2 minimal lines of a minimal cone, vertex α, β, γ, δ.	III. The ∞^2 surface-elements of the line $x = -(\gamma + i\delta)y - (\alpha - i\beta)$, $y = -(\alpha + i\beta)y + \gamma - i\delta$, $q = (\gamma + i\delta)p - (\alpha + i\beta)$.
IV. The ∞^1 element bands of a sphere whose point-loci are the rectilinear generators of the sphere.	IV. The ∞^1 plane pencils of a minimal cone; two families, p=const., q=const.	IV. The ∞^1 surface-elements of a point, or a "hinge," and the ∞^1 surface-elements of a line (flat band).
V. A spherical band.	V. A two-dimensional minimal cone, vertex α, β, γ, δ.	V. A twisted band.
VI. A surface.	VI. A minimal or isotropic congruence.	VI. A surface.
VII. An ensemble of ∞^1 surface-elements forming a curvature band on the surface VI.	VII. A minimal developable.	VII. An ensemble of ∞^1 surface-elements forming an asymptotic band on the surface VI.
VIII. The double family of curvature bands on VI.	VIII. The ∞^1 minimal developables contained in an isotropic congruence VI. (A double family, since each generator determines two developables.)	VIII. The double family of asymptotic lines on VI.

IX. A congruence of ∞^2 spheres, the so-called curvature spheres, belonging to the surface VI.	IX. The minimal congruence considered as an aggregate of ∞^2 minimal cones, each cone having a line-element in common with a line of the congruence on an edge of regression.	IX. A congruence of ∞^2 lines consisting of the ∞^2 linear tangents along the asymptotic lines.
X. System of ∞^3 spheres cutting a fixed sphere at a constant angle. (See LIE-SCHEFFERS, loc. cit., vol. I, pp. 651–654.)	X. System of ∞^3 minimal cones having their vertices on a fixed sphere.	X. Linear complex, or null-system.

WEST VIRGINIA UNIVERSITY,
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VOLTERRA'S INTEGRAL EQUATION OF THE SECOND KIND, WITH DISCONTINUOUS KERNEL, SECOND PAPER

BY

GRIFFITH C. EVANS

In an earlier paper† we have considered integral equations of the second kind with variable upper limit‡ whose kernels in spite of discontinuities were absolutely integrable, and by change of variable we were able to show the existence of solutions in some special cases of equations§ where the kernels belonged to a rather general type and were not absolutely integrable. In this paper we take up that type in more detail. In particular, we study the equation

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)g(x)} u(\xi)\, d\xi \tag{24}$$

where $K(x, \xi)$, the numerator in the expression for the kernel, is continuous in the triangular region

$$T: \qquad a \leqq \xi \leqq x \leqq b,$$

and $f(x)$, $g(x)$, comprising the denominator, are continuous in the interval

$$t: \qquad a \leqq x \leqq b.$$

References to the Earlier Paper.

$$u(x) = \phi(x) + \int_a^x K(x, \xi) u(\xi)\, d\xi. \tag{1}$$

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)} u(\xi)\, d\xi. \tag{9}$$

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)g(x)} u(\xi)\, d\xi. \tag{16}$$

* Presented to the Society, September 13, 1909.

† Transactions of the American Mathematical Society, vol. 11 (1910), p. 393. The numbering of equations and sections is continued from that paper, and reference to it in the present paper is made merely by letter and number. For the more important references see below.

‡ Equation (1).

§ Equations (9), (16), (17).

$$u(x)=\phi(x)+\int_a^x \frac{K(x,\xi)}{g(x)f(\xi)\prod_{i=1}^{b}[\xi-\psi_i(x)]^{\lambda_i}}u(\xi)\,d\xi. \tag{17}$$

Condition (A). A real function of the two variables x, ξ is to be continuous in the triangle $T: a \leqq \xi \leqq x \leqq b$, $b > a \geqq 0$, except on a finite number of curves, each composed of a finite number of continuous pieces with continuously turning tangents. Any vertical portion is to be considered a separate piece, and of such pieces there are to be merely a finite number, $x=\beta_1, x=\beta_2, \cdots, x=\beta_r$. On the other portions of the system of curves there are to be only a finite number of vertical tangents. Hence by any line $x=x_0, x_0 \neq \beta_1, \cdots, \beta_r$, the system of curves will be cut in only a finite number of points.

Condition (B). In the region $t: a \leqq x \leqq b$, a real function of the single variable x is to be continuous except at a finite number of points, $\gamma_1, \cdots, \gamma_s$, and is to remain finite.

Problem from Hydrostatics. Suppose we are given a tube, lying in a vertical plane along a curve of arbitrary shape, $s=u(x)$, where s is the distance along the curve and x the altitude. Let us fill this tube with a liquid of variable linear density ν, and then regulate its height x in the tube by allowing various amounts to flow out through the bottom. Let us then regard ν as an analytic function of the depth in the liquid, i. e., $\nu=\nu(x-\xi)$.

If

$$h(x)=\frac{\int_0^x \nu(x-\xi)u'(\xi)\,d\xi}{\int_0^x u'(\xi)\,d\xi}$$

is the average linear density and $v(x)=u'(x)$, the equation to determine $v(x)$ from $h(x)$ is

$$v(x)=\int_0^x \frac{G(x-\xi)-g'(x)}{g(x)}v(\xi)\,d\xi,$$

where

$$\nu(x-\xi)=\alpha+G(x-\xi), \qquad G(0)=0,$$

and

$$h(x)=\alpha+g(x), \qquad g(0)=0.$$

9. *Existence theorem for equation* (24).

We have the following existence theorem:

THEOREM. *If in equation* (24)

$$u(x)=\phi(x)+\int_a^x \frac{K(x,\xi)}{f(\xi)g(x)}u(\xi)\,d\xi, \tag{24}$$

1°. (*a*) *$K(x, \xi)$ is continuous in T, and $f(x), g(x)$ and their first derivatives are continuous in t;*

(*b*) *$\partial K(x, \xi)/\partial x$ and $\partial K(x, \xi)/\partial \xi$ satisfy condition (A) and are finite in T;*

(*c*) *$\phi(x)$ is continuous in t except at $x = a$, and is such that the function $\phi(x)g(x)$ and its first derivative satisfy B in t;*

2°. *the function $f(x)g(x)$ is greater than zero in t except for the value $x = a$, and for the value $x = a$ vanishes in such a way that*

$$\int_a^x \frac{dx}{f(x)g(x)}$$

*is not convergent;**

3°. $$K(a, a) \neq 0;$$

4°. $$\lim_{x=0} \phi(x)g(x) = 0:$$

then under the foregoing conditions,

(*i*) *if $K(a, a) < 0$ there exists one solution of (24) continuous throughout t, and*

(*ii*) *if $K(a, a) > 0$ there exists a one parameter family of solutions of (24) continuous throughout t except possibly for the value $x = a$. As x approaches a each solution remains less than some constant times $f(x)/(x - a)^\nu$ where ν is any number that satisfies the two conditions*

(*a*) $$1 > \nu > 0,$$

(*b*) $$\nu > 1 - \frac{1}{\left\{\frac{d}{dx}[f(x)g(x)]\right\}_{x=a}}.$$

In regard to ν it is easy to see that the two conditions (*a*), (*b*) under (*ii*) can always be satisfied. For since $f'(x)$ and $g'(x)$ are supposed to be continuous in t and not negative, since that would make $f(x)$ or $g(x)$ negative, $\{d/dx[f(x)g(x)]\}_{x=a}$ will be finite and not negative, and

$$1 - \frac{1}{\left\{\frac{d}{dx}[f(x)g(x)]\right\}_{x=a}}$$

will always be less than 1.

* Otherwise by change of variable $v(x) = u(x)/g(x)$ equation (23) is changed to one that satisfies the conditions of the theorem of section 1. And in that case there would then be one and only one finite $v(x)$.

10. *A theorem that implies the theorem of section* 9.

A theorem that lies closer to the method of proof that we have adopted, and which implies the theorem just stated, is obtained by substituting in the hypothesis for 1° (*b*) and for 3° the conditions

1′. (*b*) $\partial K(x, \xi)/\partial x$ *satisfies A and is finite in* T;

3′. $K(a, a) \neq 0$; $\lim_{x=a}[K(x, x) - K(a, a)]/(x-a)^{\nu}$ *exists, where* ν *is defined as in section* 9.

Moreover as the method of proof will show, if $K(a, a) < 0$, we may always put $\nu = 0$ without other change in the theorem. Also if $K(a, a) > 0$ and $\{d/dx(f(x)g(x)\}_{x=a} < 1$ we may take $\nu = 0$. So that for most cases the theorem holds if $\nu = 0$ be substituted throughout.

11. *Approximate Equations.*

The results of this theorem are obtained by means of a method of approximation, for the purpose of which the solution of the simpler equations

$$u(x) = \phi(x) \pm \int_a^x \frac{u(\xi)}{f(\xi)}\, d\xi \tag{25}$$

is necessary. If (25) has a solution, continuous except for a finite number of points, even when the discontinuities are not finite its derivative will exist and be continuous except for a finite number of points, and will be given by

$$u'(x) = \phi'(x) \pm \frac{u(x)}{f(x)}. \tag{26}$$

Hence any solution of the integral equation (25) will be a solution of the differential equation (26). It remains to be seen under what conditions the solution of the differential equation will be a solution of the integral equation.

Let us write the equations

$$u(x) = \phi(x) - \int_a^x \frac{u(\xi)}{f(\xi)}\, d\xi, \tag{25′}$$

$$u(x) = \phi(x) + \int_a^x \frac{x(\xi)}{f(\xi)}\, d\xi. \tag{25″}$$

In the same way we rewrite (26) as

$$u'(x) = \phi'(x) - \frac{u(x)}{f(x)}, \tag{26′}$$

$$u'(x) = \phi'(x) + \frac{u(x)}{f(x)}. \tag{26″}$$

The general solution of (26′) is

$$u(x) = e^{\int_x^b \frac{dx}{f(x)}}\left[\int_{a'}^{x} \phi'(\xi) e^{-\int_\xi^b \frac{d(\xi)}{f(\xi)}} d\xi + C\right], \tag{27′}$$

and of (26″)

$$u(x) = e^{-\int_x^b \frac{dx}{f(x)}}\left[\int_{a'}^{x} \phi'(\xi) e^{+\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi + C\right], \tag{27″}$$

where a' is arbitrarily chosen in the neighborhood of a.

In (27′), since

$$\lim_{\xi=a} e^{-\int_\xi^b \frac{d\xi}{f(\xi)}} = 0,$$

we can choose $a' = a$ without loss of generality. So that instead of (27′) we may write

$$u(x) = e^{\int_x^b \frac{dx}{f(x)}}\left[\int_{a}^{x} \phi'(\xi) e^{-\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi + C\right]. \tag{28}$$

To see under what conditions this is a solution of (25′) we first notice that the first term of the expression for $u(x)$ in (28) is itself a solution of the equation, if $\phi(a) = 0$.

In fact, substituting in the second member of (25′) we have

$$\phi(x) - \int_a^x \frac{u(\xi)}{f(\xi)} d\xi = \phi(x) - \int_a^x \frac{e^{\int_\xi^b \frac{d\xi}{f(\xi)}}}{f(\xi)} \left\{\int_a^\xi \phi'(\xi') e^{-\int_{\xi'}^b \frac{d\xi'}{f(\xi')}} d\xi'\right\} d\xi.$$

If we integrate by parts we obtain

$$\phi(x) + e^{\int_x^b \frac{dx}{f(x)}} \int_a^x \phi'(\xi) e^{-\int_\xi^b \frac{dx}{f(\xi)}} d\xi$$

$$- \lim_{x=a}\left[e^{\int_x^b \frac{dx}{f(x)}} \int_a^x \phi'(\xi) e^{-\int_\xi^b \frac{d\xi}{f(x)}} d\xi\right] - \int_a^x \phi'(\xi)\, d\xi = u(x) + \phi(a);$$

for, since $|\phi'(\xi)| \leqq N$,

$$\left| e^{\int_x^b \frac{dx}{f(x)}} \int_a^x \phi'(\xi) e^{-\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi \right| \leqq N e^{\int_x^b \frac{d\xi}{f(\xi)}} \int_a^x e^{-\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi \leqq N(x-a),$$

and

$$\lim_{x=a} e^{\int_x^b \frac{dx}{f(x)}} \int_a^x \phi'(\xi) e^{-\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi = 0.$$

Hence $\phi(a) = 0$ is a necessary and sufficient condition that the first term of (28) be a solution of (25′).

On the other hand, if $C \neq 0$ the expression (28) cannot be a solution of the equation. For it is necessary that

$$\int_a^x \frac{u(\xi)}{f(\xi)}\,d\xi$$

be convergent in the neighborhood of a, and since the integral formed from the first part of $u(\xi)$ is convergent, it is necessary that

$$\int_a^x \frac{Ce^{\int_\xi^a \frac{d\xi}{f(\xi)}}}{f(\xi)}\,d\xi$$

be convergent in the neighborhood of a. But this is obviously not the case.

Hence that (25′) have a solution it is necessary and sufficient that $\phi(a) = 0$. The solution is then unique, and is given by

$$u(x) = e^{+\int_x^b \frac{dx}{f(x)}} \int_a^x \phi'(\xi)\, e^{-\int_\xi^b \frac{d\xi}{f(\xi)}}\,d\xi. \tag{29}$$

In (27″) we cannot take $a' = a$ because in general to do this would make the integral divergent. Let us then rewrite (27″) as

$$u(x) = -e^{-\int_x^b \frac{dx}{f(x)}} \left[\int_x^{a'} \phi'(\xi)\, e^{\int_\xi^b \frac{d\xi}{f(\xi)}}\,d\xi + C \right] \tag{30}$$

and substitute this in the integral equation (25′). We know that the two members resulting have identical derivatives; it remains merely to show that at a given point they take on the same values to make them identical throughout. But to show this is no easier than to substitute directly into the integral equation. The result of the substitution is the same as that obtained by substituting the first term of the expression for $u(x)$, since the second term is obviously a solution of the homogeneous equation. The second member of (25) then becomes

$$\phi(x) + \int_a^x \frac{-e^{-\int_\xi^b \frac{d\xi}{f(\xi)}}}{f(\xi)} \left\{ \int_\xi^{a'} \phi'(\xi')\, e^{\int_{\xi'}^b \frac{d\xi'}{f(\xi')}}\,d\xi' \right\} d\xi.$$

If we integrate this by parts we obtain the expression

$$\phi(x) - e^{-\int_x^b \frac{dx}{f(x)}} \int_x^{a'} \phi'(\xi)\, e^{\int_\xi^b \frac{d\xi}{f(\xi)}}\,d\xi$$

$$+ \lim_{x=a} \left\{ e^{-\int_x^b \frac{dx}{f(x)}} \int_x^{a'} \phi'(\xi)\, e^{\int_\xi^b \frac{d\xi}{f(\xi)}}\,d\xi \right\} - \int_a^x \phi'(\xi)\,d\xi$$

$$= u_1(x) + \phi(a) - \lim_{x=a} u_1(x),$$

where $u_1(x)$ is the first term of $u(x)$, i. e.,

$$u_1(x) = -e^{-\int_x^b \frac{dx}{f(x)}} \int_x^{a'} \phi'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi.$$

We can show that $\lim_{x=a} u_1(x) = 0$ by showing that a quantity η can be determined small enough so that when $(x - a) \leqq \eta$, $|u_1(x)|$ becomes less than or equal to an arbitrarily given H. We have

$$|u_1(x)| \leqq N \left| e^{-\int_x^b \frac{dx}{f(x)}} \int_x^{a'} e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi \right|$$

$$\leqq N e^{-\int_x^b \frac{dx}{f(x)}} \int_x^s e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi + N e^{-\int_x^b \frac{dx}{f(x)}} \int_s^{a'} e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi,$$

where $x \leqq s$. Then

$$|u_1(x)| \leqq N(s - x) + N e^{-\int_x^b \frac{dx}{f(x)}} F(s),$$

where we represent by $F(s)$ the integral

$$\int_s^{a'} e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi.$$

Take $s = H/2N$ thus fixing s by the value of H. This fixes also the value of $NF(s)$. Take now $\eta \leqq s$ and so small that if $x \leqq \eta$,

$$N e^{-\int_x^b \frac{dx}{f(x)}} \leqq \frac{H}{2}.$$

We then have

$$|u_1(x)| \leqq H \qquad (a \leqq x \leqq a + \eta);$$

which was to be proved.

We see therefore that a necessary and sufficient condition that (25″) have a solution is that $\phi(a) = 0$. The general solution is then given by

$$u(x) = -e^{-\int_x^b \frac{dx}{f(x)}} \left[\int_x^{a'} \phi'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi + C \right]. \tag{31}$$

If the function $\phi(x)$ has a zero at a so intense that the expression

$$\int_x^{a'} \phi'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi$$

converges when $x = a$, we have apparently the state of affairs which was treated in section 6 of the earlier paper. To show the agreement of the two results we notice first that the function $f(\xi)$ of (25″) is not the same as that of (9), for in (25″) the function corresponding to $K(x, \xi)$ of (9) being equal to 1 is *not* less

than 1, and that in order to reduce our kernel to one that can be treated by the method of sections 1–4 we must replace $f(\xi)$ in (25″) by

$$\bar{f}(\xi) = Hf(\xi), \qquad H < 1,$$

so that the kernel in (25″) becomes $H/\bar{f}(\xi)$. With this change it is seen that the only solution with sufficient intensity of vanishing is

$$u(x) = e^{-\int_x^b \frac{dx}{f(x)}} \int_a^x \phi'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi.$$

This solution, provided that $\phi'(\xi)$ vanishes as above, continues to hold if we replace $f(x)$ by $\alpha f(x)$ and let α change continually from $+1$ to -1, thus changing (25″) into (25′).

12. *The Method of Approximation.*

Let us return now to equations with kernels of the form $K(x, \xi)/f(\xi)$ still assuming $g(x) \equiv 1$.

By taking the numerical value of $K(a, a)$ into $f(\xi)$ we can write the kernel as

$$\pm \frac{1 + \Gamma(x, \xi)}{f(\xi)},$$

and we are led to the two equations

$$(32') \qquad u(x) = \phi(x) - \int_a^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} u(\xi) d\xi,$$

$$(32'') \qquad u(x) = \phi(x) + \int_a^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} u(\xi) d\xi,$$

according as $K(a, a)$ is < 0 or > 0 respectively. Here condition 3° of the hypothesis becomes

$$\lim_{x=a} \frac{\Gamma(x, \xi)}{(x - a)^\nu} \text{ exists, } \nu > 0, \ \nu > 1 - \frac{1}{f'(a)},$$

and condition 4° becomes

$$\lim_{x=a} \phi(x) = 0.$$

In (32′) let us write

$$u(x) = u_0(x) + u^{(1)}(x),$$

where

$$u_0(x) = \phi(x) - \int_a^x \frac{u_0(\xi)}{f(\xi)} d\xi$$

and consequently

$$u^{(1)}(x) = -\int_a^x \frac{\Gamma(x,\xi)}{f(\xi)} u_0(\xi)\,d\xi - \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)} u^{(1)}(\xi)\,d\xi,$$

or

$$u^{(1)}(x) = \phi_1(x) - \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)} u^{(1)}(\xi)\,d\xi,$$

where

$$\phi_1(x) = -\int_a^x \frac{\Gamma(x,\xi)}{f(\xi)} u_0(\xi)\,d\xi.$$

In the same way we may write

$$u^{(1)}(x) = u_1(x) + u^{(2)}(x),$$

$$u_1(x) = \phi_1(x) - \int_a^x \frac{u_1(\xi)}{f(\xi)}\,d\xi,$$

$$u^{(2)}(x) = \phi_2(x) - \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)} u^{(2)}(\xi)\,d\xi,$$

where

$$\phi_2(x) = -\int_a^x \frac{\Gamma(x,\xi)}{f(\xi)} u_1(\xi)\,d\xi.$$

By continuing the process indefinitely, we have

$$u^{(m)}(x) = u_m(x) + u^{(m+1)}(x),$$

$$u_m(x) = \phi_m(x) - \int_a^x \frac{u_m(\xi)}{f(\xi)}\,d\xi,$$

(33)
$$u^{(m+1)}(x) = \phi_{m+1}(x) - \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)} u^{(m+1)}(\xi)\,d\xi,$$

$$\phi_{m+1}(x) = -\int_a^x \frac{\Gamma(x,\xi)}{f(\xi)} u_m(\xi)\,d\xi,$$

and (29) gives us

(34)
$$u_m(x) = e^{\int_x^b \frac{dx}{f(x)}} \int_a^x \phi'_m(\xi)\, e^{-\int_\xi^b \frac{d\xi}{f(\xi)}}\,d\xi.$$

We have also

(35)
$$\phi'_m(x) = -\frac{\Gamma(x,x)}{f(x)} u_{m-1}(x) - \int_a^x \frac{\Gamma_1(x,\xi)}{f(\xi)} u_{m-1}(\xi)\,d\xi.$$

It is quite easy to find restrictions on the absolute values of $\phi'_m(x)$ and

$u_m(x)$.* If

$$|\phi_m'(x)| \leqq N_m(x-a)^{i_m},$$

it follows that

$$|u_m(x)| \leqq N_m(x-a)^{i_m} e^{\int_x^b \frac{dx}{f(x)}} \int_a^x e^{-\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi.$$

If we integrate by parts and remember that $f(a)=0$, we have

$$|u_m(x)| \leqq N_m(x-a)^{i_m}\left[f(x) + e^{\int_x^b \frac{dx}{f(x)}} \int_a^x f'(\xi) e^{-\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi\right].$$

Since

$$e^{-\int_\xi^b \frac{d\xi}{f(\xi)}} \leqq e^{-\int_x^b \frac{d\xi}{f(\xi)}},$$

the last expression in the brackets is less than

$$\int_a^x f'(\xi)\, d\xi = f(x).$$

Hence

$$|u_m(x)| \leqq 2N_m(x-a)^{i_m} f(x). \tag{36}$$

Since a constant A may be chosen so that

$$|\Gamma(x,x)| \leqq A(x-a)^\nu, \qquad |\Gamma_1(x,\xi)| \leqq A, \qquad a \leqq x \leqq R,$$

we have from (35) and (36)

$$\begin{aligned}|\phi_m'(x)| &\leqq A(x-a)^\nu \frac{|u_{m-1}(x)|}{f(x)} + A\int_a^x \frac{|u_{m-1}(\xi)|}{f(\xi)}\, d\xi \\ &\leqq 2N_{m-1}A(x-a)^{i_{m-1}+\nu} + 2N_{m-1}A\int_a^x (x-a)^{i_{m-1}}\, dx \\ &\leqq 4N_{m-1}A(x-a)^{i_m+\nu},\end{aligned}$$

provided that we take R so that $R-a \leqq 1$.

Now $N_0 \leqq N$, where $|\phi'(x)| \leqq N_0$, and therefore

$$\begin{aligned}&&|u_0| &\leqq 2Nf(x),\\ N_1 = 4NA,\ i_1 = \nu, &&|u_1| &\leqq 2Nf(x)\{4A(x-a)^\nu\},\\ N_2 = (4A)^2 N,\ i_2 = 2\nu, &&|u_2| &\leqq 2Nf(x)\{4A(x-a)^\nu\}^2,\\ \cdots\cdots\cdots &&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &\\ N_m = (4A)^m N,\ i_m = m\nu, &&|u_m| &\leqq 2Nf(x)\{4A(x-a)^\nu\}^m.\end{aligned}$$

* In the cases where we can take $\nu = 0$, we define a constant Γ so that if $a \leqq x \leqq R$, $|\Gamma(x,x)| \geqq \Gamma$. We then have $|\phi_m'(x)| \leqq N_m \Gamma^{i_m}$, etc. By taking R small enough, the series in powers of Γ is shown to be convergent.

13. *A Solution of Equation* (32′).

Let us now form the series

$$U(x) = u_0(x) + u_1(x) + u_2(x) + \cdots. \tag{37}$$

This series is absolutely and uniformly convergent through a region $a \leqq x \leqq R \leqq a+1$, where R is so small that $4A(R-a)^\nu < 1$. In fact

$$|U(x)| \leqq 2Nf(x)\frac{1}{1-4A(R-a)^\nu},$$

and $U(x)$ vanishes at least to the order of vanishing of $f(x)$.

The continuous function $U(x)$, defined by (37), is a solution of the integral equation (32′). For, write

$$U(x) = u_0(x) + u_1(x) + \cdots + u_n(x) + U^{(n+1)}(x)$$

and substitute in the expression

$$L(x) = u(x) - \phi(x) + \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)} u(\xi)\,d\xi;$$

the result is

$$\begin{aligned} L(x) = U^{(n+1)}(x) - \phi(x) &+ \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)} U^{(n+1)}(\xi)\,d\xi \\ &+ u_0(x) + \int_a^x \frac{\Gamma(x,\xi)}{f(\xi)} u_0(\xi)\,d\xi + \int_a^x \frac{u_0(\xi)}{f(\xi)}\,d\xi \\ &+ u_1(x) + \int_a^x \frac{\Gamma(x,\xi)}{f(\xi)} u_1(\xi)\,d\xi + \int_a^x \frac{u_1(\xi)}{f(\xi)}\,d\xi \\ &\cdots\cdots\cdots\cdots\cdots \\ &+ u_n(x) + \int_a^x \frac{\Gamma(x,\xi)}{f(\xi)} u_n(\xi)\,d\xi + \int_a^x \frac{u_n(\xi)}{f(\xi)}\,d\xi, \end{aligned}$$

or

$$\begin{aligned} L(x) = U^{(n+1)}(x) - \phi_{n+1}(x) &+ \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)} U^{(n+1)}(\xi)\,d\xi \\ &+ u_0(x) - \phi(x) + \int_a^x \frac{u_0(\xi)}{f(\xi)}\,d\xi + u_0(x) - \phi_1(x) + \int_a^x \frac{u_1(\xi)}{f(\xi)}\,d\xi \\ &\cdots\cdots\cdots\cdots\cdots \\ &+ u_n(x) - \phi_n(x) + \int_a^x \frac{u_n(\xi)}{f(\xi)}\,d\xi, \end{aligned}$$

or

$$L(x) = U^{(n+1)}(x) - \phi_{n+1}(x) + \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)} U^{n+1}(\xi)\,d\xi.$$

Now

$$|\phi_{n+1}(x)| \leqq \{4A(x-a)^\nu\}^{n+1} N \leqq \{4A(R-a)^\nu\}^{n+1} N,$$

and

$$U^{(n+1)}(x) \leqq 2Nf(x) \sum_{m=n+1}^{\infty} \{4A(x-a)^\nu\}^m \leqq 2Nf(x) \sum_{m=n+1}^{\infty} \{4A(R-a)^\nu\}^m$$

$$\leqq 2Nf(x)\{4A(R-a)^\nu\}^{n+1} \frac{1}{1-4A(R-a)^\nu}.$$

Also

$$\left|\int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)} U^{(n+1)}(\xi)\,d\xi\right| \leqq (A+1)\int_a^x \frac{|U^{(n+1)}(\xi)|}{f(\xi)}\,d\xi$$

$$\leqq \frac{2N(A+1)\{4A(R-a)^\nu\}^{n+1}}{1-4A(R-a)^\nu}\int_a^x d\xi$$

$$\leqq \frac{2N(R-a)(A+1)\{4A(R-a)^\nu\}^{n+1}}{1-4A(R-a)^\nu}.$$

From these inequalities it follows that

$$|L(x)| \leqq \{4A(R-a)^\nu\}^{n+1}\left[N + \frac{2NB + 2N(R-a)(A+1)}{1-4A(R-a)^\nu}\right],$$

where $B \geqq f(x)$. But the right hand member may be made as small as we please, since $4A(R-a)^\nu < 1$, by taking n large enough. Hence $L(x) \equiv 0$, and $U(x)$ is a solution of the given integral equation (32′).

In order to extend this solution beyond the point R we may write the equation as

$$u(x) = \phi(x) - \int_a^R \frac{1+\Gamma(x,\xi)}{f(\xi)} u(\xi)\,d\xi + \int_R^x \frac{1+\Gamma(x,\xi)}{f(\xi)} u(\xi)\,d\xi,$$

or

$$u(x) = \phi_R(x) + \int_R^x \frac{1+\Gamma(x,\xi)}{f(\xi)} u(\xi)\,d\xi.$$

This is an equation with a continuous kernel, and since

$$\phi_R(x) = \phi(x) - \int_a^R \frac{1+\Gamma(x,\xi)}{f(\xi)} U(\xi)\,d\xi,$$

is also continuous, there is one and only one solution of the equation continuous except for a finite number of discontinuities, and finite, for $R \leqq x \leqq b$. This solution on account of its uniqueness must be independent of the choice of the point R,

$$a < R < a + \frac{1}{(4A)^{1/\nu}}, \qquad a < R < a+1.$$

We have arrived then by this approximation method at a single continuous solution of the integral equation (32′), which vanishes as x approaches a at least as strongly as $f(x)$.

14. *A Solution of Equation* (32″).

For the solution of (32″) we define

$$U(x) = u_0(x) + u_1(x) + u_2(x) + \cdots,$$

where

$$u_m(x) = \phi_m(x) + \int_a^x \frac{u_m(\xi)}{f(\xi)}\, d\xi, \qquad \phi_m(x) = \int_a^x \frac{\Gamma(x, \xi)}{f(\xi)} u_{m-1}(\xi)\, d\xi.$$

From (31) we have as a particular solution

$$u_m(x) = -e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R \phi_m'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi. \tag{38}$$

We have also

$$\phi_m'(x) = \frac{\Gamma(x, x)}{f(x)} u_{m-1}(x) + \int_a^x \frac{\Gamma_1(x, \xi)}{f(\xi)} u_{m-1}(\xi)\, d\xi, \tag{39}$$

if the various integrals are convergent. We can in this case also quite easily get restrictions on the absolute values of $\phi_m'(x)$ and $u_m(x)$.

If

$$\phi_m'(x) \leqq N_m (R - a)^{i_m}$$

$$|u_m(x)| \leqq N_m (R - a)^{i_m} \left| e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi \right|$$

$$\leqq N_m (R - a)^{i_m} |\bar{u}(x)|,$$

where

$$\bar{u}(x) = -e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi.$$

Let us assume for the present, and show later, that ν' and C can be found so that

$$|\bar{u}(x)| \leqq C f(x)(x - a)^{-\nu'} \qquad (a \leqq x \leqq R',\ 0 \leqq \nu' < \nu),$$

R' to be determined later. When $|f'(a)| < 1$ we can take $\nu' = 0$. We may write, assuming this and taking $R \leqq R'$,

$$|u_m(x)| \leqq C N_m (R - a)^{i_m} (x - a)^{-\nu'} f(x),$$

and from (39)

$$|\phi_m'(x)| \leqq A C N_{m-1} (R - a)^{i_{m-1}} (x - a)^{\nu - \nu'} + A C N_{m-1} (R - a)^{i_m - 1} \int_a^x \frac{d\xi}{(\xi - a)^{\nu'}},$$

where $\nu - \nu' > 0$. We shall take R near enough to a so that $(R-a)<1$. Hence

$$|\phi_m'(x)| \leqq A\,CN_{m-1}(R-a)^{i_{m-1}}(x-a)^{\nu-\nu'}\left(1+\frac{1}{1-\nu'}\right)$$

$$\leqq DCN_{m-1}(R-a)^{i_{m-1}}(x-a)^{\nu-\nu'}$$

$$\leqq DCN_{m-1}(R-a)^{i_{m-1}+\nu-\nu'},$$

where

$$D = A\left(1+\frac{1}{1-\nu}\right).$$

We have then

$$N_m = DCN_{m-1}, \qquad i_m + i_{m-1} + (\nu - \nu').$$

But since $N_0 = N$, $i_0 = 0$, we have

$$|u_0^{(x)}| \leqq CN(x-a)^{-\nu'}f(x),$$

$$|\phi_1'(x)| \leqq DCN(R-a)^{\nu-\nu'}, \qquad |u_1(x)| \leqq DC^2N(R-a)^{\nu-\nu'}(x-a)^{-\nu'}f(x),$$

$$|\phi_2'(x)| \leqq D^2C^2N(R-a)^{2(\nu-\nu')}, \qquad |u_2(x)| \leqq D^2C^3(R-a)^{2(\nu-\nu')}N(x-a)^{-\nu'}f(x),$$

$$|\phi_m'(x)| \leqq \{DC(R-a)^{\nu-\nu'}\}^m N, \quad |u_m(x)| \leqq \{DC(R-a)^{\nu-\nu'}\}^m CN(x-a)^{-\nu'}f(x).$$

The series

$$U(x) = u_0(x) + u_1(x) + \cdots + u_m(x) + \cdots \tag{40}$$

can now be treated. For if

$$(x-a) \leqq R - a < \left(\frac{1}{DC}\right)^{\frac{1}{\nu-\nu'}}$$

the series for $(x-a)^{\nu'}U(x)$ is absolutely and uniformly convergent, and represents a continuous function of x, $a \leqq x \leqq R$. And

$$|U(x)| \leqq N(x-a)^{-\nu'}f(x)\frac{C}{1-DC(R-a)^{\nu-\nu'}}.$$

$U(x)$ is shown to be a solution of the equation in the same way as for equations of the other type. The inequalities are but slightly different. We have

$$L(x) = U(x) - \phi(x) - \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)}U(\xi)\,d\xi$$

$$= U^{(n+1)}(x) - \phi_{n+1}(x) - \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)}U^{(n+1)}(\xi)\,d\xi,$$

where

$$|\phi_{m+1}(x)| \leqq \{DC(R-a)^{\nu-\nu'}\}^{n+1} N,$$

$$|U^{(n+1)}(x)| \leqq N(x-a)^{-\nu'} f(x) C \frac{\{DC(R-a)^{\nu-\nu'}\}^{n+1}}{1-DC(R-a)^{\nu-\nu'}},$$

and

$$\left| \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)} U^{(n+1)}(\xi)\, d\xi \right| \leqq (A+1) \int_a^x \frac{|U^{(n+1)}(\xi)|}{f(\xi)}\, d\xi$$

$$\leqq \frac{\{DC(R-a)^{\nu-\nu'}\}^{n+1} N(A+1) C}{1-DC(R-a)^{\nu-\nu'}} \frac{(x-a)^{1-\nu'}}{1-\nu'},$$

so that

$$|L(x)| \leqq \{DC(R-a)^{\nu-\nu'}\}^{n+1} \left[N + \frac{N(x-a)^{-\nu'} f(x)}{1-DC(R-a)^{\nu-\nu'}} + \frac{N(A+1)C(x-a)^{1-\nu'}}{(1-\nu')\{1-DC(R-a)^{\nu-\nu'}\}} \right].$$

The right hand member of this inequality can be made as small as we please for any value x_0 of x, $a \leqq x \leqq R$, by taking n large enough. Hence $L(x) \equiv 0$, $a \leqq x \leqq R$, and $U(x)$ is a solution of (32″).

15. *The Function* $\bar{u}(x)$.

Let us now return and justify the inequalities stated for the function

$$\bar{u}(x) = -e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R e^{\int_\xi^b \frac{d\xi}{f(\xi)}}\, d\xi$$

in section 14. This function is a solution of the equation

$$\bar{u}(x) = x + \int_a^x \frac{\bar{u}(\xi)}{f(\xi)}\, d\xi.$$

We have assumed that $f(x)$ is positive for $a < x \leqq R$ and that

$$\int_a^R \frac{dx}{f(x)}$$

is not convergent. We shall show that $|\bar{u}(x)| \leqq Cf(x)$ when $|f'(a)| < 1$.

Consider first the value of *

$$\frac{1}{f(x)} e^{-\int_x^b \frac{dx}{f(x)}}.$$

* The method used here is essentially the same as that used in evaluating $\lim_{x=a} f(x)/g(x)$ when $\lim_{x=a} f(x) = \infty$, $\lim_{x=a} g(x) = \infty$. See Osgood: *Calculus*, p. 236.

This remains finite as x approaches a. We have, in fact,

$$\frac{\dfrac{1}{f(x)}-\dfrac{1}{f(x')}}{e^{\int_x^{b'}\frac{dx}{f(x)}}-e^{\int_{x'}^{b'}\frac{dx}{f(x)}}}=\frac{\dfrac{d}{dx}\left(\dfrac{1}{f(X)}\right)}{\dfrac{d}{dx}\left(e^{\int_x^b\frac{dx}{f(x)}}\right)},\qquad a<x<X'<x',$$

or

$$\frac{1}{\dfrac{f(x)}{e^{\int_x^b\frac{dx}{f(x)}}}}\left[\frac{1-\dfrac{f(x)}{f(x')}}{1-e^{\int_x^{x'}\frac{dx}{f(x)}}}\right]=\frac{\dfrac{f'(X)}{f(X)}}{e^{\int_X^x\frac{dx}{f(x)}}}.$$

Choose $x'=R'$ close enough to the point a so that

$$|f'(x)|<H,\qquad a<x\leqq x',$$

where H is a constant such that $f'(a)<H<1$. There is then a point $x_0<x'$ such that if $a<x\leqq x_0$ the bracket may be written $1+\lambda(x)$, where $|\lambda(x)|\leqq\eta$, η arbitrarily assigned $<1-H$. We have then

$$\frac{\dfrac{1}{f(x)}}{e^{\int_x^b\frac{dx}{f(x)}}}(1-\eta)\leqq H\frac{\dfrac{1}{f(X)}}{e^{\int_X^b\frac{dx}{f(x)}}},$$

and

$$\frac{\dfrac{1}{f(x)}}{e^{\int_x^b\frac{dx}{f(x)}}}<\frac{\dfrac{1}{f(X)}}{e^{\int_X^b\frac{dx}{f(x)}}}.$$

Hence for every point $x\leqq x_0$ there must be a point X, $x_0<X\leqq R'$, for which

$$\frac{\dfrac{1}{f(X)}}{e^{\int_X^b\frac{dx}{f(x)}}}$$

is greater; otherwise we should get a sequence of points $X_1, X_2, \cdots$ having a limit point $\bar{x}\leqq x_0$. This point, however, determines in the same way a point $\bar{X}$, for which $\bar{X}>\bar{x}$. So there is no limiting point in this sense.

But for all points beyond x_0 the given function

$$\frac{\dfrac{1}{f(x)}}{e^{\int_x^b\frac{dx}{f(x)}}}$$

is less than or equal to a certain constant J. Hence

$$\frac{1}{f(x)} e^{-\int_x^b \frac{dx}{f(x)}} \leqq J,$$

and

$$e^{-\int_x^b \frac{dx}{f(x)}} \leqq J f(x) \qquad (a \leqq x \leqq R').$$

Consider now the function

$$\bar{u}(x) = -e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi,$$

and perform an integration by parts. Then

$$\bar{u}(x) = f(x) - e^{-\int_x^b \frac{dx}{f(x)}} \left[f(R) e^{\int_R^b \frac{dx}{f(x)}} \right] + e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R f'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi$$

$$\leqq f(x)\left(1 + J f(R) e^{\int_R^b \frac{dx}{f(x)}}\right) + e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R |f'(\xi)| e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi.$$

We have assumed $R \leqq R'$, whence $|f'(\xi)| < H$.

Defining

$$J' = J f(R) e^{\int_R^b \frac{dx}{f(x)}},$$

we have

$$|\bar{u}(x)| \leqq (1 + \zeta') f(x) + H |\bar{u}(x)|,$$

and since $H < 1$,

$$|\bar{u}(x)| \leqq \frac{(1 + J') f(x)}{1 - H},$$

and this defines the constant C.

We now consider $\bar{u}(x)$ when $f(a) = c$, $|c| \geqq 1$. The constant c must be positive; otherwise $f(x)$ would be negative in the neighborhood of a. We have

$$(c - \delta)(x - a) \leqq f(x) \leqq (c + \delta)(x - a), \qquad a \leqq x \leqq a + \eta,$$

where δ is small, so chosen that $c - \delta > 0$ and neither $c - \delta$ nor $c + \delta$ is equal to 1. Hence

$$\frac{1}{(c + \delta)(x - a)} \leqq \frac{1}{f(x)} \leqq \frac{1}{(c - \delta)(x - a)},$$

and

$$|\bar{u}(x)| = e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi = e^{-\int_x^\eta \frac{dx}{f(x)}} \int_x^R e^{\int_\xi^\eta \frac{d\xi}{f(\xi)}} d\xi$$

$$\leqq e^{-\int_x^\eta \frac{d\xi}{(c+\delta)(x-a)}} \int_x^R e^{\int_\xi^\eta \frac{d\xi}{(c-\delta)(\xi-a)}} d\xi.$$

If we now perform the indicated integrations, we have

$$|\bar{u}(x)| \leqq \left(\frac{x-a}{\eta-a}\right)^{\frac{1}{c+\delta}} \int_x^R \left(\frac{\xi-a}{\eta-a}\right)^{\frac{-1}{c-\delta}} d\xi.$$

If we write

$$c' = \frac{1}{c}, \qquad c' - \delta' = \frac{1}{c+\delta}, \qquad c' + \delta'' = \frac{1}{c-\delta},$$

the right hand member becomes

$$\left(\frac{x-a}{\eta-a}\right)^{c'-\delta'} \int_x^R \left(\frac{\xi-a}{\eta-a}\right)^{-(c'+\delta'')} d\xi,$$

where $c' - \delta' > 0$, $c' - \delta' \neq 1$, $c' + \delta'' \neq 1$. Hence

$$\bar{u}(x) \leqq (\eta-a)^{\delta'+\delta''}(x-a)^{c'-\delta'} \frac{(R-a)^{1-c'-\delta''} + (x-a)^{1-c'-\delta''}}{|1-c'-\delta''|}$$

$$\leqq \frac{(\eta-a)^{\delta'+\delta''}}{|1-c'-\delta''|} [(x-a)^{c'-\delta'}(R-a)^{1-c'-\delta''} + (x-a)^{1-\delta'-\delta''}],$$

where we may take δ' and δ'' as small as we please. And so if $c = c' = 1$ we may take $\nu' > 0$ as near zero as we please; if $c > 1$, $c' < 1$ we may take $\nu' > 1 - c'$ as near $1 - c'$ as we please. Hence for the general case $c \geqq 1$, we take

$$\nu' = 1 - \frac{1}{c} + \epsilon = 1 - \frac{1}{f'(a)} + \epsilon$$

where ϵ is as small as we please. And thus we prove the inequalities of § 14.

16. *The One Parameter Family of Solutions of Equation* (32″).

The question now is presented to us, since we have obtained a particular solution of (32″) by one choice of R, whether it is not possible to get other solutions by means of different values of R, $R = R_1, R_2, \cdots, R_j$, and whether there is any sort of dependence among the solutions thus resulting. The answer is that all the solutions corresponding to values of R satisfying the conditions laid down for it belong to a linear system

$$V(x) = U(x) + CW_0(x).$$

The difference of any two solutions of the non-homogeneous equation (25″) or the non-homogeneous equation (32″) is a solution of the corresponding homogeneous equation. Now the complete solution of the homogeneous equation correspond- to (25″),

$$w(x) = \int_a^x \frac{w(\xi)}{f'(\xi)} d\xi,$$

is

$$w(x) = e^{-\int_x^R \frac{d\xi}{f(\xi)}}.$$

If we let $W(x) = w_1(x) + W^{(1)}(x)$, where $w_1(x) = w(x)$ and substitute in

$$W(x) = \int_a^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} W(\xi) d\xi, \tag{41}$$

which is the homogeneous equation corresponding to (32″), the equation to determine $W^{(1)}(x)$ will be

$$W^{(1)}(x) = \int_a^x \frac{\Gamma(x, \xi)}{f(\xi)} w(\xi) d\xi + \int_a^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} W^{(1)}(\xi) d\xi,$$

or

$$W^{(1)}(x) = \psi(x) + \int_a^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} W^{(1)}(\xi) d\xi,$$

where the function

$$\psi(x) = \int_a^x \frac{\Gamma(x, \xi)}{f(\xi)} w(\xi) d\xi$$

vanishes when $x = a$, and $\psi'(x)$ remains finite as $x \doteq a$. Hence we get a solution for this equation, keeping the same value for R that was used in getting $U(x)$ in § 14, by forming a function $U_\psi(x)$ based on $\psi(x)$ in the same way that the $U(x)$ of § 14 was based on $\phi(x)$. $U_\psi(x)$ will contain C as a factor to the first degree. Hence

$$W(x) = w_1(x) + U_\psi(x)$$

will be a solution of the homogeneous equation (41), and will contain C as a factor to the first degree. $W(x)$ is not identically zero since it takes on the value C when $x = R$. We may write $W(x)$ as $CW_0(x)$ where $W_0(x)$ takes on the value 1 when $x = R$; and we have thus a system of solutions of the non-homogeneous equation (32″) given by

$$V(x) = U(x) + CW_0(x).$$

Now we may write the general solution of (25″), as we have seen, in the form

$$v(x) = e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R \phi'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi + Ce^{-\int_x^b \frac{dx}{f(x)}}.$$

Suppose we form a series $V(x)$ in the same way we formed the series $U(x)$ of § 17, except that we use the expressions

$$v_m(x) = e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R \phi'_m(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi + c_m e^{-\int_x^b \frac{dx}{f(x)}} = \bar{u}_m(x) + c_m e^{-\int_x^b \frac{dx}{f(x)}},$$

instead of the terms $u_m(x)$. If we let w_i be the general term in the expansion of $W_0(x)$ we may write the sum of the first m terms of $V(x)$ as

$$\sum_{i=1}^{m} u_i(x) + c_1 \sum_{i=1}^{m} w_i(x) + c_2 \sum_{i=1}^{m-1} w_i(x) + \cdots + c_{m-1} \sum_{i=1}^{2} w_i(x) + c_m w_1(x).$$

We have already shown the existence of the limits

$$\lim_{m=\infty} \sum_{i=1}^{m} u_i(x), \qquad \lim_{m=\infty} \sum_{i=1}^{m} |w_i(x)|.$$

We now investigate

$$(a) \qquad \lim_{m=\infty} \left\{ \sum_{i=1}^{m} c_i \sum_{i=1}^{m-i+1} w_j(x) \right\}.$$

The expression (a) may be rewritten in the form

$$\lim_{m=\infty} \sum_{p=1}^{m} \sum_{i=1}^{p} c_i w_{p+1-i}.$$

Consider now $c_1 + c_2 + \cdots$ and $w_1 + w_2 + \cdots$. Since $w_1 + w_2 + \cdots$ is absolutely convergent the product of these two series will be the series whose general term is

$$\sum_{i=1}^{p} c_i w_{p+1-i},$$

i. e., the expression (a), provided that the series of c's is convergent.* We see then that if the series of c's is convergent, the series of v's is convergent, V represents a function belonging to the linear system already discovered, and is hence a solution of the integral equation (32″).

On the other hand if the series of v's is convergent for one value of x, $x = R_1 \leqq R$, V is again of the specified linear system. For we may write $v_m(x)$ in the form

$$v_m(x) = e^{-\int_x^{R_1} \frac{dx}{f(x)}} \int_x^{R_1} \phi'_{m-1}(\xi)\, e^{\int_\xi^{R_1} \frac{d\xi}{f(\xi)}} d\xi + c'_m e^{-\int_x^{R_1} \frac{dx}{f(x)}},$$

so that

$$v_m(R_1) = c'_m, \qquad V(R_1) = \sum_0^\infty c'_m,$$

and the series

$$\sum_0^\infty c'_m = \bar{C}$$

is convergent. The series for $V(x)$ is therefore convergent for $a \leqq x \leqq R_1$,

* *Encyklopädie der Mathematischen Wissenschaften*, vol. I, page 96.

and the function $V(x)$ may be written in the form

$$\bar{U}(x) + \bar{C}\,\bar{W}(x).$$

Since, however, all solutions of the form

$$U(x) + CW_0(x)$$

are convergent at $x + R'$ they all belong to this family. In fact $W_0(x)$ is a constant times $\bar{W}(x)$, both being solutions of the homogeneous equation, since by the above treatment, putting $\phi(x) = 0$, the family $CW_0(x)$ is contained in the family $\bar{C}\bar{W}(x)$; and $\bar{U}(x)$ is therefore given by $U(x) + C_1 W_0(x)$, where C_1 is some constant. So the two families are identical. We have thus shown that all solutions obtained by taking different values of R in the development of § 14 are members of this family.

17. *Extension of the One Parameter Family beyond R.*

We have still to see how we can extend our solutions beyond the farthest point R. We may write (32″) as

$$u(x) = \phi(x) + \int_a^R \frac{1 + \Gamma(x, \xi)}{f(\xi)} u(\xi)\, d\xi + \int_R^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} u(\xi)\, d\xi$$

or

$$u(x) = \psi_c(x) + \int_R^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} u(\xi)\, d\xi,$$

where

$$\psi_c(x) = \phi(x) + \int_a^R \frac{1 + \Gamma(x, \xi)}{f(\xi)} u(\xi)\, d\xi$$

$$= \phi(x) + \int_a^R \frac{1 + \Gamma(x, \xi)}{f(\xi)} \{ U(\xi) + CW_0(\xi) \}\, d\xi.$$

This equation, when the value of C is specified, gives rise to a single solution $u(x)$, since in the region $R \leqq x \leqq b$ the kernel and the function $\psi_c(x)$ are continuous. Moreover this solution has at R the value $\psi_c(R)$ or C and hence joins on continuously to the particular solution that was assumed from a to R.

In particular the solutions $U(x)$ and $U(x) + W_0(x)$ can be so extended, and there will be a one-parameter family of solutions $U(x) + CW_0(x)$, arising from the fact that $W_0(x)$ is a solution of the homogeneous equation. But this family is identical with that obtained by extending each solution separately, since there is only one solution that takes on the value C when $x = R$. Thus the solutions are extended uniquely and continuously to the point b.

18. *The Theorem of section* 10. *A more General Theorem.*

The theorem of § 10 is now easily made complete. For given the equation

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)g(x)} u(\xi)\, d\xi \tag{20}$$

satisfying the conditions of that theorem, we succeed, by making the substitution

$$w(x) = u(x)g(x), \qquad u(x) = \frac{w(x)}{g(x)},$$

in transforming the given equation into the equation

$$w(x) = \phi(x)g(x) + \int_a^x \frac{K(x, \xi)}{[f(\xi)g(\xi)]} w(\xi)\, d\xi, \tag{42}$$

so that if $u(x)$ is a solution of the first, $w(x)$ is a solution of the second, and vice versa. But this latter equation is of the type of section 12, for which we have solutions continuous in the neighborhood of a and vanishing at a as strongly as $f(x)g(x)$ if $K(a, a) < 0$, or $f(x)g(x)/(x-a)^{\nu'}$ if $K(a, a) > 0$. Hence the solution of the given equation for $K(a, a) < 0$ is continuous at a and in its neighborhood, and for $K(a, a) > 0$ the solutions of the given equation constitute a one-parameter family, continuous in the neighborhood of a except at a and, as we approach a, remaining less in absolute value than $f(x)/(x-a)^{\nu'}$ (since $\nu > \nu'$). It is possible, if $K(a, a) > 0$, to have solutions that become infinite at a.

It is worth noting that by the same sort of substitution,

$$w(x) = \frac{g(x)}{g_1(x)} u(x),$$

we can treat the slightly more general equation

$$u(x) = \phi(x) + \int_a^x \frac{f_1(\xi)g_1(x)}{f(\xi)g(x)} K(x, \xi)u(\xi)\, d\xi.$$

In fact we have the following theorem:

THEOREM. *In the equation*

$$u(x) = \phi(x) + \int_a^x \frac{f_1(\xi)g_1(x)}{f(\xi)g(x)} K(x, \xi)u(\xi)\, d\xi,$$

if

1°. (*a*) $K(x, \xi)$ *is continuous in* T, *and* $f(x)$, $g(x)$, $f_1(x)$, $g_1(x)$ *and the first derivative of the function* $f(x)g(x)/f_1(x)g_1(x)$ *are continuous in* t;

(*b*) $\partial K(x, \xi)/\partial x$ *and* $\partial K(x, \xi)/\partial \xi$ *satisfy condition* (*A*) *and are finite in* T;

(*c*) $\phi(x)$ *is continuous in* t *except at* $x = a$, *and is such that the function* $\phi(x) \cdot g(x)/g_1(x)$ *and its first derivatives satisfy condition* (*B*) *in* t;

2°. *the function* $f(x)g(x)$ *is greater than zero in* t *except for the value* $x = a$, *and for the value* $x = a$ *vanishes in such a way that*

$$\int_a^x \frac{f_1(\xi)g_1(x)}{f(\xi)g(x)}\,dx$$

is not convergent;

3°. $K(a, a) \neq 0$;

4°. $\lim_{x=a} \phi(x)\dfrac{g(x)}{g_1(x)} = 0$:

then under the foregoing conditions,

(*i*) *if* $K(a, a) < 0$ *there exists one solution of the equation continuous throughout* t *except perhaps at* a, *and this solution as* x *approaches* a *remains less in absolute value than some constant times* $f(x)/f_1(x)$; *and*

(*ii*) *if* $K(a, a) < 0$ *there exists a one parameter family of solutions continuous throughout* t *except perhaps at* a, *and each solution as* x *approaches* a *remains less in absolute value than some constant times* $f(x)/[f_1(x)(x-a)^\nu]$, *where* ν *is any number that satisfies the conditions*

$$1 > \nu > 0, \qquad \nu > 1 - \frac{1}{\left\{\dfrac{d}{dx}\left[\dfrac{f(x)g(x)}{f_1(x)g_1(x)}\right]\right\}_{x=a}}.$$

19. *Extension of Solutions Beyond a Second Discontinuity of the Kernel.*

The question arises as to whether the solutions can be extended beyond a second vanishing point of $f(x)$ or $g(x)$. In general they cannot, as the following example shows. The equation

$$u(x) = \phi(x) - \int_0^x \frac{(a-\xi)}{\xi(2a-\xi)}\,u(\xi)\,d\xi$$

has the single solution

$$u(x) = e^{\int_x^a \frac{a-x}{x(2a-x)}\,dx} \int_0^x \phi'(\xi)\, e^{-\int_\xi^a \frac{(a-\xi)}{\xi(2a-\xi)}\,d\xi}\,d\xi$$

which becomes infinite as x approaches $2a$.

Still, there are large classes of solutions that can be extended. A condition is that the solution itself vanish at the point where $f(x)$ or $g(x)$ vanishes.

20. *Discontinuity at the End of the Interval.*

We have confined ourselves to integral equations (1) where $a \leqq x \leqq b$. When, instead, we have $a \geqq x \geqq b$, so that the discontinuity in the kernel is at the end of the interval, if we make the substitutions

$$\begin{aligned} x' &= a - x, \qquad x = a - x', \\ \xi' &= a - \xi, \qquad \xi = a - \xi', \\ d\xi' &= -\, d\xi, \end{aligned} \qquad (0 < x' \leqq a - b),$$

the equation

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{g(x) f(\xi)} u(\xi)\, d\xi$$

is reduced to

$$u(a - x') = \phi(a - x') - \int_0^{x'} \frac{K(a - x', a - \xi')}{f(a - \xi') g(a - x')} u(a - \xi')\, d\xi',$$

or

$$\bar{u}(x') = \bar{\phi}(x') - \int_0^{x'} \frac{\bar{K}(x', \xi')}{\bar{f}(\xi') \bar{g}(x')} \bar{u}(\xi')\, d\xi'.$$

Hence an equation where $K(a, a) > 0$ is changed into an equation of type (32′), and one where $K(a, a) < 0$ into one of type (32″).

21. *A Different Point of View.*

An instructive point of view as regards the nature of the solutions of (32″) developed in section 16, is obtained in the following way. If we had written (32″) in the form

$$u(x) = \phi(x) + \int_a^x \frac{1 + \lambda \Gamma(x, \xi)}{f(\xi)} u(\xi)\, d\xi,$$

thus introducing the parameter λ,* and sought for a solution as a power series in λ,

$$V(x) = v_0(x) + \lambda v_1(x) + \lambda^2 v_2(x) \cdots,$$

the conditions determining $v_m(x)$ would be precisely those that we have already found, and would yield

$$v_m(x) = \bar{u}_m(x) + c_m e^{-\int_x^b \frac{dx}{f(x)}}.$$

Hence all solutions analytic in λ belong to the linear family

$$U(x) + C W_0(x).$$

* This is analogous to the point of view of M. T. LALESCO in regard to equations of the first kind. T. LALESCO: *Sur l'équation de Volterra*, Journal de Mathematiques, ser. 6, vol. 4 (1908), p. 125. See Transactions of the American Mathematical Society, vol. 11 (1910), p. 395.

The same investigation applied to the equation (32′) shows that there is one and only one solution of the equation that is analytic in λ.

22. *A Restrictive Theorem upon a Class of Solutions.*

We can go, however, somewhat farther in limiting the number of solutions of equations (32′) and (32″). The solutions so far developed satisfy the condition

$$|V(x)| \leqq \text{const}\,\frac{f(x)}{(x-a)^{\nu}}.$$

We can show that any solution which is continuous in the neighborhood of the origin, except possibly at the origin, and which satisfies this condition, is among the solutions already found. We have the following theorem:

THEOREM. *Given the hypotheses of* § 10, 1°–4°, *the solutions there specified are the only ones continuous in the neighborhood of the origin, except possibly at the origin, such that in the neighborhood of the origin each solution satisfies the condition*

$$|u(x)| \leqq \text{const}\,\frac{f(x)}{(x-a)^{\nu}},$$

or, more generally, if

$$\Phi(x) = \int_a^x \frac{K(x,\xi) - K(a,a)}{f(\xi)}\,u(\xi)\,d\xi,$$

such that

(*a*) $\Phi(x)$ *approaches zero as* x *approaches* a;

(*b*) $\Phi'(x)$ *remains finite as* x *approaches* a.

There are no such solutions if 1°–3° *of section* 10 *hold, but not* 4°.

By the transformation

$$u(x) = \frac{w(x)}{g(x)}$$

the given equation is reduced to the equation (42)

$$w(x) = \phi(x)g(x) + \int_a^x \frac{K(x,\xi)}{f(\xi)g(x)}\,w(\xi)\,d\xi, \tag{42}$$

and the condition becomes $\lim \Phi(x) = 0$, and $\Phi'(x)$ remains finite, as x approaches a, where

$$\Phi(x) = \int_a^x \frac{K(x,\xi) - K(a,a)}{f(\xi)g(\xi)}\,w(\xi)\,d\xi.$$

If we assume the first part of the theorem of this section, the second part follows immediately. For since

$$w(x) = \phi(x)g(x) + \Phi(x) + \int_a^x \frac{w(\xi)}{a\,\dfrac{1}{K(a,a)}f(\xi)g(\xi)}\,d\xi$$

$w(x)$ must satisfy an equation of the type of (25′) or (25″). But we have seen that it is necessary, in order that there be a solution of those equations, that the term not involving the dependent variable have zero as a limit as x approaches a. Hence

$$\lim_{x=a} [\phi(x)g(x) + \Phi(x)] = 0.$$

But

$$\lim_{x=a} \Phi(x) = 0,$$

so that

$$\lim_{x=a} [\phi(x)g(x)] = 0,$$

which was to be proved.

If in (42) we put $\phi(x)$ for $\phi(x)g(x)$, $f(\xi)$ for $f(\xi)g(\xi)$, and $u(x)$ for $w(x)$, we have the equation of § 12, which is then reducible to (32′) or (32″). The condition then becomes

$$\lim_{x=a} \Phi(x) = 0, \qquad |\Phi'(x)| \leqq N, \qquad a \leqq x \leqq a' \qquad (a' > a),$$

where

$$\Phi(x) = \int_a^x \frac{\Gamma(x, \xi)}{f(\xi)} u(\xi) d\xi.$$

Let us investigate this theorem in regard to equation (32″). Replacing $1 + \Gamma(x, \xi)$ by $K(x, \xi)$ we write (32″) as

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)} u(\xi) d\xi. \tag{43}$$

The equation (25″) then becomes

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi) - \Gamma(x, \xi)}{f(\xi)} u(\xi) d\xi, \tag{44}$$

and we can obtain a solution of it by a method of approximation from (43).

For, let $U_0^0(x)$ be any chosen solution of (43) that satisfies the conditions of the theorem, and let us try to find a solution of (44) in the form

$$u_0(x) = U_0^0(x) + U_1^0(x) + U_2^0(x) + \cdots$$

where

$$U_m^0(x) = \Phi_m^0(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)} U_m^0(\xi) d\xi, \tag{45}$$

and

$$\Phi_m^0(x) = -\int_a^x \frac{\Gamma(x, \xi)}{f(\xi)} U_{m-1}^0(\xi) d\xi, \tag{46}$$

$$\Phi_0^0(x) = \phi(x).$$

Then $\Phi_1^0(x) = -\Phi(x)$, as defined in the hypothesis, so that $\Phi_1^0(a) = 0$, and $d\Phi_1^0(x)/dx$ remains finite. For U_1^0 take the solution developed in section 14, on $\Phi_1^0(x)$ as a basis. That analysis gives us

$$|U_1^0(x)| \leqq (x-a)^{-\nu'} \frac{CNf(x)}{1-DC(R-a)^{\nu-\nu'}}^{*} \qquad (R \leqq a'),$$

so that

$$|\Phi_2^{0\prime}(x)| \leqq N \frac{DC(R-a)^{\nu-\nu'}}{1-DC(R-a)^{\nu-\nu'}},$$

since

$$\Phi_2^{0\prime}(x) = -\frac{\Gamma(x,x)}{f(x)} U_1^0(x) - \int_a^x \frac{\Gamma_1(x,\xi)}{f(\xi)} U_1^0(\xi)\, d\xi.$$

In the same way

$$(\alpha) \quad |U_m^0(x)| \leqq (x-a)^{-\nu'} \left\{ \frac{DC(R-a)^{\nu-\nu'}}{1-DC(R-a)^{\nu-\nu'}} \right\}^{m-1} \frac{CNf(x)}{1-DC(R-a)^{\nu-\nu'}},$$

and

$$(\beta) \qquad |\Phi_m^{0\prime}(x)| \leqq N \left\{ \frac{DC(R-a)^{\nu-\nu'}}{1-DC(R-a)^{\nu-\nu'}} \right\}^{m-1}.$$

If we denote this parenthesis by ρ we have

$$\sum_{m=1}^{\infty} |\Phi_m^{0\prime}(x)| \leqq \frac{N}{1-\rho}$$

and the series for u_0 is uniformly convergent for $a \leqq x \leqq a'$, since taking R near enough to a we can make ρ as small as we please.

The function u_0 thus developed is shown in the same way as in § 14 to be a solution of the equation (44), which is, of course, equation (25″). Let us now pursue a method of approximation from (44) back again to (43), defining each solution of (44) by a method of approximation from solutions of (43). We define

$$\phi_1(x) = \int_a^x \frac{\Gamma(x,\xi)}{f(\xi)} u_0(\xi)\, d\xi,$$

and in general

$$\phi_n(x) = \int_a^x \frac{\Gamma(x,\xi)}{f(\xi)} u_{n-1}(\xi)\, d\xi, \qquad u_n(x) = \sum_{m=0}^{\infty} U_m^n(x),$$

$$U_m^n(x) = \Phi_m^n(x) + \int_a^x \frac{K(x,\xi)}{f(\xi)} U_m^n(\xi)\, d\xi,$$

$$\Phi_m^n(x) = -\int_a^x \frac{\Gamma(x,\xi)}{f(\xi)} U_{m-1}^n(\xi)\, d\xi, \qquad \Phi_0^n(x) = \phi_n(x).$$

* Or $CNf(x)/(1-DC\Gamma)$ when $\nu = 0$.

The series for $u_n(x)$ is a solution of the equation

$$u_n(x) = \phi_n(x) + \int_a^x \frac{u(\xi)}{f(\xi)}\, d\xi,$$

as may be proved by a method similar to that used in section 14.

We have in fact,

$$\phi_1(x) = \int_a^x \frac{\Gamma(x, \xi)}{f(\xi)} \sum_{m=0}^{\infty} U_m^0(\xi)\, d\xi$$

$$= \sum_{m=0}^{j} \int_a^x \frac{\Gamma(x, \xi)}{f(\xi)} U_m^0(\xi)\, d\xi + \int_a^x \frac{\Gamma(x, \xi)}{f(\xi)} \sum_{j+1}^{\infty} U_m^0(\xi)\, d\xi,$$

and the limit of this as j becomes infinite gives us, the last term disappearing on account of the inequalities (α), (β),

$$\phi_1(x) = \sum_{m=0}^{\infty} \int_a^x \frac{\Gamma(x, \xi)}{f(\xi)} U_m^0(\xi)\, d\xi,$$

or

$$\phi_1(x) = -\sum_{m=1}^{\infty} \Phi_m^0(x).$$

We see then that if we define

$$U_0^1(x) = -\sum_{m=1}^{\infty} U_m^0(x),$$

it will be a solution of the equation

$$U_0^1(x) = \Phi_0^1(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)} U_0^1(\xi)\, d\xi,$$

and will satisfy the inequality

$$|U_0^1(x)| \leqq (x-a)^{-\nu} f(x) \frac{N\rho}{D(1-\rho)}, \qquad \text{since} \qquad (x-a)^{-\nu'} \leqq \frac{(R-a)^{\nu-\nu'}}{(x-a)^{\nu}}.$$

Now if we determine $U_m^1(x)$, $m > 0$, as the function U of section 14 based on $\Phi_m^1(x)$, we have

$$|\Phi_m^{1\prime}(x)| \leqq \frac{N\rho^m}{(1-\rho)},$$

and

$$\sum_{m=1}^{\infty} |\Phi_m^{1\prime}(x)| \leqq \frac{N}{1-\rho} \cdot \frac{\rho}{1-\rho}.$$

And if in general we put

$$U_0^n(x) = -\sum_{m=1}^{\infty} U_m^{n-1}(x),$$

$U_0^n(x)$ will be a solution of its proper equation

$$U_0^n(x) = \Phi_0^n(x) + \int_a^x \frac{K(x,\xi)}{f(\xi)} U_0^n(\xi)\, d\xi,$$

and

$$(\gamma)\quad |U_0^n(x)| \leqq (x-a)^{-\nu} f(x) \frac{N}{D} \cdot \left(\frac{\rho}{1-\rho}\right)^n, \qquad |\Phi_m^{n'}(x)| \leqq \frac{N}{1-\rho}\left(\frac{\rho}{1-\rho}\right)^{n-1} \rho^m,$$

and

$$(\delta)\qquad |\phi'_{n+1}(x)| \leqq \sum_{m=1}^{\infty} |\Phi_m^{n'}(x)| \leqq \frac{N}{1-\rho}\left(\frac{\rho}{1-\rho}\right)^n;$$

formulas that are verified by mathematical induction.

Let us now consider the function

$$V(x) = \lim_{n=\infty} [\, u_0(x) + u_1(x) + u_2(x) + \cdots + u_n(x)\,].$$

We have

$$\begin{aligned} u_0(x) + u_1(x) + \cdots + u_n(x) &= U_0^0(x) + U_1^0(x) + U_2^0(x) + \cdots \\ &\quad \cdots \\ &\quad + U_0^n(x) + U_1^n(x) + U_2^n(x) + \cdots, \end{aligned}$$

and this reduces to

$$U_0^0(x) - U_0^{n+1}(x).$$

But by (γ)

$$|U_0^{n+1}(x)| \leqq (x-a)^{-\nu} f(x) \frac{N}{D}\left(\frac{\rho}{1-\rho}\right)^{n+1},$$

whence if we take R so small that $\rho < \frac{1}{2}$, i. e., $\rho/(1-\rho) < 1$, we shall have

$$\lim_{n=\infty} |U_0^{n+1}(x)| \equiv 0.$$

Hence

$$V(x) = U_0^0(x).$$

But since

$$\sum_{n=0}^{\infty} u_n(x)$$

is convergent for $x = R$, $V(x)$ must be one of the linear family

$$V(x) = U(x) + C\, W_0(x).*$$

Hence any solution that satisfies the conditions of the present theorem is a member of that one-parameter family.

This same analysis applies to equation (32′). In the case of (32′), since each $u_n(x)$ is uniquely determined, there is only one possible solution.

* See section 16.

23. *A Restrictive Theorem upon the Totality of Solutions.*

So far we have considered only finite solutions, or at most solutions that become infinite at $x = a$ to an order not greater than the first. It is possible, however, to limit the totality of solutions as to character.

THEOREM. *Let the kernel of* (1) *be in the form*

$$\frac{K(x, \xi) f_1(\xi)}{f(\xi) g(x)} *$$

where

1°. (*a*) $K(x, \xi)$ *is continuous in* T,† *and* $f(x)$, $g(x)$ *and* $f_1(x)$ *are continuous in* t;

(*b*) $\partial K(x, \xi)/\partial \xi = K_2(x, \xi)$ *satisfies* A, *and is finite in* T;

(*c*) $\phi(x)$ *is continuous in* t *except perhaps at* a, *and is such that the function* $\phi(x) g(x)$ *is continuous at* a.

2°. *The function* $f(x) g(x)$ *vanishes at most a finite number of times in* t.

3°. *On any horizontal line* $\xi = \xi_0$ *cutting* T *there is at least one point in* T *for which* $K(x, \xi) \neq 0$.‡

Then all the solutions of (1) *continuous in* t *except for a finite number of points are such that the function* $u(x) g(x)$ *remains continuous in* t, *except possibly at* $x = b$.§

This conclusion is equivalent to the statement that there can be no solutions, continuous in t except possibly for a finite number of points, of the equation

$$u(x) = \phi(x) + \int_a^x K(x, \xi) \frac{f_1(\xi)}{f(\xi)} u(\xi) d\xi,$$

and not continuous in all points of t except possibly $x = b$. In fact equations of the first form may be reduced to those of the second by change of independent variable (see section 18).

For the proof of this theorem we need the following lemma, which is stated in slightly more general form than necessary:

Let $h(x)$ *be a function of* x *continuous for* $A < x \leqq B$ *except for a finite number of finite discontinuities, such that*

$$\int_A^B h(x) dx$$

converges.

* This redundant form of expression for the kernel is used on account of condition 3°.

† If we replace the triangle T by the square S: $a \leqq \xi \leqq b$, $a \leqq x \leqq b$ this theorem holds for the equation with constant limits, $u(x) = \phi(x) + \int_a^b K(x, \xi) u(\xi) d\xi$.

‡ An example that does not satisfy the condition 3° is given by BÔCHER, *Introduction to the Study of Integral Equations*, p. 17. The equation is $u(x) = \int_0^x \xi^{x-\xi} u(\xi) d\xi$, which has as solutions $u(x) = kx^{x-1}$, which are non-integrable.

§ This exception was not noted in a preliminary paper, Bulletin of the American Mathematical Society, 16, (1909), p. 130.

Let $r(x)$ be a function such that $r(x)$ and $r'(x)$ are continuous for $A \leqq x \leqq B$ except for a finite number of finite discontinuities.

Then

$$\int_A^B r(x)h(x)dx$$

converges.

The statement is easily verified by integration by parts. If $B' > A$ is any point before the second discontinuity of $r(x)$,

$$\int_A^B r(x)h(x)dx = \int_{B'}^B r(x), h(x)dx + r(B')\int_A^{B'} h(x)dx - \int_A^{B'} r'(x)\int_A^x h(s)dsdx,$$

whence if in the region $A \leqq x \leqq B$ we define P, P' and Q by the relations

$$P \geqq |r(x)|, \qquad P' \geqq |r'(x)|, \qquad Q \begin{cases} \geqq \left| \int_A^x h(s)ds \right|, \\ \geqq \left| \int_{B'}^B |h(x)|dx \right|, \end{cases}$$

we shall have

$$\left| \int_A^B r(x)h(x)dx \right| \leqq PQ + P'Q|B' - A|.$$

To apply this lemma to the conditions of the theorem it is convenient to extend our integrations outside the oblique boundary of T. For this purpose let us define

$$K(x, \xi) = K(x, x), \qquad \xi > x,$$

by which means a continuous extension across the boundary line $\xi = x$ is provided. Beyond this boundary we shall have $K_2(x, \xi) = 0$.

Let us suppose that $x = a$ is the first point where $u(x)$ is not continuous. On the line $\xi = a$ in T there is a point in the neighborhood of which $K(x, a) \neq 0$. Let $x = x_0$ be a point for which $K(x, a) \neq 0$ and for which

$$\int_a^{x_0} K(x_0\xi)\frac{f_1(\xi)}{f(\xi)}u(\xi)d\xi$$

is convergent. Then also

$$\int_a^R K(x_0, \xi)\frac{f_1(\xi)}{f(\xi)}u(\xi)d\xi$$

is convergent, $a \leqq R \leqq x_0$. Let us fix R less than a_1, any second supposed discontinuity of $u(x)$, and near enough to a so that $K(x_0, \xi) \neq 0$, $a \leqq \xi \leqq R$, and $f(\xi) \neq 0$, $a < \xi \leqq R$.

In the integral

$$\int_a^R K(x_0\xi)\frac{f_1(\xi)}{f(\xi)}u(\xi)$$

take the integrand as the function $h(\xi)$ of the lemma. As $r(x)$ choose the function

$$\frac{K(x,\xi)}{K(x_0\xi)} \qquad (a\leqq x\leqq R),$$

which for a fixed value of x is a continuous function of ξ. The value of $r'(\xi)$ is

$$r'(\xi)=\frac{K(x_0\xi)K_2(x,\xi)-K(x,\xi)K_2(x_0\xi)}{[K(x_0,\xi)]^2},$$

and accordingly remains finite. Consequently the conditions of the lemma are satisfied. Since

$$r(\xi)h(\xi)=K(x,\xi)\frac{f_1(\xi)}{f(\xi)}u(\xi),$$

we have the convergence of the integral

$$\int_a^R \frac{K(x,\xi)f_1(\xi)}{f(\xi)}u(\xi)\,d\xi \qquad (a\leqq x\leqq R),$$

which carries with it the convergence of the integral

$$u(x)-\phi(x)=\int_a^x K(x,\xi)\frac{f_1(\xi)}{f(\xi)}u(\xi)\,d\xi \qquad (a\leqq x\leqq R).$$

Since the numbers P, P', Q of the lemma can be defined as independent of x, it follows that

$$\lim_{x=a}\int_a^x K(x,\xi)\frac{f_1(\xi)}{f(\xi)}u(\xi)\,d\xi=0,$$

and therefore, since $\phi(x)$ is continuous at $x=a$, $u(x)$ is also.

In the same way the integral

$$\int_{a_1}^x K(x,\xi)\frac{f_1(\xi)}{f(\xi)}u(\xi)\,d\xi$$

may be treated, where x is in the neighborhood of a_1 $(a_1\neq b)$, the next succeeding possible discontinuity of u. The integrals

$$\int_a^{a_1} K(x,\xi)\frac{f_1(\xi)}{f(\xi)}u(\xi)\,d\xi, \qquad \int_a^x K(x,\xi)\frac{f_1(\xi)}{f(\xi)}u(\xi)\,d\xi$$

are continuous functions of x at $x=a_1$. And therefore, since $\phi(x)$ is continuous, $u(x)$ must be continuous at $x=a_1$ also.

24. *Continuity of Certain Functions Represented by Definite Integrals.*

In particular the above method of proof establishes the continuity throughout t except at b of any integral of the form

$$\int_a^x G(x, \xi) \frac{f_1(\xi)}{f(\xi)} u(\xi)\, d\xi,$$

where $G(x, \xi)$ is continuous in T, $\partial G/\partial \xi$ satisfies (A) and is finite in T, and $u(x)$ is a solution of the equation

$$u(x) = \phi(x) + \int_a^x K(x, \xi) \frac{f_1(\xi)}{f(\xi)} u(\xi)\, d\xi,$$

and therefore where $u(x)$ is a solution of the equation

$$u(x) = \phi(x) + \int_a^x K(x, \xi) \frac{f_1(\xi)}{f(\xi) g(x)} u(\xi)\, d\xi,$$

under the conditions imposed in section 23. Also if $G(x, \xi)$ and $\partial G/\partial \xi$ merely satisfy (A) and are finite in T, the integral is continuous in t except at a finite number of points, and remains finite except possibly at $x = b$.

25. *Application to Analytic Kernels.*

If we let $g(x) \equiv 1, f(x) \equiv 1$ the equation (1) becomes

$$u(x) = \phi(x) + \int_a^x f_1(\xi) K(x, \xi) u(\xi)\, d\xi.$$

We know that there is *one and only one continuous solution of this equation.* The theorem of *section* 23 tells us that *this is the only solution continuous except for a finite number of points.*

Under this special case is included the integral equation with analytic kernel; for any function of the two variables analytic in T can be written in the form

$$f_1(\xi) K(x, \xi)$$

where $K(x, \xi)$ satisfies the conditions of section 23. If $\phi(x)$ is also analytic the unique solution is the known analytic solution.*

26. *Equations for which section 9 gives the Complete Solution.*

Let us make the hypothesis:

The kernel of (1) *shall be in the form*

$$\frac{K(x, \xi)}{f(\xi) g(x)},$$

* If the kernel of the integral equation with constant coefficients is analytic in S and if $\phi(x)$ is continuous in t, the continuous solutions are the only solutions continuous except for a finite number of points.

where

1°. (*a*) $K(x, \xi)$ *is continuous in* T, *and* $f(x)g(x)$ *and first derivatives are continuous in* t;

(*b*) $\partial K(x, \xi)/\partial x$ *and* $\partial K(x, \xi)/\partial \xi$ *satisfy* A, *and are finite in* T;

(*c*) $\phi(x)$ *is continuous in* t *except perhaps at* a, *and is such that the function* $\phi(x)g(x)$ *and its first derivative satisfy* (B);

2°. *the function* $f(x)g(x)$ *is greater than zero in the neighborhood of* a, *and vanishes at* a *in such a way that*

$$\int_a^x \frac{dx}{f(x)g(x)}$$

is not convergent;

3°. $$K(a, a) \neq 0;$$

4°. $$\lim_{x=a} \phi(x)g(x) = 0.$$

Under these conditions the requirements of the theorems of sections 9 or 10 and 23 are satisfied in the neighborhood of the origin. There are some cases where the solutions specified in section 9 are the only possible solutions of the equation continuous except for a finite number of points. To investigate this let us impose the further condition that

$$K_{12}(x, \xi) = \frac{\partial^2 K}{\partial x \partial \xi}$$

shall satisfy A *and be finite in* T.* It is easy to show that if in addition (*i*) $K(x, \xi)$ *is constant when* $\xi = x$ *or* (*ii*) *it is known that the derivative of* $u(x)g(x)$ *remains finite as* x *approaches* a, *then all solutions continuous in* t *except for a finite number of points belong to the system of section* 9.

We have the equations (see section 22),

$$\Phi(x) = \int_a^x \frac{K(x, \xi) - K(a, a)}{f(\xi)} u(\xi)\, d\xi,$$

$$\Phi'(x) = \frac{K(x, x) - K(a, a)}{f(x)} u(x) + \int_a^x \frac{K_1(x, \xi)u(\xi)}{f(\xi)} d\xi \quad (x \neq a),$$

where

$$K_1(x, \xi) = \partial K(x, \xi)/\partial x.$$

In (*i*)

$$K(x, x) - K(a, a) \equiv 0,$$

and since the integrals

$$\int_a^x \frac{K(x, \xi) - K(a, a)}{f(\xi)} u(\xi)\, d\xi, \qquad \int_a^x \frac{K_1(x, \xi)}{f(\xi)} u(\xi)\, d\xi,$$

* This condition was left out in the preliminary paper mentioned before.

are both convergent and remain finite at $x = a$ (see section 24), $\Phi(x)$ and $\Phi'(x)$ both have the limit 0 as x approaches a. Hence the conditions of the theorem of section 22 are satisfied.

In (*ii*) we have

$$u(x)g(x) = \phi(x)g(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)} u(\xi)d\xi,$$

$$\frac{d}{dx}\{u(x)g(x)\} = \frac{d}{dx}\{\phi(x)g(x)\} + \frac{K(x, x)}{f(x)} u(x) + \int_a^x \frac{K_1(x, \xi)}{f(\xi)} u(\xi)d\xi \quad (x \neq a),$$

$$\frac{d}{dx}\{u(x)g(x)\} - \frac{d}{dx}\{\phi(x)g(x)\} - \int_a^x \frac{K_1(x, \xi)}{f(\xi)} u(\xi)d\xi = \frac{K(x, x)}{f(x)} u(x) \quad (x \neq a),$$

where

$$K_1(x, \xi) = \frac{\partial K(x, \xi)}{\partial x}.$$

The first member of this last equation remains finite as x approaches a; hence the second member must remain finite. And since $K(x, \xi)$ is continuous and $K(a, a) \neq 0$, the function $u(x)/f(x)$ and therefore

$$[K(x, x) - K(a, a)] \frac{u(x)}{f(x)}$$

remains finite as x approaches a. Hence by section 24

$$\lim_{x=a} \Phi(x) = 0,$$

and $\Phi'(x)$ remains finite as x approaches a. So the conditions of the theorem of section 22 are satisfied.

The solutions specified in section 9 are therefore the only solutions continuous except for a finite number of points.

27. *Equations Whose Kernels are Analytic Functions of x, ξ Divided by a Power of ξ. Treatment by Power Series.*

Let us now consider in the neighborhood of the origin equations where the kernel is in the form of an analytic function,

$$K(x, \xi) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} x^i \xi^j,$$

divided by an integral power of ξ, ξ^μ. This kernel is no less general than that in which the denominator is of the form $x^\nu \xi^\mu$, since the second case can be reduced to the first by a change of dependent variable. We shall suppose $\phi(x)$ also to be analytic at $x = 0$.

Let us seek analytic solutions of this equation,

$$u(x) = \phi(x) + \int_0^x \frac{K(x, \xi)}{\xi^\mu} u(\xi) d\xi, \tag{48}$$

in the neighborhood of the origin. Substitute

$$u(x) = \sum_{i=0}^{\infty} b_i x^i, \qquad \phi(x) = \sum_{i=0}^{\infty} g_i x^i,$$

$$\frac{K(x, \xi)}{\xi^\mu} = \frac{1}{\xi^\mu} \sum_{p=\lambda}^{\infty} \sum_{i=0}^{p} a_{i, p-i} x^i \xi^{p-i} \qquad (\mu > \lambda \geqq 0).$$

The equation then becomes

$$\sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} g_i x^i + \int_0^x \sum_{r=\lambda}^{\infty} \sum_{p=\lambda}^{r} b_{r-p} \xi^{r-p-\mu} \sum_{i=0}^{p} a_{i, p-i} x^i \xi^{p-i} d\xi.$$

Let us formally determine the coefficients b_i. Our operations will be justified by a proof of the convergence of the resulting series for $u(x)$.

We have

$$\sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} g_i x^i + \int_0^x \sum_{r=\lambda}^{\infty} \sum_{r=\lambda}^{r} b_{r-p} \sum_{i=0}^{p} a_{i, p-1} x^i \xi^{r-\mu-i} d\xi.$$

We may assume that $K(x, \xi)$ contains no factor ξ. Hence in order that the integral be convergent it is necessary that $u(x)$ contain x^μ as a factor. Under these conditions we get

$$\sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} g^i x^i + \sum_{r=\mu}^{\infty} \sum_{p=\lambda}^{r-\mu} b_{r-p} \sum_{i=0}^{p} \frac{a_{i, p-i}}{r - \mu - i + 1} x^{r-\mu+1}, \tag{48'}$$

and

$$b_{r-\mu+1} = g_{r-\mu+1} + \sum_{p=\lambda}^{r-\mu} b_{b-p} \sum_{i=0}^{p} \frac{a_{i, p-i}}{r - \mu - i + 1}. \tag{48''}$$

Or, writing

$$r - \lambda = m, \qquad r = \lambda + m,$$

we have

$$b_{m+\lambda-\mu+1} = g_{m+\lambda-\mu+1} + \sum_{p=\lambda}^{m+\lambda-\mu} b_{m+\lambda-p} \sum_{i=0}^{p} \frac{a_{i, p-i}}{m + \lambda - \mu - i + 1}.$$

To determine b_m we have the equation

$$-b_m \left(\sum_{i=0}^{\lambda} \frac{a_{i, \lambda-1}}{m + \lambda - \mu - i + 1} \right) = g_{m+\lambda-\mu+1} - b_{m+\lambda-\mu+1}$$

$$+ \sum_{p=\lambda+1}^{m+\lambda-\mu} b_{m+\lambda-p} \sum_{i=0}^{p} \frac{a_{i, p-1}}{m + \lambda - \mu - i + 1},$$

or, on writing $s = m + \lambda - p$,

$$(49)\qquad -b_m\left(\sum_{i=0}^{\lambda}\frac{a_{i,\,\lambda-i}}{m+\lambda-\mu-i+1}\right) = g_{m+\lambda-\mu+1} - b_{m+\lambda-\mu+1} + \sum_{s=\mu}^{m-1} b_s \sum_{i=0}^{m+\lambda-s}\frac{a_{i,\,m+\lambda-s-1}}{m+\lambda-\mu-i+1}.$$

Since $b_m = 0$ for $m < \mu$, g_m must be zero for $i \leqq \lambda$.

We have seen that if $\lambda = 0$, i. e., $K(0,0) \neq 0$ the solutions are developable into a series of functions which have *essential singularities* at the origin. And so in general we may expect the above series for $u(x)$ to have no circle of convergence; this in fact can be shown by particular cases, the coefficients b_m being comparable to $m!$ in magnitude. In the special case when $\mu - \lambda = 1$ however, the convergence of the series may be shown. The equation to determine b_m becomes

$$(50)\qquad b_m\left(1-\sum_{i=0}^{\lambda}\frac{a_{i,\,\lambda-i}}{m-i}\right) = g_m + \sum_{s=\mu}^{m-1} b_s \sum_{i=0}^{m+\lambda-s}\frac{a_{i,\,m+\lambda-s-i}}{m-i},$$

or

$$b_m = \frac{g_m}{1-\sum_{i=0}^{\lambda}\frac{a_{i,\,\lambda-i}}{m-i}} + \frac{\sum_{s=0}^{m-1} b_s \sum_{i=0}^{m+\lambda-s}\frac{a_{i,\,m+\lambda-i}}{m-i}}{1-\sum_{i=0}^{\lambda}\frac{a_{i,\,\lambda-i}}{m-i}}.$$

Let us take $m \geqq m_0$ great enough so that

$$\sum_{i=0}^{\lambda}\frac{|a_{i,\,\lambda-i}|}{m_0-i} \leqq \frac{1}{m_0-\lambda}\sum_{i=0}^{\lambda}|a_{i,\,\lambda-i}| \leqq \tfrac{1}{2}.$$

Then we shall have

$$|b_m| < 2|g_m| + 2\sum_{s=\mu}^{m-1}|b_s|\sum_{i=0}^{m+\lambda-s}\frac{|a_{i,\,m+\lambda-s-i}|}{m-i} \qquad (m \geqq m_0),$$

and therefore

$$|b_m| < 2|g_m| + 2\sum_{s=\mu}^{m-1}|b_s|\sum_{i=0}^{m+\lambda-s}|a_{i,\,m+\lambda-s-i}|.$$

Let us define

$$c_s = \sum_{i=0}^{\lambda+s}|a_{i,\,\lambda+s-i}|.$$

Then c_s is the sum of the absolute values of the terms of the sth diagonal in the rectangular array of $K(x,\xi)$, and

$$\sum_{s=0}^{\infty} c_s x^s$$

is a power series with a radius of convergence $\rho \neq 0$. With this definition we

can write our last inequality as

$$|b_m| < 2|g_m| + 2\sum_{s=\mu}^{m-1} |b_s| c_{m-s} \qquad (m > m_0),$$

or, since $b_0 = b_1 = \cdots = b_{\mu-1} = 0$,

$$|b_m| < 2|g_m| + 2\sum_{s=0}^{m-1} |b_s| c_{m-s} \qquad (m > m_0).$$

Let us define

$$b'_m = 2|g_m| + 2\sum_{s=0}^{m-1} b'_s c_{m-s},$$

$$b'_0 = b'_1 = \cdots = b'_{m_0} > |b_j| \qquad (j = 0, 1, \cdots, \mu, \cdots, m_0),$$

assuming for the present that the coefficients in the series for b_m up to the m_0th term have been defined. They are, of course, defined uniquely beyond that.

Then $|b_m| < b'_m$. In fact $|b_0| < b'_0$, $|b_1| < b'_1$, $\cdots$, $|b_{m_0}| < b'_{m_0}$, and if $|b_0|, \cdots, |b_{m-1}|$ are respectively less than $b'_0, \cdots, b'_m$ then $|b_m| < b'_m$, as we see from the explicit formulas.

Accordingly we have merely to show the convergence of the power series

$$\sum_{m=0}^{\infty} b'_m x^m.$$

But we shall show, and the convergence of the given series follows from this fact, that if

$$\sum_{m=0}^{\infty} \gamma_m x^m \qquad \text{and} \qquad \sum_{m=0}^{\infty} \kappa_m x^m \qquad (\gamma_m > 0, \kappa_m > 0),$$

are convergent throughout the circle of radius $\rho > 0$, there is a circle of radius $\rho' > 0$ within which the series

$$\sum_{m=0}^{\infty} \beta_m x^m$$

is convergent, a series whose coefficients are defined by the relation

$$\beta_m = \gamma_m + \sum_{s=0}^{m-1} \beta_s \kappa_{m-s}.$$

For if $R < \rho$ and $R < 1$, then

$$\gamma_m < \frac{M}{R^m},$$

where M is some constant, and there exists a constant N such that

$$\kappa_m < \frac{N}{R^m}, \qquad \kappa_{m-s} < \frac{N}{R^m},$$

whence

$$\beta_m \leqq \frac{M}{R^m} + \frac{N}{R^m} \sum_{s=0}^{m-1} \beta_s.$$

Now if we define

$$\beta'_0 = \beta_0, \qquad \beta'_m = \frac{M}{R^m} + \frac{N}{R^m}\sum_{s=0}^{m-1}\beta'_s, \tag{51}$$

we shall have $\beta_m < \beta'_m$. But by (51)

$$\beta'_m = \left(\frac{1+N}{R}\right)\beta'_{m-1}.$$

Therefore if $\rho' < R/(1+N)$, the power series defined by (51) is convergent for $|x| \leqq \rho'$, and the series

$$\sum_{m=0}^{\infty}\beta_m x^m$$

is convergent throughout the same circle.

It remains to see under what conditions the terms b_i, $i < m_0$, in (50) will not be defined. This will be when for some value of m

$$1 - \sum_{i=0}^{\lambda}\frac{a_{i,\lambda-1}}{m-i} = 0.$$

Otherwise, namely in the general case, there is one and only one analytic solution. Since $\mu - \lambda = 1$, it is necessary that

$$g_0 = g_1 = , \cdots, = g_{\mu-1} = 0.$$

There cannot be more than $\lambda + 1$ integral values of m for which

$$1 - \sum_{i=0}^{\lambda}\frac{a_{i,\lambda-1}}{m-1} = 0.$$

Let these k values be $m_1, \cdots, m_k$. Then there is a one-parameter family of solutions of the homogeneous equation if we take $b_0, b_1, \cdots, b_{m_k-1} = 0$ and choose b_{m_k} arbitrarily. In general these are all the analytic solutions of the homogeneous equation. For suppose b_l $(l < m_k)$ is the first coefficient not zero; let b_{m_j} be the next coefficient for which

$$1 - \sum_{i=0}^{\lambda}\frac{a_{i,\lambda-1}}{m-1} = 0.$$

Then substituting in the equations (50) $g_m = 0$, we see that the condition

$$\sum_{i=0}^{m-1} b_s \sum_{s=\mu}^{m+\lambda-s}\frac{a_{i,\,m_j+\lambda-s-i}}{m_j - 1} = 0$$

must be satisfied. If this condition is satisfied for one value of b_s it is satisfied for all, and if it is not satisfied for any chosen value it is satisfied for none, since the succeeding coefficients up to b_{m_j} are uniquely determined from b_l and linearly dependent on it. Kernels can be chosen so that this condition shall be satisfied

and equally well so that the condition shall fail to be satisfied. Of course if $\mu > m_k$ there can be no analytic solution of the homogeneous equation, other than $u \equiv 0$, since $b_0 \cdots b_{\mu-1} = 0$.

If there is a particular analytic solution of the non-homogeneous equation, the general analytic solution will be the sum of this particular solution and the solutions developed above for the homogeneous equation. A necessary and sufficient condition that there should be a particular solution is that b's can be found satisfying (50). The equations (50) include among them the equations

$$g_{m_j} + \sum_{i=0}^{m_j-1} b_s \left(\sum_{s=\mu}^{m_j+\lambda-s} \frac{a_{i,\, m_j+\lambda-s-i}}{m_j - i} \right) = 0 \qquad (j = 1, 2, \cdots, k),$$

which can be regarded as equations restricting certain coefficients in the development of $\phi(x)$. It is sufficient for a particular solution if, in addition to the general restriction $g_0, g_1, \cdots, g_\lambda = 0$, also $g_0, g_1, \cdots, g_{m_k} = 0$.

Then $b_0, b_1, \cdots, b_\mu, \cdots, b_{m_k}$ may all be chosen zero, and a particular solution thereby uniquely determined.

28. *Case where $\mu - \lambda > 1$; Asymptotic Developments.*

As we have already noticed, the series are in general divergent when $\mu - \lambda > 1$. In the cases that fall under the treatment of the present paper, that is, in all cases where $\lambda = 0$, these divergent series still have meaning.

To interpret these divergent series let us put

$$u(x) = \sum_{i=0}^{n} b_i x^i + u_n(x),$$

where the b's have the meaning already assigned to them, and where $u_n(x)$ is to be determined. Substituting in the equation

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{\xi^\mu} u(\xi)\, d\xi, \tag{48}$$

we get

$$\sum_{i=0}^{n} b_i x^i + u_n(x) = \sum_{i=0}^{\infty} g_i(x) + \int_a^x \frac{K(x, \xi)}{\xi^\mu} \sum_{i=0}^{n} b_i \xi^i\, d\xi + \int_a^x \frac{K(x, \xi)}{\xi^\mu} u_n(\xi)\, d\xi,$$

or, to determine $u_n(x)$,

$$u_n(x) = \phi_n(x) + \int_a^x \frac{K(x, \xi)}{\xi^\mu} u_n(\xi)\, d\xi, \tag{52}$$

where

$$\phi_n(x) = -\sum_{i=0}^{n} b_i x^i + \sum_{i=0}^{\infty} g_i x^i + \int_a^x \frac{K(x, \xi)}{\xi^\mu} \sum_{i=0}^{n} b_i \xi^i\, d\xi.$$

But from (48′) and (48″) we know, putting $\lambda = 0$, that the function

$$\psi_n(x) = -\sum_{i=0}^{n-\mu+1} b_i x^i + \sum_{i=0}^{n-\mu+1} g_i x^i + \sum_{r=\mu}^{n} \sum_{p=0}^{r-\mu} b_{r-p} \sum_{i=0}^{p} \frac{a_{i,\, p-i}}{r - \mu - i + 1} x^{r-\mu+1} \cdots$$

is identically zero. But the function

$$\phi_n(x) - \psi_n(x) = \sum_{i=n-\mu+2}^{\infty} g_i x^i + \sum_{r=n+1}^{\infty} \sum_{p=0}^{r-n-1} b_{r-p} \sum_{i=0}^{p} \frac{a_{i,\,p-i}}{r-\mu-i+1} x^{r-\mu+1}$$

vanishes to the order $n-\mu+2$. Hence $\phi_n(x)$ vanishes to the order $n-\mu+2$, and since we have a simple power series, $\phi_n'(x)$ vanishes to the order $n-\mu+1$. Now by the development of sections 11–15, we know that there are solutions of (52). In that development we may take $\nu = 1$; and any solution, since $|\phi_n'(x)| \leqq Nx^{n-\mu+1}$, will satisfy a condition

$$|u_n(x)| \leqq Nx^{n-\mu+1} Cx^{\mu-\nu'} \leqq CNx^{n+1-\nu'} \leqq CNx^n \qquad \text{(since } \nu' < 1\text{)}.$$

Now

$$u(x) = \sum_{i=0}^{n-1} b_i x^i + b_n x^n + u_n x = \sum_{i=0}^{n-1} b_i x^i + u_{n-1}(x).$$

Hence

$$u_{n-1}(x) = b_n x^n + u_n(x),$$

and

$$|u_{n-1}(x)| \leqq \text{const } x^n.$$

Hence

$$u(x) - \sum_{i=0}^{n} b_i x^i \leqq \text{const } x^{n+1}.$$

This is the meaning of the power series development. It is asymptotic.*

29. *Equations of the First Kind.*

The treatment of integral equations of the first kind

$$0 = \phi(x) + \int_0^x K(x, \xi) u(\xi)\, d\xi \tag{53}$$

with analytic kernels, is analogous to that of section 28. There is no gain in generality in having a denominator ξ^n in the kernel, because such a denominator may be removed by the transformation $u(\xi) = \xi^n v(\xi)$.

Equation (53) can be reduced to an equation of the form of (48) where, with some exceptions, $\mu - \lambda = 1$. Or from (48) we get at once the equations for the determination of the coefficients of the analytic expansion of the solution by putting $\mu = 0$, and omitting the 1 in the parenthesis of the left hand member. The equations then are

$$-b_m\left(\sum_{i=0}^{a_{i,\,\lambda-i}} \frac{a_{i,\,\lambda-i}}{m+\lambda+1-i}\right) = g_{m+\lambda+1} + \sum_{s=0}^{m-1} b_s \left(\sum_{i=0}^{m+\lambda-s} \frac{a_{i,\,m+\lambda-s+i}}{m+\lambda+1-i}\right). \tag{54}$$

* Borel, *Leçons sur les séries divergentes*, page 35.

Consider now the equation

$$\sum_{i=0}^{\lambda}\left(\frac{a_{i,\,\lambda-i}}{m+\lambda+1-i}\right)=0, \tag{55}$$

which cannot have more than λ integral roots. If the left hand member does not vanish for any integral value of m, and if its product by m does not vanish as m becomes infinite, the power series defined by (54) is convergent, as may be easily shown, and a necessary and sufficient condition that it be a solution of the equation (53) is that

$$g_0, g_1, \cdots, g_\lambda = 0.$$

If there are integral values of m that satisfy (55) the considerations that enter are those that we have treated on pages 467–8. The condition that

$$\lim_{m=\infty} m\sum_{i=0}^{\lambda}\frac{a_{i,\,\lambda-i}}{m+\lambda+1-i}\neq 0$$

is equivalent to the condition

$$\sum_{i=0}^{\lambda} a_{i,\,\lambda-1}\neq 0.* \tag{56}$$

30. *A Problem in Hydrostatics.*

Let us now return to the example from hydrostatics given in the earlier paper. The equation to be solved is

$$v(x)=\int_0^x \frac{G_1(x-\xi)-g'(x)}{g(x)}v(\xi)\,d\xi,$$

where $g(x)$ vanishes at least to the first order when $x=0$. For convenience let us consider $g(x)$ as an analytic function, and let us suppose $\alpha_1 - g'(0)\neq 0$, where

$$G_1(x-\xi)=\alpha_1+2\alpha_2(x-\xi)+3\alpha_3(x-\xi)^2+\cdots.$$

By the transformation

$$w(x)=g(x)v(x),$$

the above equation is reduced to

$$w(x)=\int_0^x \frac{G_1(x-\xi)-g'(x)}{g(\xi)}w(\xi)\,d\xi.$$

Since we are considering only continuous solutions $v(x)$, i. e., only tubes lying on curves which do not become horizontal at the bottom, we must have $w(x)/g(x)$ continuous. Hence both $w(x)$ and $w'(x)$ are continuous, and the

*See first paper, page 395–396 ; $a_{i,\,\lambda-i}=A_i$ in the condition of T. Lalesco.

only possible solutions will be given by those of the theorem of section 9. Hence if in the neighborhood of $x=0$, $g(x)$ has the opposite sign from $\alpha_1 - g'(0)$ there will be no solutions; and if $g(x)$ has the same sign as $\alpha_1 - g'(0)$ there will be an infinite number of solutions.

A problem leading to an equation somewhat similar to this is the question of filling the tube with a second liquid, whose density at a point $h(\xi)$ is a function of its height *from the bottom*, so that the tube shall have the same weight filled with the second liquid as it has filled with the first, to the same height. The equation then is

$$\int_A^x h(\xi)\,ds = \int_A^x \nu(x-\xi)\,ds,$$

or

$$\int_0^x h(\xi)u'(\xi)\,d\xi = \int_0^x \nu(x-\xi)u'(\xi)\,d\xi,$$

which yields by differentiation

$$h(x) = \frac{1}{u'(x)}\frac{d}{dx}\int_0^x \nu(x-\xi)u'(\xi)\,d\xi,$$

or

$$h(x) = \alpha + \int_0^x \frac{G_1(x-\xi)u'(\xi)}{u'(x)}\,d\xi.$$

If being given $h(x)$ we try to find the curve, the equation is

$$0 = \int_0^x \{\nu(x-\xi) - h(\xi)\}v(\xi)\,d\xi,$$

where $v(\xi) = u'(\xi)$; whence, differentiating, we find

$$[\alpha - h(x)]\,v(x) = \int_0^x G_1(x-\xi)v(\xi)\,d\xi,$$

or

$$v(x) = \int_0^x \frac{G_1(x-\xi)v(\xi)}{\alpha - h(x)}\,d\xi.$$

If $\alpha - h(x)$ is continuous, with continuous first derivative, there are no solutions of this equation continuous except at a finite number of points, except those for which $v(x)[\alpha - h(x)]$ remains finite. If $h(\alpha) \neq \alpha$, $v(x) = 0$ is the only solution possible. We assume that $\alpha_1 \neq 0$.

The equation belongs to the special class where the numerator of the kernel is the sum of a constant and a function which vanishes identically when $\xi = x$. Hence the only possible solutions are those given in section 9.

If in the neighborhood of $x = 0$, $\alpha - h(x)$ has the opposite sign from α_1 there will be no solutions except $v(x) \equiv 0$. Hence, if the density of the first

liquid increases as we descend from the surface, the density of the second must decrease as we go up from the bottom.

If $\alpha - h(x)$ has the same sign as α_1 there will be an infinite number of solutions such that

$$u(x)\,[\alpha - h(x)]$$

remains finite as x approaches 0 (§ 9). In fact, any solution is less in absolute value than const.$/(x-a)^{\nu}$, where

$$1 > \nu > 1 - \frac{1}{|h'(x)|}$$

(§ 14). And if $|h'(x)| < 1$ all the solutions are finite, i. e., the curves start up from the bottom with a positive slope. If $h(x)$ is an analytic function, the power series development given in section 28 will have all its coefficients zero, since $\alpha_1 \neq 0$.

Of course, if $v(x)$ vanishes when $x = a$, the solution will represent an imaginary curve; for with real curves $|ds| \geqq |dx|$, i. e.,

$$\left|\frac{ds}{dx}\right| \geqq 1.$$

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ONE-PARAMETER FAMILIES AND NETS OF PLANE CURVES*

BY

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Introduction.

In most treatments of the theory of one-parameter families of plane curves attention is concentrated altogether upon the discussion of the envelope. More recently, some authors, especially LILIENTHAL† and SCHEFFERS,‡ have considered some other problems associated with such families of curves, but all of these are of a metric nature. In the present paper, the point of view is that of projective differential geometry and the analytic treatment is based upon the invariant theory of a certain completely integrable system of partial differential equations of the second order. Incidentally a simple new geometrical interpretation is found for the well-known Laplacian transformation of a linear partial differential equation of the second order, which enables one to make use of the results of that theory for the purpose of studying nets of plane curves. The osculating conics of the curves of the net are also determined and their properties investigated for the first time.

§ 1. *The differential equations of the problem. Integrability conditions.*

Let the homogeneous coördinates $y^{(k)}$ of a point P_y of the plane be given as analytic functions of two independent variables;

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3). \tag{1}$$

Clearly the equations $u = \text{const.}$ and $v = \text{const.}$ will, in general, give rise to two one-parameter families of plane curves, such that, if we consider merely the portions of these curves in a certain domain, one and only one curve of each family will pass through every point of the domain. We shall speak of these two families together as a *net*, so that equations (1) define a net. This net will degenerate if and only if the ratios $y^{(1)} : y^{(2)} : y^{(3)}$ can be expressed as functions of a single independent variable, i. e., if and only if there exist three non-vanishing

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† LILIENTHAL, *Grundlage einer Krümmungslehre der Kurvenscharen*, Leipzig, 1896; *Vorlesungen über Differentialgeometrie*, Leipzig, 1908.

‡ SCHEFFERS, *Kgl. Sächsische Gesellschaft der Wissenschaften*, Berichte, vol. 57 (1905); Mathematische Annalen, vol. 60 (1905).

functions α, β, γ of u and v such that

$$\alpha y_u^{(k)} + \beta y_v^{(k)} + \gamma y^{(k)} = 0 \qquad (k = 1, 2, 3). \tag{2}$$

Consequently, *the net defined by equations* (1) *is degenerate if and only if the determinant*

$$D = \begin{vmatrix} y_u^{(1)}, & y_u^{(2)}, & y_u^{(3)} \\ y_v^{(1)}, & y_v^{(2)}, & y_v^{(3)} \\ y^{(1)}, & y^{(2)}, & y^{(3)} \end{vmatrix} \tag{3}$$

vanishes identically.

In particular, in order that the net may not degenerate, $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ must be linearly independent.

Let us assume that equations (1) define a non-degenerate net. Then D does not vanish, and we can always find a system of partial differential equations of the form

$$\begin{aligned} y_{uu} &= a y_u + b y_v + c y, \\ y_{uv} &= a' y_u + b' y_v + c' y, \\ y_{vv} &= a'' y_u + b'' y_v + c'' y, \end{aligned} \tag{4}$$

satisfied by the three functions $y^{(1)}$, $y^{(2)}$, $y^{(3)}$.

It is easy to show that, conversely, a system of form (4) with analytic coefficients has just three linearly independent analytic solutions $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, for which the determinant D does not vanish identically, provided that the integrability conditions

$$\frac{\partial y_{uu}}{\partial v} = \frac{\partial y_{uv}}{\partial u}, \qquad \frac{\partial y_{uv}}{\partial u} = \frac{\partial y_{vv}}{\partial u}$$

are satisfied *identically*, i. e., without the addition of one or more equations of the form (2) to system (4). The identical validity of these integrability conditions implies the following six equations:

$$\begin{aligned} b a'' + a_v &= a' b' + c' + a'_u, \\ a b' + b b'' + c + b_v &= a' b + b^2 + b'_u, \\ a c' + b c'' + c_v &= a' c + b' c' + c'_u, \\ a'^2 + b' a'' + a'_v &= a a'' + a' b'' + c'' + a''_u, \\ a' b' + c' + b'_v &= a'' b + b''_u, \\ a' c' + b' c'' + c'_v &= a'' c + b'' c' + c''_u. \end{aligned} \tag{5}$$

Now every non-degenerate net of plane curves determines a unique system of form (4), for which the conditions (5) must of course be satisfied. On the other hand every system of form (4), for which the integrability conditions (5) are

satisfied, has three linearly independent solutions $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, for which the determinant D does not vanish identically, and the most general solutions of such a system will be

$$\bar{y}^{(k)} = c_{k1}y^{(1)} + c_{k2}y^{(2)} + c_{k3}y^{(3)} \quad (k = 1, 2, 3), \qquad |c_{kl}| \neq 0.$$

Consequently, if we interpret $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ as the homogeneous coördinates of a point in a plane, any fundamental system of solutions of (4) defines a non-degenerate net of plane curves, and the net defined by the most general fundamental system of solutions of (4) will be a projective transformation of this.

We see, therefore, that *all projective differential properties of a net must be expressible in terms of the coefficients and variables of the system of partial differential equations* (4).

It is possible to simplify the integrability conditions (5) somewhat. From the first and fifth of these we easily deduce

$$a_v + b'_v = a'_u + b''_u,$$

so that we may write

$$a + b' = f_u, \qquad a' + b'' = f_v, \tag{6}$$

as a consequence of which (5) may be written as follows:

$$\begin{aligned} f_{uv} - b'_v - a'_u &= c' + a'b' - a''b, \\ b_v - b'_u &= -b'f_u - bf_v - c + 2b'^2 + 2a'b, \\ c_v - c'_u &= -c'f_u + 2b'c' + a'c - bc'', \\ a'_v - a''_u &= a''f_u + a'f_v + c'' - 2a'^2 - 2a''b', \\ c'_v - c''_u &= c'f_v - 2a'c' + a''c - b'c''. \end{aligned} \tag{7}$$

§ 2. *The seminvariants and the canonical form of the system.*

The analytic representation of a net by a system of equations such as (1) is not unique. The coefficients of system (4) therefore do not depend merely upon the geometric properties of the net, but also upon the particular form of analytic representation adopted. In the first place, since $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ are homogeneous coördinates, only those functions can have a geometrical significance which depend only upon the ratios of these quantities. Now, a transformation of the form

$$y^{(k)} = \lambda(u, v)y^{(k)} \qquad (k = 1, 2, 3)$$

leaves these ratios unaltered whatever value the arbitrary function $\lambda(u, v)$ may have. Consequently it will be necessary to find *those combinations of the coefficients of* (4) *and of their derivatives, which are left unchanged by any transformation of the form*

$$y = \lambda(u, v)\bar{y}. \tag{8}$$

We shall call them seminvariants. The corresponding invariant functions which depend also upon y, y_u, y_v, shall be called *semicovariants.*

We proceed to carry out the transformation (8). We find immediately

$$
\begin{aligned}
&y_u = \lambda \bar{y}_u + \lambda_u \bar{y}, \qquad y_v = \lambda \bar{y}_v + \lambda_v \bar{y},\\
&y_{uu} = \lambda \bar{y}_{uu} + 2\lambda_u \bar{y}_u + \lambda_{uu} \bar{y},\\
&y_{uv} = \lambda \bar{y}_{uv} + \lambda_v \bar{y}_u + \lambda_u \bar{y}_v + \lambda_{uv} \bar{y},\\
&y_{vv} = \lambda \bar{y}_{vv} + 2\lambda_v \bar{y}_v + \lambda_{vv} \bar{y}.
\end{aligned} \tag{9}
$$

If these values be substituted in system (4), and the new coefficients be denoted by $\bar{a}$, $\bar{b}$, etc., we shall find:

$$
\begin{aligned}
&\bar{a} = a - 2\frac{\lambda_u}{\lambda}, \qquad \bar{b} = b, \qquad \bar{c} = c + a\frac{\lambda_u}{\lambda} + b\frac{\lambda_v}{\lambda} - \frac{\lambda_{uu}}{\lambda},\\
&\bar{a}' = a' - \frac{\lambda_v}{\lambda}, \qquad \bar{b}' = b' - \frac{\lambda_u}{\lambda}, \qquad \bar{c}' = c' + a'\frac{\lambda_u}{\lambda} + b'\frac{\lambda_v}{\lambda} - \frac{\lambda_{uv}}{\lambda},\\
&\bar{a}'' = a'', \qquad \bar{b}'' = b'' - 2\frac{\lambda_v}{\lambda}, \qquad \bar{c}'' = c'' + a''\frac{\lambda_u}{\lambda} + b''\frac{\lambda_v}{\lambda} - \frac{\lambda_{vv}}{\lambda}.
\end{aligned} \tag{10}
$$

According to (6) we have

$$a + b' = f_u, \qquad a' + b'' = f_v.$$

If we write similarly

$$\bar{a} + \bar{b}' = \bar{f}_u, \qquad \bar{a}' + \bar{b}'' = \bar{f}_v,$$

we shall find

$$\bar{f}_u = f_u - 3\frac{\lambda_u}{\lambda}, \qquad \bar{f}_v = f_v - 3\frac{\lambda_v}{\lambda}.$$

We may, therefore, transform (4) *into another system of the same form for which*

$$\bar{a} + \bar{b}' = 0, \qquad \bar{a}' + \bar{b}'' = 0, \tag{11}$$

if we choose $\lambda(u, v)$ *in the transformation* (8) *subject to the conditions*

$$\frac{\lambda_u}{\lambda} = \tfrac{1}{3} f_u, \qquad \frac{\lambda_v}{\lambda} = \tfrac{1}{3} f_v, \qquad \lambda = \text{const. } e^{\frac{1}{3}f}. \tag{12}$$

Let us actually make this particular transformation, and let us denote the coefficients of the resulting system by capital letters. We shall have

$$
\begin{aligned}
&A = a - \tfrac{2}{3} f_u, \qquad B = b, \qquad C = c + \tfrac{1}{3} a f_u + \tfrac{1}{3} b f_v - \tfrac{1}{3} f_{uu} - \tfrac{1}{9} f_u^2,\\
&A' = a' - \tfrac{1}{3} f_v, \qquad B' = b' - \tfrac{1}{3} f_u, \qquad C' = c' + \tfrac{1}{3} a' f_u + \tfrac{1}{3} b' f_v - \tfrac{1}{3} f_{uv} - \tfrac{1}{9} f_u f_v,\\
&A'' = a'', \qquad B'' = b'' - \tfrac{2}{3} f_v, \qquad C'' = c'' + \tfrac{1}{3} a'' f_u + \tfrac{1}{3} b'' f_v - \tfrac{1}{3} f_{vv} - \tfrac{1}{9} f_v^2,\\
&\qquad\qquad A + B' = 0, \qquad A' + B'' = 0.
\end{aligned} \tag{13}
$$

An integrable system of form (4) *can always be transformed into another*

one whose uniquely determined coefficients are given by equations (13), *and satisfy the characteristic relations*

$$A + B' = 0, \qquad A' + B'' = 0.$$

The most general transformation which effects this reduction is

$$y = \text{const. } \bar{y} e^{\frac{1}{3}\int}.$$

We shall say that, as a result of this transformation, the system has been reduced to its *canonical form.*

The coefficients $A, B, C, \cdots$ *of a system in its canonical form are seminvariants of system* (4). This follows at once from the uniqueness of the canonical form, and may also be verified for each of the quantities $A, B, C, \cdots$ by direct calculation. Now let $f(a, b, \cdots; a_u, b_u, \cdots)$ be any seminvariant depending not merely upon the coefficients of (4) but also upon their derivatives to any order. Since a seminvariant does not change its value if $a, b, c, \cdots$ in its expression be replaced by $\bar{a}, \bar{b}, \bar{c}, \cdots$, the coefficients of any system obtained from (4) by a transformation of the form $y = \lambda\bar{y}$, we shall have in particular

$$f(a, b, c, \cdots; a_u, b_u, \cdots) = f(A, B, C, \cdots; A_u, B_u, \cdots);$$

i. e., it is a function of $A, B, C, \cdots$ and of their derivatives. Conversely, it is evident that any function of $A, B, C, \cdots$ and of their derivatives is a seminvariant.

Consequently, *every seminvariant of system* (4) *is a function of the seven fundamental ones,* $B, C, A', B', C', A'', C''$, *and of the various derivatives of these quantities. And every function of these variables is a seminvariant.*

Since, by means of the equations (4), any function of y, y_u, y_v, and the higher derivatives of y may be reduced to one of y, y_u and y_v alone, we may confine our search for semi-covariants to those depending upon these three variables only. Of course y itself is a relative semi-covariant. It is easy to verify that

$$\rho = y_u - b'y, \qquad \sigma = y_v - a'y$$

are also semi-covariants. *Any relative semi-covariant may be expressed as a homogeneous function of these three and of seminvariants.**

* It is easy to show that an absolute semi-covariant must be a homogeneous function of degree zero of y, y_u and y_v. Any such function C may be written as a function of the two ratios ρ/y, σ/y and of the coefficients of (4);

$$C = f\left(\frac{\rho}{y}, \frac{\sigma}{y}, a, b, c, \cdots\right).$$

But, as in the argument for seminvariants, this reduces to

$$C = f\left(\frac{\rho}{y}, \frac{\sigma}{y}, A, B, C, \cdots\right),$$

whence follows easily the theorem as stated.

The integrability conditions of system (4) must, of course, be left invariant by the transformation $y = \lambda \bar{y}$; i. e., they must be capable of expression in terms of seminvariants. We obtain them by writing the conditions that the system in its canonical form must be integrable. *This gives the integrability conditions of* (4) *in the form:*

$$
\begin{aligned}
&B'_v + A'_u + C' + A'B' - A''B = 0,\\
&B_v - B'_u = -C + 2B^2 + 2A'B,\\
&C_v - C'_u = 2B'C' + A'C - BC'',\\
&A'_v - A''_u = C'' - 2A^2 - 2A''B',\\
&C'_v - C''_u = -2A'C' + A''C - B'C''.
\end{aligned}
\tag{14}
$$

§ 3. *The invariants.*

We have not yet found an adequate analytic equivalent for the geometric properties of the net. To be sure, the seminvariants depend merely upon the ratios of $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, and not upon their absolute values; but they also depend upon the parametric representation which has been adopted for the net. Clearly the net remains unchanged by any transformation of the form

$$
\bar{u} = U(u), \qquad \bar{v} = V(v), \tag{15}
$$

where U and V are arbitrary functions of the single variables indicated, while any more general transformation of these variables would change the net. We must therefore find those combinations of the seminvariants and semi-covariants which remain unaltered as a result of a transformation of form (15). We shall call them *invariants* and *covariants* respectively. These functions are *intrinsically* connected with the projective differential properties of the net. Strictly speaking we ought to adjoin the further transformation

$$
\bar{u} = v, \qquad \bar{v} = u,
$$

which merely interchanges the two one-parameter families of the net, but we shall prefer to think of these two families separately, so that formally we are constructing a theory of the two component one-parameter families rather than a theory of the net itself. From our invariants it is very easy to construct those invariants which may in all strictness be called invariants of the net. They are, of course, symmetric combinations of the invariants considered here.

Let us transform system (4) by equations (15). We find

$$
\frac{\partial y}{\partial u} = \frac{\partial y}{\partial \bar{u}} U', \qquad \frac{\partial y}{\partial v} = \frac{\partial y}{\partial \bar{v}} V',
$$

$$
\begin{aligned}
\frac{\partial^2 y}{\partial u^2} &= \frac{\partial^2 y}{\partial \bar{u}^2} U'^2 + \frac{\partial y}{\partial \bar{u}} U'', \\
\frac{\partial^2 y}{\partial u \partial v} &= \frac{\partial^2 y}{\partial \bar{u} \partial \bar{v}} U' V', \\
\frac{\partial^2 y}{\partial v^2} &= \frac{\partial^2 y}{\partial \bar{v}^2} V'^2 + \frac{\partial y}{\partial \bar{v}} V'',
\end{aligned} \tag{16}
$$

where the accents indicate differentiation in the customary way. If we substitute these values into system (4) and denote the coefficients of the new system by $\bar{a}$, $\bar{b}$, etc., we shall find:

$$
\begin{aligned}
&\bar{a} = \frac{1}{U'}(a - \eta), \qquad \bar{b} = \frac{V' b}{U'^2}, \qquad \bar{c} = \frac{c}{U'^2}, \\
&\bar{a}' = \frac{a'}{V'}, \qquad \bar{b}' = \frac{b'}{U'}, \qquad \bar{c}' = \frac{c'}{U' V'}, \\
&\bar{a}'' = \frac{U' a''}{V'^2} \qquad \bar{b}'' = \frac{1}{V'}(b'' - \zeta), \qquad \bar{c}'' = \frac{c''}{V'^2},
\end{aligned} \tag{17}
$$

where

$$
\eta = \frac{U''}{U'}, \qquad \zeta = \frac{V''}{V'}, \tag{18}
$$

so that η is a function of u, and ζ of v only. We find further

$$
\begin{aligned}
\bar{f}_u &= \frac{1}{U'}(f_u - \eta), \qquad \bar{f}_v = \frac{1}{V'}(f_v - \zeta), \\
\bar{f}_{uu} &= \frac{1}{(U')^2}(f_{uu} - \eta f_u + \eta^2 - \eta_u), \\
\bar{f}_{uv} &= \frac{1}{U' V'} f_{uv}, \\
\bar{f}_{vv} &= \frac{1}{(V')^2}(f_{vv} - \zeta f_v + \zeta^2 - \zeta_v),
\end{aligned} \tag{19}
$$

where the notations $\bar{f}_u, \bar{f}_{uu}$, etc., are abbreviations for $\partial \bar{f}/\partial \bar{u}$, $\partial^2 \bar{f}/\partial \bar{u}^2$, etc.

If now we make use of the expressions (13) of the fundamental seminvariants, and denote by $\bar{A}$, $\bar{B}$, etc., the corresponding seminvariants of the transformed system of differential equations, we shall find

$$
\bar{B} = \frac{V'}{(U')^2} B, \qquad \bar{C} = \frac{1}{(U')^2}(C - \tfrac{1}{3} A\eta - \tfrac{1}{3} B\zeta - \tfrac{1}{9}\eta^2 + \tfrac{1}{3}\eta_u),
$$

$$\bar{A}' = \frac{1}{V'}(A' + \tfrac{1}{3}\zeta), \qquad \bar{B}' = \frac{1}{U'}(B' + \tfrac{1}{3}\eta),$$

(20)

$$\bar{C}' = \frac{1}{U'V'}(C' - \tfrac{1}{3}A'\eta - \tfrac{1}{3}B'\zeta - \tfrac{1}{9}\eta\zeta),$$

$$\bar{A}'' = \frac{U'}{(V')^2}A'', \qquad \bar{C}'' = \frac{1}{(V')^2}(C'' - \tfrac{1}{3}A''\eta - \tfrac{1}{3}B''\zeta - \tfrac{1}{9}\zeta^2 + \tfrac{1}{3}\zeta_v).$$

We see that *the seminvariants B and A'' are at the same time (relative) invariants, and it is easy to verify the invariance of all of the following quantities:*

$$\mathfrak{B} = B, \qquad \mathfrak{C} = C - B'_u - 2B'^2 + A'B,$$

(21)

$$\mathfrak{A}' = A' + \frac{1}{6}\frac{A''_v}{A''}, \qquad \mathfrak{B}' = B' + \frac{1}{6}\frac{B_u}{B}, \qquad \mathfrak{C}' = C' + A'B',$$

$$\mathfrak{A}'' = A'', \qquad \mathfrak{C}'' = C'' - A'_v - 2A'^2 + A''B'.$$

In fact we have

$$\bar{\mathfrak{B}} = \frac{V'}{(U')^2}\mathfrak{B}, \qquad \bar{\mathfrak{C}} = \frac{1}{(U')^2}\mathfrak{C},$$

(22)

$$\bar{\mathfrak{A}}' = \frac{1}{V'}\mathfrak{A}', \qquad \bar{\mathfrak{B}}' = \frac{1}{U'}\mathfrak{B}', \qquad \bar{\mathfrak{C}}' = \frac{1}{U'V'}\mathfrak{C}',$$

$$\bar{\mathfrak{A}}'' = \frac{U'}{(V')^2}\mathfrak{A}'', \qquad \bar{\mathfrak{C}}'' = \frac{1}{(V')^2}\mathfrak{C}''.$$

Whenever a relative invariant $\theta^{(m,\,n)}$ satisfies the equation

$$\bar{\theta}^{(m,\,n)} = (U')^m (V')^n \theta^{(m,\,n)} \tag{23}$$

we shall say that its *weights* with respect to u and v are m and n respectively. These weights may be characterized by the symbol (m, n). Thus $\mathfrak{B}$ is of weight $(-2, 1)$, etc. We proceed to show how we may deduce from $\theta^{(m,\,n)}$ by differentiation two new invariants, which we shall denote by $(\theta^{(m,\,n)}, B')$ and $(\theta^{(m,\,n)}, A')$.

We find from (23)

$$\bar{\theta}^{(m,\,n)}_u = (U')^{m-1}(V')^n(\theta^{(m,\,n)}_u + m\eta\theta^{(m,\,n)}),$$

(24)

$$\bar{\theta}^{(m,\,n)}_v = (U')^m(V')^{n-1}(\theta^{(m,\,n)}_v + n\zeta\theta^{(m,\,n)}),$$

where, of course, $\bar{\theta}^{(m,\,n)}_u$ and $\bar{\theta}^{(m,\,n)}_v$ denote $\partial\bar{\theta}^{(m,\,n)}/\partial\bar{u}$ and $\partial\bar{\theta}^{(m,\,n)}/\partial\bar{v}$ respectively. Since we have from (20)

$$\bar{A}' = \frac{1}{V'}(A' + \tfrac{1}{3}\zeta), \qquad \bar{B}' = \frac{1}{U'}(B' + \tfrac{1}{3}\eta),$$

we shall find that the expressions

$$(25)\quad (\theta^{(m,n)}, B') = \theta_u^{(m,n)} - 3mB'\theta^{(m,n)}, \qquad (\theta^{(m,n)}, A') = \theta_v^{(m,n)} - 3nA'\theta^{(m,n)}$$

are two new invariants of weights $(m-1, n)$ *and* $(m, n-1)$ *respectively.* It is also very easy to verify that *if* θ *is an invariant (relative or absolute),* $\partial^2 \log \theta / \partial u \partial v$ *is also an invariant, of weight* $(-1, -1)$.

Clearly we may obtain from (22), by the repeated application of the two processes defined by equations (25), an infinite number of invariants. We wish to show that all invariants of the net will be functions of those obtained in this manner.

In the first place we recall the fact that every invariant is a function of seminvariants, i. e., of B, C, etc., and of the derivatives of these quantities. But (21) shows that B, C, $\cdots$ may be expressed as functions of $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{A}'$, $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{A}''$, $\mathfrak{C}''$, and of *their* derivatives. Consequently every invariant I must be a function of the seven fundamental invariants and of their derivatives, i. e.,

$$I = F(\mathfrak{B}, \mathfrak{C}, \cdots; \mathfrak{B}_u, \cdots; \mathfrak{B}_v, \cdots; \mathfrak{B}_{uu}, \cdots).$$

Let us introduce infinitesimal transformations by putting

$$U = u + \phi(u)\delta t, \qquad V = v + \psi(v)\delta t,$$

where δt is an infinitesimal. Then

$$(26)\quad U' = 1 + \phi'\delta t, \qquad V' = 1 + \psi'\delta t, \qquad \eta = \phi''\delta t, \qquad \zeta = \psi''\delta t$$

and

$$\delta\mathfrak{B} = (-2\phi' + \psi')\mathfrak{B}\delta t, \qquad \delta\mathfrak{C} = -2\phi'\mathfrak{C}\delta t,$$

$$(27)\quad \delta\mathfrak{A}' = -\psi'\mathfrak{A}'\delta t, \qquad \delta\mathfrak{B}' = -\phi'\mathfrak{B}'\delta t, \qquad \delta\mathfrak{C}' = -(\phi' + \psi')\mathfrak{C}'\delta t,$$

$$\delta\mathfrak{A}'' = (\phi' - 2\psi')\mathfrak{A}''\delta t, \qquad \delta\mathfrak{C}'' = -2\psi'\mathfrak{C}''\delta t.$$

In general, for an invariant $\theta^{(m,n)}$, of weight (m, n), we shall find

$$\delta\theta^{(m,n)} = (m\phi' + n\psi')\theta^{(m,n)}\delta t,$$

$$(28)\quad \delta\theta_u^{(m,n)} = [\{(m-1)\phi' + n\psi'\}\theta_u^{(m,n)} + m\theta^{(m,n)}\phi'']\delta t,$$

$$\delta\theta_v^{(m,n)} = [\{m\phi' + (n-1)\psi'\}\theta_v^{(m,n)} + n\theta^{(m,n)}\psi'']\delta t.$$

The latter formulæ enable us to calculate $\delta\mathfrak{B}_u$, $\delta\mathfrak{B}_v$, etc. The results may be tabulated as follows:

	$\mathfrak{B}_u$	$\mathfrak{B}_v$	$\mathfrak{C}_u$	$\mathfrak{C}_v$	$\mathfrak{A}'_u$	$\mathfrak{A}'_v$	$\mathfrak{B}'_u$	$\mathfrak{B}'_v$	$\mathfrak{C}'_u$	$\mathfrak{C}'_v$	$\mathfrak{A}''_u$	$\mathfrak{A}''_v$	$\mathfrak{C}''_u$	$\mathfrak{C}''_v$
ϕ'	$-3\mathfrak{B}_u$	$-2\mathfrak{B}_v$	$-3\mathfrak{C}_u$	$-2\mathfrak{C}_v$	$-\mathfrak{A}'_u$		$-2\mathfrak{B}'_u$	$-\mathfrak{B}'_v$	$-2\mathfrak{C}'_u$	$-\mathfrak{C}'_v$		$+\mathfrak{A}''_v$	$-\mathfrak{C}''_u$	
ψ'	$+\mathfrak{B}_u$			$-\mathfrak{C}_v$	$-\mathfrak{A}'_u$	$-2\mathfrak{A}'_v$		$-\mathfrak{B}'_v$	$-\mathfrak{C}'_u$	$-2\mathfrak{C}'_v$	$-2\mathfrak{A}''_u$	$-3\mathfrak{A}''_v$	$-2\mathfrak{C}''_u$	$-3\mathfrak{C}''_v$
ϕ''	$-2\mathfrak{B}$		$-2\mathfrak{C}$				$-\mathfrak{B}'$		$-\mathfrak{C}'$		$+\mathfrak{A}''$			
ψ''		$+\mathfrak{B}$				$-\mathfrak{A}'$				$-\mathfrak{C}'$		$-2\mathfrak{A}''$		$-2\mathfrak{C}''$

Let I be an invariant (absolute) which depends only upon the seven fundamental invariants and their *first* derivatives. Then δI must vanish for arbitrary values of ϕ', ψ', ϕ'', ψ''. We thus obtain a system of four partial differential equations for I, involving 21 independent variables. The matrix of the coefficients of these four differential equations is obtained by adding the seven columns corresponding to the infinitesimal transformations of $\mathfrak{B}$, $\mathfrak{C}$, $\cdots$, $\mathfrak{C}''$ to the table just calculated. The form of the matrix makes it evident at once that the four equations are independent. Moreover, according to Lie's theory they must form a complete system. Therefore, there must exist $21 - 4 = 17$ independent absolute invariants of the kind in question. Five of these may be chosen as independent of the derivatives. In fact we have

$$\overline{\mathfrak{A}''\mathfrak{B}^2} = \frac{1}{(U')^3}\mathfrak{A}''\mathfrak{B}^2, \qquad \overline{\mathfrak{A}''^2\mathfrak{B}} = \frac{1}{(V')^3}\mathfrak{A}''^2\mathfrak{B},$$

so that

$$\frac{\mathfrak{C}^3}{\overline{\mathfrak{A}''^2\mathfrak{B}^4}}, \quad \frac{\mathfrak{A}^3}{\overline{\mathfrak{A}''^2\mathfrak{B}}}, \quad \frac{\mathfrak{B}^3}{\overline{\mathfrak{A}''\mathfrak{B}^2}}, \quad \frac{\mathfrak{C}'^3}{\overline{\mathfrak{A}''^3\mathfrak{B}^3}}, \quad \frac{\mathfrak{C}''^3}{\overline{\mathfrak{A}''^4\mathfrak{B}^2}}$$

are five independent absolute invariants. From the fundamental seven relative invariants we can form fourteen others of the form $(\theta^{(m,n)}, B')$ and $(\theta^{(m,n)}, A')$, but, of course, only twelve of these can be independent. In fact the following two,

$$(\mathfrak{B}, B'), \qquad (\mathfrak{A}'', A')$$

reduce to $6\mathfrak{B}\mathfrak{B}'$ and $6\mathfrak{A}'\mathfrak{A}''$ respectively, while the other twelve are independent of each other and of the fundamental seven.

We have proved that all invariants involving only the first derivatives of $\mathfrak{B}$, $\mathfrak{C}$, etc., may be obtained from the fundamental seven by the processes (25). Let us speak of these invariants as being of the first order. Now let I be an invariant of the second order. The system of differential equations satisfied by such invariants contains two more equations than the corresponding system for invariants of the first order. The matrix of the coefficients of $\partial I/\partial\mathfrak{B}$, $\partial I/\partial\mathfrak{A}''$, $\partial I/\partial\mathfrak{B}_v$, $\partial I/\partial\mathfrak{A}''_u$, $\partial I/\partial\mathfrak{B}_{vv}$, $\partial I/\partial\mathfrak{A}''_{vv}$ may be represented by means of the matrices

$$M = \begin{Vmatrix} -2\mathfrak{B}, & +\mathfrak{A}'' \\ +\mathfrak{B}, & -2\mathfrak{A}'' \end{Vmatrix}, \qquad N = \begin{Vmatrix} 0, & +\mathfrak{A}'' \\ +\mathfrak{B}, & 0 \end{Vmatrix}$$

in the form

$$\begin{Vmatrix} M, & P, & Q \\ 0, & N, & R \\ 0, & 0, & N \end{Vmatrix}$$

where 0 denotes a square matrix of the second order with vanishing elements, and where the values of the coefficients of the second order matrices P, Q, R

are immaterial. The determinant of this matrix is

$$|M|\,|N|^2,$$

where

$$|M| = 3\mathfrak{A}''\mathfrak{B}, \qquad |N| = -\mathfrak{A}''\mathfrak{B},$$

and therefore, in general, different from zero. The two new equations of the system are therefore independent. The same proof applies to the two new equations obtained when we pass to the invariants of the third order, etc.

The system of differential equations for the invariants of the second order contains 21 more independent variables and 2 more independent equations than that for invariants of the first order. There must exist, therefore, 19 independent invariants of the second order. The application of our two processes to the twelve independent invariants of the first order gives 24 invariants of the second order. We proceed to show that just nineteen of these are independent, so that our process actually furnishes them all.

We have

$$(\mathfrak{B}, A') = \mathfrak{B}_v - 3A'\mathfrak{B}, \qquad (\mathfrak{A}'', B') = \mathfrak{A}''_u - 3B'\mathfrak{A}'',$$

where

$$[(\mathfrak{C}, A'), B'] = \mathfrak{B}_{uv} - 3A'\mathfrak{B}_u - 3A'_u\mathfrak{B} + 6B'(\mathfrak{B}, A'),$$
$$[(\mathfrak{A}'', B'), A'] = \mathfrak{A}''_{uv} - 3B'\mathfrak{A}''_v - 3B'_v\mathfrak{A}'' + 6A'(\mathfrak{A}'', B').$$

If we remember that

$$A' = \mathfrak{A}' - \frac{1}{6}\frac{\mathfrak{A}''_v}{\mathfrak{A}''}, \qquad B' = \mathfrak{B}' - \frac{1}{6}\frac{\mathfrak{B}_u}{\mathfrak{B}},$$

we see that these invariants contain the second derivatives of the fundamental invariants only in the combinations

$$\mathfrak{B}_{uv} + \frac{1}{2}\frac{\mathfrak{B}}{\mathfrak{A}''}\mathfrak{A}''_{uv} \qquad \text{and} \qquad \mathfrak{A}''_{uv} + \frac{1}{2}\frac{\mathfrak{A}''}{\mathfrak{B}}\mathfrak{B}_{uv}$$

respectively, and are, therefore, independent. The other two invariants of the second order which can be formed from $\mathfrak{A}''$ and $\mathfrak{B}$, viz.,

$$[(\mathfrak{B}, A'), A'] \qquad \text{and} \qquad [(\mathfrak{A}'', B'), B']$$

contain the further variables $\mathfrak{B}_{vv}\mathfrak{A}''_{vv}$ and $\mathfrak{B}_{uu}\mathfrak{A}''_{uu}$ respectively, so that these four are certainly independent.

Each of the remaining five pairs of invariants of the first order gives rise to only three instead of four independent invariants of the second order. We have, for example,

$$(\mathfrak{C}, B') = \mathfrak{C}_u + 6B'\mathfrak{C}, \qquad (\mathfrak{C}, A') = \mathfrak{C}_v$$

so that $[(\mathfrak{C}, B'), B']$ and $[(\mathfrak{C}, A'), A']$ are obviously independent. But

$$[(\mathfrak{C}, B'), A'] = \mathfrak{C}_{uv} + 6B'\mathfrak{C}_v + 6B'_v\mathfrak{C},$$
$$[(\mathfrak{C}, A'), B'] = \mathfrak{C}_{uv} + 6B'\mathfrak{C}_v,$$

so that their difference involves, of second order derivatives, only $\mathfrak{B}_{uv}$ which by our previous consideration, is capable of expression in terms of $[(\mathfrak{B}, A), B']$, $[(\mathfrak{A}'', B'), A']$ and invariants of lower order. Our process actually supplies us, therefore, with $2\cdot 2 + 5\cdot 3 = 19$ independent invariants of the second order.

It is clear that there must exist $7n + 5$ independent invariants of the nth order (not counting those of order lower than n). Let us assume that n of these, those obtained from $\mathfrak{B}$ by our differentiation processes, are of the form

$$\mathfrak{B}_{n-k,\,k} = \frac{\partial^n \mathfrak{B}}{\partial u^{n-k}\,\partial v^k} + \frac{\mathfrak{B}}{2\mathfrak{A}''}\frac{\partial^n \mathfrak{A}''}{\partial u^{n-k}\,\partial v^k} + \cdots \qquad (k = 1, 2, \cdots, n),$$

the terms not written being of order lower than n. Then none of these n invariants involves $\partial^n \mathfrak{B}/\partial u^n$ or $\partial^n \mathfrak{A}''/\partial u^n$. Let n other invariants be of the form

$$\mathfrak{A}''_{n-k,\,k} = \frac{\partial^n \mathfrak{A}''}{\partial u^{n-k}\,\partial v^k} + \frac{\mathfrak{A}''}{2\mathfrak{B}}\frac{\partial^n \mathfrak{B}}{\partial u^{n-k}\,\partial v^k} + \cdots \qquad (k = 0, 1, 2, \cdots, n-1),$$

so that none of these involves the variables $\partial^n \mathfrak{A}''/\partial v^n$ or $\partial^n \mathfrak{B}/\partial v^n$. Let the invariants formed from $\mathfrak{C}$, $n + 1$ in number, be

$$\mathfrak{C}_{n-k,\,k} = \frac{\partial^n \mathfrak{C}}{\partial u^{n-k}\,\partial v^k} + \cdots \qquad (k = 0, 1, 2, \cdots, n),$$

where the omitted terms may involve, of quantities of the nth order, only the derivatives of $\mathfrak{A}''$ and $\mathfrak{B}$. Let there be $n + 1$ invariants of this same character for each of the remaining fundamental invariants $\mathfrak{A}'$, $\mathfrak{B}'$, $\mathfrak{C}'$ and $\mathfrak{C}''$. All of these conditions are actually satisfied for $n = 2$.

Let us consider the invariants of order $n + 1$,

$$\begin{aligned}
&(\mathfrak{B}_{n-k,\,k}, B'), \quad k = 1, 2, \cdots, n, \qquad (\mathfrak{B}_{0,\,n}, A'),\\
&(\mathfrak{A}''_{n-k,\,k}, A'), \quad k = 0, 1, 2, \cdots, n-1, \qquad (\mathfrak{A}''_{n,\,0}, B'),\\
&(\mathfrak{C}_{n-k,\,k}, B'), \quad k = 0, 1, 2, \cdots, n, \qquad (\mathfrak{C}_{0,\,n}, A'),
\end{aligned}$$

and the corresponding ones formed from $\mathfrak{A}'$, $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{C}''$. They will be $2(n + 1) + 5(n + 2) = 7n + 12$ in number; they will obviously be independent and will have all of the properties which were assumed for the invariants of order n. Since there exists a system of invariants of the second order with these properties, there exists a system of invariants of the nth order with the same properties, as our induction proof clearly shows. Since, moreover, our process gives us precisely the right number of independent invariants of order $n + 1$, we have proved the following theorem.

Every invariant of the system of partial differential equations (4) *is a function of the fundamental seven, and of those others which can be obtained from them by repeated application of the two differentiation processes* (25).

The system of equations (14), the integrability conditions for (4), must of

course be an invariant system of equations. We find, more specifically, that each one of them is invariant by itself. It must therefore be possible to rewrite these conditions in such a form as to exhibit their invariant character. In order to do this in a convenient form we shall introduce the two further invariants

$$H = C' + A'B' - A'_u, \qquad K = C' + A'B' - B'_v, \tag{29}$$

which, of course, may be expressed in terms of our fundamental invariants, but which merit a special notation on account of their great importance, and because their expression in terms of the fundamental invariants is rather complicated. We shall first show how these latter expressions may be obtained. Since

$$A'_u = \mathfrak{A}'_u - \frac{1}{6}\frac{\partial^2 \log \mathfrak{A}''}{\partial u \partial v}, \qquad B'_v = \mathfrak{B}'_v - \frac{1}{6}\frac{\partial^2 \log \mathfrak{B}}{\partial u \partial v}$$
$$\mathfrak{A}'_u = (\mathfrak{A}', B'), \qquad \mathfrak{B}'_v = (\mathfrak{B}', A'), \tag{30}$$

we have

$$H = \mathfrak{C}' - (\mathfrak{A}', B') + \frac{1}{6}\frac{\partial^2 \log \mathfrak{A}''}{\partial u \partial v}, \qquad K = \mathfrak{C}' - (\mathfrak{B}', A') + \frac{1}{6}\frac{\partial^2 \log \mathfrak{B}}{\partial u \partial v}, \tag{31}$$

so that it remains to find the expressions for $\partial^2 \log \mathfrak{A}''/\partial u \partial v$ and $\partial^2 \log \mathfrak{B}/\partial u \partial v$. Now we have

$$(\mathfrak{B}, A') = \mathfrak{B}_v - 3A'\mathfrak{B}, \qquad [(\mathfrak{B}, A'), B'] = \mathfrak{B}_{uv} - 3A'\mathfrak{B}_u - 3A'_u\mathfrak{B} + 6B'(\mathfrak{B}, A').$$

From these equations we find

$$2\frac{\partial^2 \log \mathfrak{B}}{\partial u \partial v} + \frac{\partial^2 \log \mathfrak{A}''}{\partial u \partial v} = \frac{1}{\mathfrak{B}}\{2[(\mathfrak{B}, A'), B'] - 12\mathfrak{B}'(\mathfrak{B}, A') + 6\mathfrak{B}(\mathfrak{A}', B')\}. \tag{32}$$

In the same way we may prove the formula

$$\frac{\partial^2 \log \mathfrak{B}}{\partial u \partial v} + 2\frac{\partial^2 \log \mathfrak{A}''}{\partial u \partial v} = \frac{1}{\mathfrak{A}''}\{2[(\mathfrak{A}'', B'), A'] - 12\mathfrak{A}'(\mathfrak{A}'', B') + 6\mathfrak{A}''(\mathfrak{B}', A')\}, \tag{33}$$

and these two equations suffice to solve our problem.

The integrability conditions may now be written as follows:

$$-\mathfrak{A}''\mathfrak{B} - H - K = 0, \qquad \mathfrak{C} + (\mathfrak{B}, A') = 0,$$
$$\mathfrak{C}'' + (\mathfrak{A}'', B') = 0, \qquad (\mathfrak{C}, A') - (K, B') + \mathfrak{B}\mathfrak{C}'' = 0, \tag{34}$$
$$(\mathfrak{C}'', B') - (H, A') + \mathfrak{A}''\mathfrak{C} = 0,$$

and in this form their invariant character is obvious.

We have already observed that the seminvariants are capable of being expressed in terms of the fundamental invariants and of their derivatives. But if the seminvariants are known as functions of u and v the corresponding net of plane curves is determined except for a projective transformation. We see, therefore, that the following theorem is true.

If the seven fundamental invariants of a net are given as functions of u and

v so as to satisfy the five conditions (34), *the net is determined except for a projective transformation.*

§ 4. *The covariants of the net and its Laplacian Transformations.*

Let

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3)$$

be the equations of the net of plane curves under consideration. For given values of u and v, $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ are the coördinates of a point y* through which pass the two curves $u = \text{const.}$ and $v = \text{const.}$ of our net. The tangent t to the curve $v = \text{const.}$ will be obtained by joining y to the point y_u whose coördinates are

$$y_u^{(k)} = \frac{\partial f^{(k)}(u, v)}{\partial u} \qquad (k = 1, 2, 3),$$

and the coördinates of any point of this tangent (distinct from y) may be expressed in the form

(35) $$y_u^{(k)} + \alpha y^{(k)} \qquad (k = 1, 2, 3).$$

Now let y move along a curve $u = \text{const.}$, and in each of its positions construct the line t tangent to that one of the curves $v = \text{const.}$ which passes through it. These lines t will have an envelope, which we now proceed to determine. The expressions (35) are the coördinates of an arbitrary point Q of t. As y moves along the curve $u = \text{const.}$, v alone changes, so that the tangent of the curve described by Q will be obtained by joining Q to the point R whose coördinates are

$$\frac{\partial}{\partial v}(y_u^{(k)} + \alpha y^{(k)}) = y_{uv}^{(k)} + \alpha y_v^{(k)} + \alpha_v y^{(k)}$$

$$= a' y_u^{(k)} + (b' + \alpha) y_v^{(k)} + (c' + \alpha_v) y^{(k)} \qquad (k = 1, 2, 3).$$

Now if Q is the point where t meets its envelope, t must be tangent to the envelope at Q, i. e., the line QR must coincide with t. This will be so if and only if $\alpha = -b'$, assuming of course that the curves $u = \text{const.}$ and $v = \text{const.}$ are not tangent to each other at the point y.

We see, therefore, that *the tangents to the curves $v = \text{const.}$ constructed at the points where these curves meet a fixed curve $u = \text{const.}$ have as their envelope the locus of the point P_ρ whose coördinates are*

$$\rho^{(k)} = y_u^{(k)} - b' y^{(k)} \qquad (k = 1, 2, 3).$$

Similarly, *the tangents to the curves $u = \text{const.}$ along a fixed curve $v = \text{const.}$, have as their envelope the locus of the point P_σ whose coördinates are given by*

$$\sigma^{(k)} = y_v^{(k)} - a' y^{(k)} \qquad (k = 1, 2, 3).$$

* This is, of course, the same point which has previously been denoted by P_y.

It follows from these considerations, and may easily be verified analytically that *the semi-covariants*

$$\rho = y_u - b'y, \qquad \sigma = y_v - a'y \tag{36}$$

are also covariants of system (4). *The same thing is, of course, true of* y *itself.*

All other relative covariants are homogeneous functions of these three and of invariants. For in the first place we need consider only covariants which involve, necessarily in homogeneous fashion, y, y_u and y_v and no derivatives of y of a higher order. In fact, if a covariant did involve higher derivatives we could express them in terms of these three by means of equations (4) and those derived from them by differentiation. If we were to set up the system of partial differential equations satisfied by the absolute covariants, we should find the same system as that considered before for the absolute invariants with the additional terms arising from the presence of the two new variables, viz., the two ratios of the quantities y, y_u and y_v. This system will therefore have two more independent solutions than that for the invariants, and these two may be chosen to be ρ/y and σ/y since they are clearly independent. All absolute covariants, then, are functions of ρ/y, σ/y and of absolute invariants, whence follows at once the theorem as stated.

The covariants ρ and σ, as we have seen, establish a correspondence between the point P_y and two other points P_ρ and P_σ. As u and v vary, each of these two points P_ρ and P_σ, in general, describes a new net of plane curves which shall be called Laplacian transforms of the net determined by P_y. More specifically we shall call the net determined by P_σ the first, and that determined by P_ρ the minus first Laplacian transform of the original net, for the reason that the first Laplacian transform of the net determined by P_ρ is the original net. We proceed to set up the partial differential equations of the net determined by P_σ.

We may assume that the system (4) has been reduced to its canonical form

$$\begin{aligned} y_{uu} &= -B'y_u + By_v + Cy, \\ y_{uv} &= A'y_u + B'y_v + C'y, \\ y_{vv} &= A''y_u - A'y_v + C''y. \end{aligned} \tag{37}$$

We shall find from (36), by differentiation and making use of (37),

$$\sigma_u = B'y_v + (C' - A'_u)y, \qquad \sigma_v = A''y_u - 2A'y_v + (C'' - A'_v)y, \tag{38}$$

and further

$$\begin{aligned} \sigma_{uu} &= (C' - A'_u + A'B')y_u + (B'_u + B'^2)y_v + (C'_u - A'_{uu} + B'C')y, \\ \sigma_{uv} &= A''B'y_u + (C' - A'_u + B'_v - A'B')y_v + (C'_v - A'_{uv} + B'C'')y, \\ \sigma_{vv} &= (A''_v - A'A'')y_u + (C'' - 3A'_v + A''B' - 2A'B'')y_v \\ &\qquad + (C''_v - A'_{vv} + A''C' - 2A'C'')y. \end{aligned} \tag{39}$$

Now from (36) and (38) we find

$$(40)\qquad \begin{aligned} y_v &= A'y + \sigma, \\ \sigma_u &= Hy + B'\sigma, \qquad \sigma_v = (C'' - A'_v - 2A'^2)y + A''y_u - 2A'\sigma, \end{aligned}$$

whence

$$(41)\qquad \begin{aligned} A''Hy &= A''\sigma_u - A''B'\sigma, \\ A''Hy_u &= -(C'' - A'_v - 2A'^2)\sigma_u + H\sigma_v + [2HA' + B'(C'' - A'_v - 2A'^2)]\sigma, \\ A''Hy_v &= A'A''\sigma_u + A''(H - A'B')\sigma. \end{aligned}$$

We shall therefore obtain for σ a system of partial differential equations of the form

$$(42)\qquad \begin{aligned} \sigma_{uu} &= a_1\sigma_u + b_1\sigma_v + c_1\sigma, \\ \sigma_{uv} &= a'_1\sigma_u + b'_1\sigma_v + c'_1\sigma, \\ \sigma_{vv} &= a''_1\sigma_u + b''_1\sigma_v + c''_1\sigma, \end{aligned}$$

whose coefficients may be expressed as follows:

$$(43)\qquad \begin{aligned} A''Ha_1 &= A''_uH + A''H_u - A''HB', \qquad A''Hb_1 = H^2, \\ A''Hc_1 &= -A''B'H_u + 2A'H^2 + H(A''B'_u - B'A''_u + 2A''B'^2), \\ A''Ha'_1 &= A''(A'H + H_v), \qquad A''Hb'_1 = A''B'H, \\ A''Hc'_1 &= -A''B'H_v + A''H^2 + A''H(B'_v - A'B'), \\ A''Ha''_1 &= A''^2K + A''\mathfrak{C}''_v - \mathfrak{C}''A''_v, \qquad A''Hb''_1 = (A''_v - A'A'')H, \\ A''Hc''_1 &= A''H\mathfrak{C} - B'(A''\mathfrak{C}''_v - \mathfrak{C}''A''_v + A''^2K) + 2H(A'A''_v - A''A'_v + A'^2A''). \end{aligned}$$

The system (42) is not in its canonical form. In order to reduce it to the canonical form, we put

$$\sigma = \lambda y_1,$$

where λ is subject to the conditions

$$\frac{\lambda_u}{\lambda} = \tfrac{1}{3}(a_1 + b'_1) = \frac{1}{3}\left(\frac{A''_u}{A''} + \frac{H_u}{H}\right), \qquad \frac{\lambda_v}{\lambda} = \tfrac{1}{3}(a'_1 + b''_1) = \frac{1}{3}\left(\frac{A''_v}{A''} + \frac{H_v}{H}\right),$$

so that

$$\lambda = \text{const.}\ \sqrt[3]{A''H}.$$

We shall omit the details of the calculation. *A system of form* (37) *is transformed into another system likewise in its canonical form by the transformation*

$$(44)\qquad y_1 = \frac{y_v - A'y}{\sqrt[3]{A''H}}.$$

Its coefficients, the seminvariants of the first Laplacian transformed net, are as

follows:

$$A_1 = -B' + \frac{1}{3}\left(\frac{A''_u}{A''} + \frac{H_u}{H}\right), \qquad B_1 = \frac{H}{A''},$$

$$C_1 = B'_u + 2B'^2 + 2\frac{A'H}{A''} - \tfrac{4}{3}B'\left(\frac{A''_u}{A''} + \frac{H_u}{H}\right) + \frac{1}{3}\frac{H}{A''}\left(\frac{A''_v}{A''} + \frac{H_v}{H}\right)$$
$$+ \frac{5}{9}\left(\frac{A''_u}{A''} + \frac{H_u}{H}\right)^2 - \frac{2}{3}\frac{A''_u H_u}{A''H} - \frac{1}{3}\left(\frac{A''_{uu}}{A''} + \frac{H_{uu}}{H}\right),$$

$$A'_1 = A' + \frac{H_v}{H} - \frac{1}{3}\left(\frac{A''_v}{A''} + \frac{H_v}{H}\right), \qquad B'_1 = B' - \frac{1}{3}\left(\frac{A''_u}{A''} + \frac{H_u}{H}\right),$$

$$C'_1 = C' + B'_v - A'_u - B'\frac{H_v}{H} + \frac{1}{3}\left(A' + \frac{H_v}{H}\right)\left(\frac{A''_u}{A''} + \frac{H_u}{H}\right) + \tfrac{1}{3}B'\left(\frac{A''_v}{A''} + \frac{H_v}{H}\right)$$
(45)
$$- \frac{1}{9}\left(\frac{A''_u}{A''} + \frac{H_u}{H}\right)\left(\frac{A''_v}{A''} + \frac{H_v}{H}\right) - \frac{1}{3}\left(\frac{A''_{uv}}{A''} + \frac{H_{uv}}{H} - \frac{A''_u A''_v}{A''^2} - \frac{H_u H_v}{H^2}\right),$$

$$A''_1 = \frac{A''K}{H} + \frac{1}{H}\left(\mathfrak{C}''_v - \mathfrak{C}''\frac{A''_v}{A''}\right), \qquad B''_1 = -A' + \frac{A''_v}{A''} - \frac{2}{3}\left(\frac{A''_v}{A''} + \frac{H_v}{H}\right),$$

$$C''_1 = \mathfrak{C}'' - \frac{B'}{H}\left(\mathfrak{C}''_v - \mathfrak{C}''\frac{A''_v}{A''} + KA''\right) + 2\left(A'\frac{A''_v}{A''} - A'_v + A'^2\right)$$
$$+ \frac{1}{3H}\left(\mathfrak{C}''_v - \mathfrak{C}''\frac{A''_v}{A''} + KA''\right)\left(\frac{A''_u}{A''} + \frac{H_u}{H}\right) + \frac{1}{3}\left(-A' + \frac{A''_v}{A''}\right)\left(\frac{A''_v}{A''} + \frac{H_v}{H}\right)$$
$$- \frac{1}{9}\left(\frac{A''_v}{A''} + \frac{H_v}{H}\right)^2 - \frac{1}{3}\left[\frac{A''_{vv}}{A''} + \frac{H_{vv}}{H} - \left(\frac{A''_v}{A''}\right)^2 - \left(\frac{H_v}{H}\right)^2\right],$$

where we have assumed A'' and H different from zero. The cases thus excluded are necessarily so, since the corresponding net is degenerate. In fact, if $A'' = 0$, we see from (37) that

$$y_{vv} = -A'y_v + C''y,$$

so that the three solutions $y^{(k)}$ will satisfy a homogeneous relation of the first degree with coefficients independent of v; i. e., the curves $u = \text{const.}$ of the original net are straight lines. The locus of P_σ in that case obviously degenerates into a curve, the envelope of these straight lines. This is also apparent from (40), for if $A'' = 0$, on account of (14), $C'' - A'_v - 2A'^2$ will also vanish, so that we shall find

$$\sigma_v = -2A'\sigma.$$

The curves $u = \text{const.}$ of the transformed net degenerate into the points of the envelope, the curves $v = \text{const.}$ of the transformed net all coincide with this same envelope.

If $H = 0$, equations (40) show that

$$\sigma_u = B'\sigma;$$

i. e., the envelope of the tangents to the curves $u = \text{const.}$ along a curve $v = \text{const.}$ degenerates into a point. In this case the curves $v = \text{const.}$ of the transformed net degenerate into points, and the curves $u = \text{const.}$ all coincide with the locus of these points.

We can now verify the following theorem. *If $A'' = 0$ or $B = 0$, one of the families of the original net is composed of straight lines and the first or minus first Laplacian transformed net degenerates. The only other cases in which one of these transformed nets degenerates are obtained by equating either H or K to zero.*

In fact, the first Laplacian transformed net will degenerate if and only if the locus of P_σ for all possible values of u and v reduces to a curve, i. e., if σ satisfies an equation of the form

$$\alpha\sigma_u + \beta\sigma_v + \gamma\sigma = 0,$$

or if

$$\alpha(Hy + B'\sigma) + \beta[(C'' - A'_v - 2A'^2)y + A''y_u - 2A'\sigma] + \gamma\sigma = 0.$$

Now the points y, y_u and σ are not collinear if the curves $u = \text{const.}$ and $v = \text{const.}$ are not tangent to each other at the point P_y. Consequently we should have

$$\alpha H + \beta(C'' - A'_v - 2A'^2) = 0, \qquad \beta A'' = 0, \qquad \alpha B' - 2\beta A' + \gamma = 0.$$

If H is not zero, we should therefore find necessarily $A'' = 0$, since $\alpha = \beta = \gamma = 0$ is not an admissible solution.

The seven fundamental invariants of the first Laplacian transformed net have the following values:

$$\mathfrak{B}_1 = \frac{H}{\mathfrak{A}''}, \qquad \mathfrak{C}_1 = \frac{1}{\mathfrak{A}''}(H_v + 3A'H),$$

$$\mathfrak{A}'_1 = A' + \frac{2}{3}\frac{H_v}{H} - \frac{1}{3}\frac{\mathfrak{A}''_v}{\mathfrak{A}''} + \frac{1}{6\mathfrak{A}''_1}\frac{\partial\mathfrak{A}''_1}{\partial v}, \qquad \mathfrak{B}'_1 = B' - \frac{1}{2}\frac{\mathfrak{A}''_u}{\mathfrak{A}''} - \frac{1}{6}\frac{H_u}{H},$$

$$(46)\quad \mathfrak{C}'_1 = \mathfrak{C}' - A'_u + B'_v - \frac{1}{3}\left(\frac{\mathfrak{A}''_{uv}}{\mathfrak{A}''} + \frac{H_{uv}}{H} - \frac{\mathfrak{A}''_u\mathfrak{A}''_v}{\mathfrak{A}''^2} - \frac{H_uH_v}{H^2}\right),$$

$$\mathfrak{A}''_1 = \frac{1}{H}\left(\mathfrak{A}''K + \mathfrak{C}''_v - \mathfrak{C}''\frac{\mathfrak{A}''_v}{\mathfrak{A}''}\right),$$

$$\mathfrak{C}''_1 = \mathfrak{C}'' + 3A'\left(\frac{\mathfrak{A}''_v}{\mathfrak{A}''} - \frac{H_v}{H}\right) + \frac{\mathfrak{A}''_v}{\mathfrak{A}''}\frac{H_v}{H} - 3A'_v - \frac{H_{vv}}{H}.$$

We find further

$$(46a)\qquad H_1 = 2H - K - \frac{\partial^2 \log H}{\partial u\partial v}, \qquad K_1 = H.$$

We now proceed to set down the corresponding equations for the minus first

Laplacian transformed net. We shall find

$$\begin{aligned} \rho_{uu} &= a_{-1}\rho_u + b_{-1}\rho_v + c_{-1}\rho, \\ \rho_{uv} &= a'_{-1}\rho_u + b'_{-1}\rho_v + c'_{-1}\rho, \\ \rho_{vv} &= a''_{-1}\rho_u + b''_{-1}\rho_v + c''_{-1}\rho, \end{aligned} \tag{47}$$

where

$$\begin{aligned} &BKa_{-1} = (B_u - BB')K, \quad BKb_{-1} = B\mathfrak{C}_u - B_u\mathfrak{C} + B^2H, \\ &BKc_{-1} = BK\mathfrak{C} - A'(B\mathfrak{C}_u - B_u\mathfrak{C} + B^2H) + 2K(B'B_u - BB'_u + BB'^2), \\ &BKa'_{-1} = A'BK, \quad BKb'_{-1} = BB'K + BK_u, \\ &BKc'_{-1} = -A'BK_u + BK^2 + BK(A'_u - A'B'), \\ &BKa''_{-1} = K^2, \quad BKb''_{-1} = B_vK + BK_v - A'BK, \\ &BKc''_{-1} = -A'BK_v + 2B'K^2 + K(BA'_v - A'B_v + 2A'^2B). \end{aligned} \tag{48}$$

In order to reduce (47) to its canonical form we put

$$y_{-1} = \frac{\rho}{\sqrt[3]{BK}}. \tag{49}$$

The coefficients of the resulting system, the seminvariants of (47), are:

$$\begin{aligned} A_{-1} &= -B' + \frac{B_u}{B} - \frac{2}{3}\left(\frac{B_u}{B} + \frac{K_u}{K}\right), \qquad B_{-1} = \frac{1}{BK}(B\mathfrak{C}_u - B_u\mathfrak{C} + B^2H), \\ C_{-1} &= \mathfrak{C} - \frac{A'}{BK}(B\mathfrak{C}_u - B_u\mathfrak{C} + B^2H) + 2\left(B'\frac{B_u}{B} - B'_u + B'^2\right) \\ &\quad + \frac{1}{3}\left(\frac{B_u}{B} - B'\right)\left(\frac{B_u}{B} + \frac{K_u}{K}\right) + \frac{1}{3BK}\left(\frac{B_v}{B} + \frac{K_v}{K}\right)(B\mathfrak{C}_u - B_u\mathfrak{C} + B^2H) \\ &\quad - \frac{1}{3}\left(\frac{B_{uu}}{B} + \frac{K_{uu}}{K} - \frac{B_u^2}{B^2} - \frac{K_u^2}{K^2}\right) - \frac{1}{9}\left(\frac{B_u}{B} + \frac{K_u}{K}\right)^2, \\ A'_{-1} &= A' - \frac{1}{3}\left(\frac{B_v}{B} + \frac{K_v}{K}\right), \qquad B'_{-1} = B' + \frac{K_u}{K} - \frac{1}{3}\left(\frac{B_u}{B} + \frac{K_u}{K}\right), \\ C'_{-1} &= C' + A'_u - B'_v - A'\frac{K_u}{K} + \tfrac{1}{3}A'\left(\frac{B_u}{B} + \frac{K_u}{K}\right) + \frac{1}{3}\left(B' + \frac{K_u}{K}\right)\left(\frac{B_v}{B} + \frac{K_v}{K}\right) \\ &\quad - \frac{1}{3}\left(\frac{B_{uv}}{B} + \frac{K_{uv}}{K} - \frac{B_uB_v}{B^2} - \frac{K_uK_v}{K^2}\right) - \frac{1}{9}\left(\frac{B_u}{B} + \frac{K_u}{K}\right)\left(\frac{B_v}{B} + \frac{K_v}{K}\right), \\ A''_{-1} &= \frac{K}{B}, \qquad B'_{-1} = -A' + \frac{1}{3}\left(\frac{B_v}{B} + \frac{K_v}{K}\right), \\ C''_{-1} &= A'_v + 2A'^2 + \frac{2B'K}{B} + \frac{1}{3}\frac{K}{B}\left(\frac{B_u}{B} + \frac{K_u}{K}\right) - \tfrac{4}{3}A'\left(\frac{B_v}{B} + \frac{K_v}{K}\right) \\ &\quad + \frac{5}{9}\left(\frac{B_v}{B} + \frac{K_v}{K}\right)^2 - \frac{2}{3}\frac{B_v}{B}\frac{K_v}{K} - \frac{1}{3}\left(\frac{B_{vv}}{B} + \frac{K_{vv}}{K}\right). \end{aligned} \tag{50}$$

The seven fundamental invariants of the minus first Laplacian transformed net have the following values:

$$\mathfrak{B}_{-1} = \frac{1}{\mathfrak{B}K}(\mathfrak{B}\mathfrak{C}_u - \mathfrak{B}_u\mathfrak{C} + \mathfrak{B}^2 H),$$

$$\mathfrak{C}_{-1} = \mathfrak{C} + 3B'\left(\frac{\mathfrak{B}_u}{\mathfrak{B}} - \frac{K_u}{K}\right) + \frac{\mathfrak{B}_u}{\mathfrak{B}}\frac{K_u}{K} - 3B'_u - \frac{K_{uu}}{K},$$

$$\mathfrak{A}'_{-1} = A' - \frac{1}{2}\frac{\mathfrak{B}_v}{\mathfrak{B}} - \frac{1}{6}\frac{K_v}{K}, \qquad \mathfrak{B}'_{-1} = B' - \frac{1}{3}\frac{\mathfrak{B}_u}{\mathfrak{B}} + \frac{2}{3}\frac{K_u}{K} + \frac{1}{6\mathfrak{B}_{-1}}\frac{\partial \mathfrak{B}_{-1}}{\partial u}, \tag{51}$$

$$\mathfrak{C}'_{-1} = \mathfrak{C}' + A'_u - B'_v - \frac{1}{3}\left(\frac{\mathfrak{B}_{uv}}{\mathfrak{B}} + \frac{K_{uv}}{K} - \frac{\mathfrak{B}_u\mathfrak{B}_v}{\mathfrak{B}^2} - \frac{K_u K_v}{K^2}\right),$$

$$\mathfrak{A}''_{-1} = \frac{K}{\mathfrak{B}}, \qquad \mathfrak{C}''_{-1} = \frac{1}{\mathfrak{B}}(K_u + 3B'K),$$

and furthermore

$$H_{-1} = K, \qquad K_{-1} = 2K - H - \frac{\partial^2 \log K}{\partial u \partial v}. \tag{52}$$

It is easy to show that these two Laplacian transformations may be regarded as inverse to each other, a fact which has been made use of in our notation. In general, we shall therefore obtain an infinity of nets of plane curves by the repetition of one or the other of these transformations. It may happen however that a certain one of these nets degenerates, as is the case for instance if either H or K vanishes. In all such cases the general integral of the equation

$$\frac{\partial^2 y}{\partial u \partial v} = A'\frac{\partial y}{\partial u} + B'\frac{\partial y}{\partial v} + C'y \tag{53}$$

can be obtained by quadratures, and we are in possession of a very elegant analytical theory of this differential equation from this point of view, a theory originated by Laplace and further developed by Darboux. To Darboux is due also a geometrical interpretation of this theory which puts it into relation on the one hand with the theory of conjugate nets of curves on curved surfaces, and on the other with the theory of congruences of straight lines. But these geometric theories as developed by Darboux are not projective theories, since a simple equation of form (53) does not suffice to characterize a surface projectively. A purely projective theory may be obtained by adding a second partial differential equation of the second order to (53), and this is essentially what has been done by the present author in his general projective theory of surfaces and congruences.*

*E. J. WILCZYNSKI. *Projective Differential Geometry of Curved Surfaces.* Five memoirs. These Transactions, 1907–1909. Compare also *Sur la théorie générale des congruences*, soon to be published by the Royal Belgian Academy.

In the present paper we have found a different geometric interpretation for Darboux's theory, which seems to have escaped notice until now, and moreover since we are considering, not a single linear partial differential equation of the second order, but a system of three such equations, our theory is a purely projective one. Every one of Darboux's analytical theorems furnishes a preliminary basis for a series of deeper investigations concerning such systems of partial differential equations or, geometrically speaking, concerning nets of plane curves. Suppose, for instance, that equation (53) is of such a character that its general solution may be obtained by quadratures. It will contain a certain number of arbitrary functions. The question remains: how to determine these arbitrary functions so as to satisfy the remaining two equations of the system, and this reduces to a problem in ordinary differential equations.

As the point y moves along a curve $u = \text{const.}$ or $v = \text{const.}$ of the original net, the corresponding point y_1 or σ describes a curve $u = \text{const.}$ or $v = \text{const.}$ of the first Laplacian transformed net. Now the lines yy_1 are the tangents of the curves $u = \text{const.}$ Let us speak of those curves of the transformed nets which are described by y_1 when y moves along one of the curves $u = \text{const.}$, or by y_2 when y moves along one of the curves $v = \text{const.}$ as *direct*, and the other curves of their respective nets as *indirect* transforms of the curves of the original net. The lines which join the points of a curve of the original net to the corresponding points of its direct Laplacian transform will then be tangents of the former curve. The lines which join the points of a curve of the original net to the corresponding point of its indirect Laplacian transform are tangent to the latter.

This terminology is convenient in enabling us to state the following simple theorem. *If a net of plane curves is obtained from another by a Laplacian transformation, its two families of curves are respectively the direct transforms of one and the indirect transforms of the other of the two families of the original net. That family of the transformed net, which corresponds indirectly to a family of the original net, can never be composed of straight lines.*

In fact, if the curves $v = \text{const.}$ in the first Laplacian transformed net are straight lines, B, and therefore, according to (45), H must vanish identically. But if $H = 0$, the transformed net is degenerate.

In a net obtained by Laplacian transformation from another net, those curves which correspond directly to curves of the original net may be straight lines. *They will be straight lines in the first transformed net if and only if*

$$\mathfrak{A}''^2 K + \mathfrak{A}''\mathfrak{C}_v'' - \mathfrak{C}''\mathfrak{A}_v'' = 0, \tag{54a}$$

and for the minus first, if and only if

$$\mathfrak{B}^2 H + \mathfrak{B}\mathfrak{C}_u - \mathfrak{C}\mathfrak{B}_u = 0. \tag{54b}$$

§ 5. *Determination of those nets whose Laplacian transforms are merely projections of the original net.*

By means of the Laplace transformation two nets are brought into what we may call a point-to-point correspondence, those being corresponding points which, in the two nets, correspond to the same pair of values (u, v). Our equations are such that both the original and the transformed nets are referred to the same independent variables. Consequently if this correspondence is to be equivalent to a projective transformation, the corresponding invariants must be equal for all values of u and v. Since, moreover, the seminvariants are expressible in terms of the invariants and their derivatives, the conditions necessary and sufficient for projective equivalence of a net and of its first Laplacian transformed net are

$$B_1 = B, \qquad C_1 = C, \qquad A_1' = A', \qquad B_1' = B', \qquad C_1' = C',$$
$$A_1'' = A'', \qquad C_1'' = C''.$$

According to (45) we find, therefore, the following conditions for A, B, C, etc.:

$$\begin{aligned}
&\frac{A_u''}{A''} + \frac{H_u}{H} = 0 \qquad \frac{H}{A''} = B,\\
&B_u' + 2\dot{B}'^2 + 2\frac{A'H}{A''} + \frac{1}{3}\frac{H}{A''}\left(\frac{A_v''}{A''} + \frac{H_v}{H}\right) - \frac{2}{3}\frac{A_u''}{A''}\frac{H_u}{H} = C,\\
&\frac{2}{3}\frac{H_v}{H} - \frac{1}{3}\frac{A_v''}{A''} = 0, \qquad B_v' - A_u' - B'\frac{H_v}{H} + \tfrac{1}{3}B'\left(\frac{A_v''}{A''} + \frac{H_v}{H}\right) = 0,\\
&\frac{A''K}{H} + \frac{1}{H}\left(\mathfrak{C}_v'' - \mathfrak{C}''\frac{A_v''}{A''}\right) = A'',\\
&\mathfrak{C}'' - \frac{B'}{H}\left(\mathfrak{C}_v'' - \mathfrak{C}''\frac{A_v''}{A''} + KA''\right) + 2\left(A'\frac{A_v''}{A''} - A_v' + A'^2\right)\\
&\quad + \frac{1}{3}\left(-A' + \frac{A_v''}{A''}\right)\left(\frac{A_v''}{A''} + \frac{H_v}{H}\right) - \frac{1}{9}\left(\frac{A_v''}{A''} + \frac{H_v}{H}\right)^2 - \frac{1}{3}\frac{\partial}{\partial v}\left(\frac{A_v''}{A''} + \frac{H_v}{H}\right) = C''.
\end{aligned} \tag{55}$$

The first and fourth of these conditions give, upon integration,

$$A''H = \psi(v), \qquad A''H^{-2} = \phi(u),$$

$\phi(u)$ and $\psi(v)$ being functions of the single arguments indicated.* Consequently

$$A'' = \phi^{\frac{1}{3}}\psi^{\frac{2}{3}}, \qquad H = \phi^{-\frac{1}{3}}\psi^{\frac{1}{3}}.$$

Now A'' and H are relative invariants, and if we put

$$\bar{u} = U(u), \qquad \bar{v} = V(v), \tag{56}$$

* Neither of these functions can vanish identically, since A'' and H must be different from zero. Otherwise the transformed net would be degenerate.

they are transformed into

$$\bar{A}'' = A'' \frac{U'}{(V')^2}, \qquad \bar{H} = \frac{H}{U'V'}$$

respectively. We may, therefore, reduce them to unity by choosing U' and V' so that

$$(U')^3 = \frac{1}{\phi(u)}, \qquad (V')^3 = \psi(v).$$

The most general transformation of the form (56) *which leaves the conditions* $A'' = H = 1$ *invariant will be conditioned by the equations*

$$(U')^3 = 1, \qquad (V')^3 = 1.$$

The conditions (55) now reduce considerably. They become

$$A'' = 1, \qquad H = C' + A'B' - A'_u = 1, \qquad B = 1,$$

$$2A' + B'_u + 2B'^2 = C, \qquad B'_v - A'_u = 0, \qquad K + \mathfrak{C}''_v = 1,$$

$$\mathfrak{C}'' - B' + 2(-A'_v + A'^2) = C'',$$

whence

$$A'' = 1, \qquad B = 1, \qquad A' = \phi_v, \qquad B' = \phi_u, \qquad C' = 1 - \phi_u\phi_v + \phi_{uv},$$

$$C = \phi_{uu} + 2\phi_v + 2\phi_u^2, \qquad C'' = \phi_{vv} + 2\phi_v^2 - \phi_u + \lambda(u).$$

Here $\lambda(u)$ is an arbitrary function of u, and ϕ is subject to the equation $\phi_{vv} = 0$, so that

$$\phi = \mu(u) + \nu(u)v,$$

$\mu(u)$, $\nu(u)$ being arbitrary functions of u. Let us denote $d\mu/du$, $d^2\mu/du^2$ by μ', μ'', etc., then we shall have

$$A = -B', \qquad B = 1, \qquad C = \mu'' + 2\nu + 2\mu'^2 + (\nu'' + 4\mu'\nu')v + 2(\nu')^2v^2,$$

$$A' = \nu, \qquad B' = \mu' + \nu'v, \qquad C' = 1 - \nu\mu' + \nu'' - \nu\nu'v,$$

$$A'' = 1, \qquad B'' = -A', \qquad C'' = \lambda + 2\nu^2 - \mu' - \nu'v.$$

These quantities must satisfy the integrability conditions (14). Imposing this condition we find that we must have

$$\lambda = 3l, \qquad \mu = k + lu, \qquad \nu = m,$$

where k, l, m are arbitrary constants, so that we find finally

$$\begin{aligned} &A = -l, && B = 1, && C = 2(l^2 + m), \\ &A' = m, && B' = l, && C' = 1 - lm, \\ &A'' = 1, && B'' = -m, && C'' = 2(l + m^2). \end{aligned} \tag{57}$$

If a net is projectively equivalent to its first Laplacian transform, its system

of differential equations can be reduced to one with the constant coefficients (57) *which involve two arbitraries.*

A system of partial differential equations of this form may be integrated as follows. The function

$$y = e^{\alpha u + \beta v}$$

will be a solution of it, if α and β can be determined so as to satisfy the three conditions

$$\begin{aligned} \alpha^2 &= -l\alpha + \beta + 2(l^2 + m), \\ \alpha\beta &= m\alpha + l\beta + 1 - lm, \\ \beta^2 &= \alpha - m\beta + 2(l + m^2), \end{aligned} \tag{58}$$

and these, it is easy to show, are consistent. The elimination of β leads to the following equation for α:

$$f(\alpha) = \alpha^3 - 3(l^2 + m)\alpha + 2l^3 + 3lm - 1 = 0. \tag{59}$$

Let α_1, α_2, α_3 be the three roots of this cubic, and let

$$\beta_k = \frac{m\alpha_k + 1 - lm}{\alpha_k - l} \qquad (k = 1, 2, 3), \tag{60}$$

be the corresponding values of β. In order that the three solutions of our system of partial differential equations

$$y^{(k)} = e^{\alpha_k u + \beta_k v} \qquad (k = 1, 2, 3), \tag{61}$$

may be linearly independent it is necessary and sufficient that the three roots α_k of (59) be distinct, as may be seen by applying the usual test. *Let us assume, therefore, that the discriminant of* (59) *or*

$$\Delta = 1 - 6lm - 3l^2m^2 - 4(l^3 + m^3) \tag{62}$$

is different from zero. Then the three quantities β_1, β_2, β_3, *will also be distinct, and the net defined by equations* (61) *will be non-degenerate.* In fact the determinant D [cf. eq. (3)] becomes

$$D = \begin{vmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix} = \Sigma(\alpha_2\beta_3 - \alpha_3\beta_2).$$

On account of (60) we may write

$$\begin{aligned} D &= \Sigma \frac{(\alpha_2 - \alpha_3)[m\alpha_2\alpha_3 + (1 - lm)(\alpha_2 + \alpha_3) - l(1 - lm)]}{(\alpha_2 - l)(\alpha_3 - l)} \\ &= \Sigma \frac{(\alpha_2 - \alpha_3)[m\alpha_2\alpha_3 + (1 - lm)(\alpha_2 + \alpha_3) - l(1 - lm)](\alpha_1 - l)}{(\alpha_1 - l)(\alpha_2 - l)(\alpha_3 - l)}. \end{aligned}$$

The denominator of this fraction is equal to

$$-f(l) = 1.$$

Consequently

$$D = \Sigma(\alpha_2 - \alpha_3)[m\alpha_1\alpha_2\alpha_3 + (1 - lm)(\alpha_1\alpha_2 + \alpha_1\alpha_3) - l(1 - lm)\alpha_1 - lm\alpha_2\alpha_3 - l(1 - lm)(\alpha_2 + \alpha_3) + l^2(1 - lm)].$$

But, according to (59),

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \qquad \alpha_2\alpha_3 + \alpha_3\alpha_1 + \alpha_1\alpha_2 = -3(l^2 + m),$$
$$\alpha_1\alpha_2\alpha_3 = 1 - 3lm - 2l^3.$$

Consequently

$$D = \Sigma(\alpha_2 - \alpha_3)[m(1 - 3lm - 2l^3) - 3(l^2 + m)(1 - lm) + l^2(1 - lm) - \alpha_2\alpha_3]$$
$$= [m(1 - 3lm - 2l^3) - 3(l^2 + m)(1 - lm) + l^2(1 - lm)]\Sigma(\alpha_2 - \alpha_3) - \Sigma(\alpha_2^2\alpha_3 - \alpha_2\alpha_3^2).$$

But obviously

$$\Sigma(\alpha_2 - \alpha_3) = 0, \qquad \Sigma(\alpha_2^2\alpha_3 - \alpha_2\alpha_3^2) = -\Delta,$$

so that

(63) $$D = \Delta$$

and is consequently different from zero. Therefore equations (61) define a non-degenerate net of plane curves if the quantities α_1, α_2, α_3 are distinct.

We proceed to discuss the case when two of the roots of (59) are equal, say

$$\alpha_3 = \alpha_2, \qquad \Delta = 0,$$

while α_1 and α_2 are distinct. We may put again

(64) $$y^{(k)} = e^{\alpha_k u + \beta_k v} \qquad (k = 1, 2),$$

but it remains to find a third solution of our system of partial differential equations. The second equation of (58) may be put into the form

$$(\alpha - l)(\beta - m) = 1,$$

so that we may write

$$\alpha - l = t, \qquad \beta - m = \frac{1}{t},$$

and consider the cubic

(65) $$t^3 + 3lt^2 - 3mt - 1 = 0$$

instead of (59). If we denote its roots by t_1, t_2, t_3 we have in our case $t_3 = t_2$. According to well-known principles, the function obtained from

$$\frac{\partial}{\partial t}[e^{(l+t)u + (m + 1/t)v}]$$

by substituting $t = t_2$ after the differentiation, will be a solution of our partial differential equations if t_2 is a double root of (65). We thus find

(66) $$y^{(3)} = [(\alpha_2 - l)u - (\beta_2 - m)v]e^{\alpha_2 u + \beta_2 v}.$$

That this function satisfies the differential equations may also be verified directly. The linear independence of the three functions $y^{(k)}$ and the non-degeneracy of the net may be verified as before, or even more simply as follows. We find

$$\frac{y^{(3)}}{y^{(2)}} = (\alpha_2 - l)u - (\beta_2 - m)v,$$

$$\log\frac{y^{(1)}}{y^{(2)}} = (\alpha_1 - \alpha_2)u + (\beta_1 - \beta_2)v,$$

whence

$$t_1 t_2^2 \log\frac{y^{(1)}}{y^{(2)}} + t_1 t_2 (t_2 - t_1)\frac{y^{(3)}}{y^{(2)}} = (t_1 - t_2)^2 v,$$

$$t_1 \log\frac{y^{(1)}}{y^{(2)}} + (t_2 - t_1)\frac{y^{(3)}}{y^{(2)}} = (t_1 - t_2)^2 u, \tag{67}$$

as the finite equations of the curves of our net. We see that neither family of curves degenerates, since $t_2 \neq t_1$. Moreover, the three functions are linearly independent, since no root of (65) can be equal to zero, and consequently the curves (67) cannot reduce to straight lines.

If all three roots of (59) are equal, their common value must be zero, and we must have

$$l^2 + m = 0, \qquad 1 - 3lm - 2l^3 = 0,$$

so that either

$$l = -1, \quad m = -1 \quad \text{or} \quad l = -\omega, \quad m = -\omega^2 \quad \text{or} \quad l = -\omega^2, \quad m = -\omega,$$

where ω is a primitive third root of unity. If we write down the three systems of differential equations which correspond to these three alternatives, we find that they may be transformed into each other by a transformation of the form

$$\bar{u} = \alpha u, \qquad \bar{v} = \beta v,$$

where α and β are constants. It suffices therefore to consider the case

$$l = m = -1,$$

which gives rise to the following system of differential equations:

$$y_{uu} = y_u + y_v, \qquad y_{uv} = -y_u - y_v, \qquad y_{vv} = y_u + y_v.$$

This system has the following three linearly independent solutions;

$$y^{(1)} = (u - v)^2 + u + v, \qquad y^{(2)} = u - v, \qquad y^{(3)} = 1.$$

Consequently the equations of the two one-parameter families of the net will be

$$(x_1 + x_2)x_3 - x_2^2 - 2ux_3^2 = 0,$$

$$(x_1 - x_2)x_3 - x_2^2 - 2vx_3^2 = 0; \tag{68}$$

i. e., both families are pencils of conics, which may be completely characterized as follows. The conics $u = \text{const.}$ all have four-pointic (third order) contact with

$$(x_1 + x_2)x_3 - x_2^2 = 0 \tag{69}$$

at the vertex $A\,(x_2 = x_3 = 0)$ of the triangle of reference. Similarly, the conics $v = \text{const.}$ have four-pointic contact with the conic

$$(x_1 - x_2)x_3 - x_2^2 = 0 \tag{70}$$

at the point A. It remains to characterize the two conics (69) and (70). The conic (69) touches the line $x_1 + x_2 = 0$ at the vertex $C(x_1 = x_2 = 0)$ of the fundamental triangle, while (70) at the same point touches the line $x_1 - x_2 = 0$, and these two lines divide harmonically the sides of the triangle AC and BC which meet in C. The third side of the triangle $BC(x_1 = 0)$ meets the two conics in two points which divide it harmonically. We may, therefore, construct our two families of conics as follows.

Choose an arbitrary triangle and denote its vertices by A, B, C. Through the vertex C draw two lines, l and m, which divide CA and CB harmonically. On the side BC of the triangle choose any two points, D and E, which divide BC harmonically. Construct the conic K_1 tangent to AB at A, tangent to l at C and passing through D. Construct also the conic K_2, tangent to AB at A, tangent to m at C, and passing through E. The two pencils of conics, which have four-pointic contact with K_1 and K_2 respectively at A, determine the most general net of conics which has the property that its Laplacian transformed nets are projectively equivalent to it.

Let us return to the general case, in which the three numbers α_1, α_2, α_3 are distinct. The curves of the net are, of course, anharmonic curves, using Halphen's terminology, since the linear differential equations of the third order which define them have constant coefficients.* For every curve of this kind there exists a one-parameter group of collineations which converts it into itself, and in the general case this collineation group has an invariant triangle. From this remark it easily follows that *the three points in which a tangent of the curve intersects the sides of the invariant triangle, together with its point of contact, will determine an anharmonic ratio, constant along the whole curve.* This constant anharmonic ratio is an absolute invariant of the curve, but its expression in terms of the coefficients of the differential equation is irrational. In fact, the six values of this anharmonic ratio, which correspond to the different possible arrangements of the four points on the tangent, are the roots of the following equation of the sixth order

$$3^9 \frac{(\lambda^2 - \lambda + 1)^3}{(\lambda - 2)^2(1 - 2\lambda)^2(\lambda + 1)^2} = \frac{\theta_8^3}{\theta_3^8}, \tag{71}$$

* Cf. equations (73) to (76) of the following paragraph. See also E. J. Wilczynski, *Projective Differential Geometry*, pp. 86–90.

where

$$\frac{\theta_8^3}{\theta_3^8} = \frac{3^9(m+l^2)^3}{(2l^3+3lm-1)^2} \tag{72a}$$

for the curves $v = \text{const.}$ and

$$\frac{\theta_8^3}{\theta_3^8} = \frac{3^9(l+m^2)^3}{(2m^3+3lm-1)^2} \tag{72b}$$

for the curves $u = \text{const.}$*

It is not difficult to show that, *when the invariant triangle and the value K of the anharmonic ratio which corresponds to a definite arrangement of the four points on the tangent are given, there exists a unique one-parameter family of anharmonic curves belonging to the given triangle and the given anharmonic ratio in the sense of the above theorem.* This being so, we can describe our net in the general case [all roots of (59) distinct] as follows.

Construct an arbitrary triangle. Let λ and λ' be two numbers different from each other, from zero, from ± 1, 2 or $\frac{1}{2}$.† Construct the two one-parameter families of anharmonic curves which have the given triangle as invariant triangle, and whose curves belong to the anharmonic ratios λ and λ' respectively. Any net constructed in this way will be projectively equivalent with those obtained from it by Laplacian transformations.

The assumption that λ and λ' shall be distinct does not imply that the invariants (72a) and (72b) shall have different values. In fact, for a given value of the absolute invariant $\theta_8^3/\theta_3^{(8)}$, (71) shows that there are six values of λ no two of which coincide, except in the cases which have been excluded.

The nature of the net in the intermediate case, when two roots of (59) coincide, but not all three, is most easily investigated as follows. Let

$$\frac{y^{(2)}}{y^{(3)}} = x, \qquad \frac{y^{(1)}}{y^{(3)}} = y$$

be cartesian coördinates. Then equations (67) may be written

$$t_2 \log y + (t_2 - t_1)x = \bar{v}, \qquad t_1 \log y + (t_2 - t_1)x = \bar{u},$$

where

$$\bar{v} = \frac{(t_1 - t_2)^2}{t_1 t_2} v, \qquad \bar{u} = (t_1 - t_2)^2 u,$$

or, more simply,

$$y = Ue^{\frac{(t_1-t_2)x}{t_1}}, \qquad y = Ve^{\frac{(t_1-t_2)x}{t_2}},$$

where

$$U = e^{\frac{\bar{u}}{t_1}}, \qquad V = e^{\frac{\bar{v}}{t_2}},$$

* Loc. cit., pp. (58), (59), (88).

† These values are excluded because they correspond to the cases in which the three quantities α_1, α_2, α_3 are not all distinct.

so that the families $u = \text{const.}$, $v = \text{const.}$, coincide with the families $U = \text{const.}$, $V = \text{const.}$, respectively.

In our case t_1 and t_2 are distinct, so that the equations may be written

$$y = UA^x, \qquad y = VB^x,$$

where A and B are two distinct numbers, which may be thought of as being given arbitrarily.

We therefore obtain a net of the desired character if we construct two exponential curves

$$y = A^x, \qquad y = B^x,$$

with arbitrary, but distinct bases, and then magnify their ordinates in all possible ratios. The most general net of the kind considered is obtained from this one by projective transformation.

It is of interest to notice the further result that *the system of differential equations with constant coefficients which has presented itself in this theory is the most general completely integrable system of form* (4) *whose invariants are constants.* This may be verified easily by making use of the integrability conditions in the form (14).

§ 6. *Osculating conics of curves of the net.*

An individual plane curve is characterized projectively by a linear differential equation of the third order. We now proceed to determine these equations for the curves $u = \text{const.}$ and $v = \text{const.}$ of a net when the latter is defined by a system of form (4).

We have

$$y_{uu} = ay_u + by_v + cy,$$

and consequently by differentiation and making use of (4)

$$y_{uuu} = (a_u + a^2 + a'b + c)y_u + (b_u + ab + bb')y_v + (c_u + ac + bc')y.$$

The elimination of y_v between these two equations gives the desired equation for the curves $v = \text{const.}$, viz.,

$$y_{uuu} + 3p_1 y_{uu} + 3p_2 y_u + p_3 y = 0, \tag{73}$$

where

$$p_1 = -\frac{1}{3}\left(\frac{b_u}{b} + a + b'\right), \qquad p_2 = -\frac{1}{3}\left(a_u - a\frac{b_u}{b} + c + a'b - ab'\right),$$

$$p_3 = -\left(c_u - c\frac{b_u}{b} + bc' - b'c\right). \tag{74}$$

Similarly we find the differential equations of the curves $u = \text{const.}$, viz.,

$$y_{vvv} + 3q_1 y_{vv} + 3q_2 y_v + q_3 y = 0, \tag{75}$$

where

$$q_1 = -\frac{1}{3}\left(\frac{a''_v}{a''} + a' + b''\right), \qquad q_2 = -\frac{1}{3}\left(b''_v - b''\frac{a''_v}{a''} + c'' + a''b' - a'b''\right),$$

$$q_3 = -\left(c''_v - c''\frac{a''_v}{a''} + a''c' - a'c''\right). \tag{76}$$

The seminvariants and semi-covariants of (73) are *

$$P_2 = p_2 - p_1^2 - \frac{\partial p_1}{\partial u}, \qquad P_3 = p_3 - 3p_1p_2 + 2p_1^3 - \frac{\partial^2 p_1}{\partial u^2},$$

$$z^{(u)} = y_u + p_1 y, \qquad \rho^{(u)} = y_{uu} + 2p_1 y_u + p_2 y; \tag{77}$$

the corresponding functions for (75) are

$$Q_2 = q_2 - q_1^2 - \frac{\partial q_1}{\partial v}, \qquad Q_3 = q_3 - 3q_1q_2 + 2q_1^3 - \frac{\partial^2 q_1}{\partial v^2},$$

$$z^{(v)} = y_v + q_1 y, \qquad \rho^{(v)} = y_{vv} + 2q_1 y_v + q_2 y. \tag{78}$$

If we substitute into the expressions for $z^{(u)}$, $y = y^{(k)}$ $(k = 1, 2, 3)$, where $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ are three linearly independent solutions of system (4), $z^{(u)}$ assumes three values which we interpret as the homogeneous coördinates of a point. This point $z^{(u)}$ will obviously be on the tangent of the curve $v = \text{const.}$ constructed at the point y. Similarly $\rho^{(u)}$ represents a third point of the plane, and these three points will not be collinear unless y is a point of inflection of the curve $v = \text{const.}$ under consideration. Any expression of the form

$$a_1 y + a_2 z^{(u)} + a_3 \rho^{(u)}$$

will then represent a point, whose coördinates, referred to the triangle y, $z^{(u)}$, $\rho^{(u)}$ may be chosen proportional to a_1, a_2, a_3 respectively. We shall prefer, however, to use the notation

$$a_k = x_k^{(u)} \qquad (k = 1, 2, 3).$$

Referred to this system of coördinates, the equation of the conic having fourth-order contact with the curve $v = \text{const.}$ at y is

$$x_2^{(u)2} - 2x_1^{(u)}x_3^{(u)} + P_2 x_3^{(u)2} = 0.\dagger \tag{79}$$

We wish to find the equation of this same conic referred to the triangle of reference y, z, ρ whose geometrical significance in connection with a given net has already been discussed. In order to do this we must find the equations of transformation between the two systems of coördinates.

Let x_1, x_2, x_3 be the coördinates of a point with respect to the triangle y, ρ,

* E. J. WILCZYNSKI, *Projective Differential Geometry*, pp. 58, 59.

† Ibid., p. 65.

σ, and let $x_1^{(u)}$, $x_2^{(u)}$, $x_3^{(u)}$ be the coördinates of the same point with respect to the triangle y, $z^{(u)}$, $\rho^{(u)}$. Then we must have

$$x_1 y + x_2 \rho + x_3 \sigma = \omega(x_1^{(u)} y + x_2^{(u)} z^{(u)} + x_3^{(u)} \rho^{(u)}),$$

where ω is a factor of proportionality. Now we find from the definition of these quantities the following relations:

$$\begin{aligned} \rho &= -(p_1 + b')y + z^{(u)}, \\ \sigma &= \frac{-a'b - c + ap_1 + 2p_1^2 - p_2}{b} y - \frac{a + 2p_1}{b} z^{(u)} + \frac{1}{b}\rho^{(u)}. \end{aligned} \tag{80}$$

Let these values be substituted into the preceding equation. The coefficients of y, $z^{(u)}$, $\rho^{(u)}$ on both sides of the equation must be equal; otherwise we should find a linear homogeneous differential equation of the form

$$Ly_{uu} + Py_u + Qy = 0$$

satisfied by $y^{(1)}$, $y^{(2)}$, $y^{(3)}$; i. e., the curve $v = \text{const.}$ considered would be a straight line, which case we shall exclude. The exceptional case suggested by (80), viz., $b = 0$, has the same significance. Obviously in this exceptional case there is no proper osculating conic.

We obtain, therefore, the following equations of transformation

$$\begin{aligned} \omega x_1^{(u)} &= x_1 - (p_1 + b')x_2 + \frac{1}{b}(-a'b - c + ap_1 + 2p_1^2 - p_2)x_3, \\ \omega x_2^{(u)} &= x_2 - \frac{a + 2p_1}{b} x_3, \\ \omega x_3^{(u)} &= \frac{1}{b} x_3. \end{aligned} \tag{81}$$

The equation of the conic (79) referred to the new triangle of reference becomes therefore

$$\begin{aligned} b^2 x_2^2 + \tfrac{1}{3} b \left(2\frac{b_u}{b} - 4a + 8b'\right) x_2 x_3 - 2b x_1 x_3 \\ + \left(3p_2 - p_1^2 - \frac{\partial p_1}{\partial u} + 2ap_1 + a^2 + 2a'b + 2c\right) x_3^2 = 0. \end{aligned} \tag{82}$$

Since our triangle of reference is a covariant triangle, the coefficients of this equation must be invariants of the net. In fact we find that *equation* (82) *may be reduced to the following:*

$$M \equiv \mathfrak{B}^2 x_2^2 + 4\mathfrak{B}\mathfrak{B}' x_2 x_3 - 2\mathfrak{B} x_1 x_3 + \phi x_3^2 = 0, \tag{83}$$

where

$$\phi = \mathfrak{C} - 4\mathfrak{B}'^2 + 2\mathfrak{B}'_u + 6\mathfrak{B}'B'. \tag{84}$$

In order to verify this statement, it is convenient to assume that system (4) is given in its canonical form, and to introduce for p_1 and p_2 their values from (74).

In the same way we may find the equation of the conic which osculates the curve $u = \text{const.}$ at the point y. We have, in this case, the relations

$$\begin{aligned} \rho &= \frac{1}{a''}\left[(-a''b' - c'' + b''q_1 + 2q_1^2 - q_2)y - (2q_1 + b'')z^{(v)} + \rho^{(v)}\right], \\ \sigma &= -(a' + q_1)y + z^{(v)}, \end{aligned} \tag{85}$$

so that

$$\begin{aligned} \omega x_1^{(v)} &= x_1 + \frac{1}{a''}(-a''b' - c'' + b''q_1 + 2q_1^2 - q_2)x_2 - (a' + q_1)x_3, \\ \omega x_2^{(v)} &= -\frac{1}{a''}(2q_1 + b'')x_2 + x_3, \\ \omega x_3^{(v)} &= \frac{1}{a''}x_2. \end{aligned} \tag{86}$$

The equation of the conic which osculates the curve $u = \text{const.}$ at the point y becomes

$$N \equiv \psi x_2^2 + 4\mathfrak{A}'\mathfrak{A}''x_2x_3 + \mathfrak{A}'^2x_3^2 - 2\mathfrak{A}''x_1x_2 = 0, \tag{87}$$

where

$$\psi = \mathfrak{C}'' - 4\mathfrak{A}'^2 + 2\mathfrak{A}'_v + 6\mathfrak{A}'A'. \tag{88}$$

Equations (83) and (87) enable us to interpret geometrically a number of invariants.

The line $x_1 = 0$, i. e., the line which joins the points P_ρ and P_σ, will intersect the conic $M = 0$ in two points which divide $P_\rho P_\sigma$ harmonically if and only if $\mathfrak{C}'$ is equal to zero. Similarly *the two points determined on $P_\rho P_\sigma$ by the conic $N = 0$ divide $P_\rho P_\sigma$ harmonically if and only if $\mathfrak{A}' = 0$.* The anharmonic ratios which these two groups of four points determine, if $\mathfrak{A}'$ and $\mathfrak{C}'$ are not zero, are easily expressible as absolute invariants. *The point P_σ is on the conic $M = 0$ if and only if $\phi = 0$*; similarly *ψ will vanish if and only if P_ρ is on the conic $N = 0$.*

Neither of the osculating conics is degenerate unless either $\mathfrak{B}$ or $\mathfrak{A}''$ vanishes, i. e., unless the point of osculation is a point of inflection of the corresponding curve $u = \text{const.}$ or $v = \text{const.}$

The line $P_\rho P_\sigma$ is tangent to $M = 0$ if and only if

$$\phi - 4\mathfrak{B}'^2 = 0; \tag{89a}$$

it is tangent to $N = 0$ *if and only if*

$$\psi - 4\mathfrak{A}'^2 = 0. \tag{89b}$$

The conics $M = 0$ and $N = 0$, of course, have the point P_y in common. The discriminant of the general conic

$$\lambda M + \mu N = 0$$

of their pencil may be reduced to

$$\mathfrak{B}^4\lambda^3 + \mathfrak{B}^2(\psi - 4\mathfrak{A}''\mathfrak{B}')\lambda^2\mu + \mathfrak{A}'^2(\phi - 4\mathfrak{A}'\mathfrak{B})\lambda\mu^2 + \mathfrak{A}''^4\mu^3, \tag{90}$$

and the values of $\lambda : \mu$ which annul this expression will determine the degenerate conics of the pencil, and consequently the remaining three points of intersection of $M = 0$ and $N = 0$. The two conics will touch each other if the discriminant of (90) vanishes.

We have found the equations of the osculating conics of the curves of the net which meet at the point P_y corresponding to the values u and v of the independent variables. We proceed to determine the osculating conics of the curves of the net in the vicinity of P_y.

Let us denote by y', ρ', σ' the vertices of the triangle which is obtained from y, ρ, σ when the infinitesimal increments δu and δv are given to u and v respectively. We shall have

$$\begin{aligned} y' &= y + y_u\delta u + y_v\delta v, \\ \rho' &= \rho + \rho_u\delta u + \rho_v\delta v, \\ \sigma' &= \sigma + \sigma_u\delta u + \sigma_v\delta v, \end{aligned} \tag{91}$$

retaining only terms of the first order in δu and δv. Now let us assume that (4) is given in its canonical form. Then we find

$$\begin{aligned} y_u &= B'y + \rho, \qquad y_v = A'y + \sigma, \\ y_{uu} &= (C - B'^2 + A'B')y - B'\rho + B\sigma, \\ y_{uv} &= (C' + 2A'B')y + A'\rho + B'\sigma, \\ y_{vv} &= (C'' - A'^2 + A''B')y + A''\rho - A'\sigma; \end{aligned} \tag{92}$$

whence

$$\begin{aligned} \rho_u &= \mathfrak{C}y - 2B'\rho + B\sigma, \\ \rho_v &= Ky + A'\rho, \\ \sigma_u &= Hy + B'\sigma, \\ \sigma_v &= \mathfrak{C}''y + A''\rho - 2A'\sigma. \end{aligned} \tag{93}$$

The first and last of these equations provide us with geometrical interpretations for the invariant equations $\mathfrak{C} = 0$ and $\mathfrak{C}'' = 0$ respectively.

If $\mathfrak{C} = 0$, *the curve described by* P_ρ *as* P_y *moves along a curve* $v = \text{const.}$ *is such that its tangent at* P_ρ *passes through* P_σ.

If $\mathfrak{C}'' = 0$, *the curve described by* P_σ *as* P_y *moves along a curve* $u = \text{const.}$ *is such that its tangent at* P_σ *passes through* P_ρ.

We substitute the values just found for $y_u, \cdots, \sigma_v$ into (91), and obtain

$$\begin{aligned} y' &= y(1 + B'\delta u + A'\delta v) + \rho\delta u + \sigma\delta v, \\ \rho' &= y(\mathfrak{C}\delta u + K\delta v) + \rho(1 - 2B'\delta u + A'\delta v) + \sigma B\delta u, \\ \sigma' &= y(H\delta u + \mathfrak{C}''\delta v) + \rho A''\delta v + \sigma(1 + B'\delta u - 2A'\delta v). \end{aligned} \tag{94}$$

Now let x_1', x_2', x_3' be the coördinates of a point with respect to the triangle y', ρ', σ', while x_1, x_2, x_3 are the coördinates of the same point with respect to y, ρ, σ. Then we shall have

$$\omega(x_1 y + x_2\rho + x_3\sigma) = x_1' y' + x_2'\rho' + x_3'\sigma',$$

whence

$$\begin{aligned} \omega x_1 &= (1 + B'\delta u + A'\delta v)x_1' + (\mathfrak{C}\delta u + K\delta v)x_2' + (H\delta u + \mathfrak{C}''\delta v)x_3', \\ \omega x_2 &= \delta u \cdot x_1' + (1 - 2B'\delta u + A'\delta v)x_2' + A''\delta v \cdot x_3', \\ \omega x_3 &= \delta v \cdot x_1' + B\delta u \cdot x_2' + (1 + B'\delta u - 2A'\delta v)x_3', \end{aligned} \tag{95}$$

or inversely

$$\begin{aligned} \omega' x_1' &= x_1 - (B'x_1 + \mathfrak{C}x_2 + Hx_3)\delta u - (A'x_1 + Kx_2 + \mathfrak{C}''x_3)\delta v, \\ \omega' x_2' &= x_2 - (x_1 - 2B'x_2)\delta u - (A'x_2 + A''x_3)\delta v, \\ \omega' x_3' &= x_3 - (Bx_2 + B'x_3)\delta u - (x_1 - 2A'x_3)\delta v. \end{aligned} \tag{96}$$

The equation of the conic which osculates the curve $v = \text{const.}$ of our net at the point $(u + \delta u, v + \delta v)$, referred to the triangle y', ρ', σ' will be

$$\begin{aligned} &(\mathfrak{B}^2 + 2\mathfrak{B}\mathfrak{B}_u\delta u + 2\mathfrak{B}\mathfrak{B}_v\delta v)x_2'^2 + 4\left(\mathfrak{B}\mathfrak{B}' + \frac{\partial(\mathfrak{B}\mathfrak{B}')}{\partial u}\delta u + \frac{\partial(\mathfrak{B}\mathfrak{B}')}{\partial v}\delta v\right)x_2'x_3' \\ &\qquad - 2(\mathfrak{B} + \mathfrak{B}_u\delta u + \mathfrak{B}_v\delta v)x_1'x_3' + (\phi + \phi_u\delta u + \phi_v\delta v)x_3'^2 = 0, \end{aligned}$$

If we introduce the values (96) of x_1', x_2', x_3' *this equation may be reduced to*

$$\left(1 + \frac{4}{3}\frac{B_u}{B}\delta u\right)M + \left(\phi_u - 2B'\phi - \frac{4}{3}\frac{\mathfrak{B}_u}{\mathfrak{B}}\phi + \frac{1}{2}\mathfrak{B}H\right)x_3^2\delta u + P\delta v = 0, \tag{97}$$

where

$$\begin{aligned} P \equiv\ & 2\mathfrak{B}x_1^2 - 4\mathfrak{B}\mathfrak{B}'x_1x_2 - 2(\mathfrak{B}_v + A'\mathfrak{B} + \phi)x_1x_3 + 2\mathfrak{B}(\mathfrak{B}_v - A'\mathfrak{B})x_2^2 \\ & + 2(2\mathfrak{B}_v\mathfrak{B}' + 2\mathfrak{B}\mathfrak{B}_v' + 2A'\mathfrak{B}\mathfrak{B}' + \mathfrak{B}K - \mathfrak{A}''\mathfrak{B}^2)x_2x_3 \\ & + (\phi_v + 4A'\phi - 4\mathfrak{A}''\mathfrak{B}\mathfrak{B}' + 2\mathfrak{B}\mathfrak{C}'')x_3^2. \end{aligned} \tag{98}$$

If we allow y to move along a curve $v = \text{const.}$, $\delta v = 0$. The osculating conic of this curve remains unchanged if and only if

$$3\mathfrak{B}\phi_u - 4\mathfrak{B}_u\phi - 6\mathfrak{B}B'\phi + 6\mathfrak{B}^2H = 0. \tag{99}$$

We see, therefore, that (99) *is the necessary and sufficient condition for a net whose curves* $v = \text{const.}$ *are conics. Similarly if*

$$3\mathfrak{A}''\psi_v - 4\mathfrak{A}'_v\psi - 6\mathfrak{A}''A'\psi + 6\mathfrak{A}''^2K = 0 \tag{100}$$

the curves $u = \text{const.}$ *will be conics.*

In general we see that the osculating conics of the curves $v = \text{const.}$ in the vicinity of the point (u, v) belong to the two-parameter family of conics

$$\lambda x_3^2 + \mu M + \pi P = 0. \tag{101}$$

The points of intersection of the conics M and P are of special interest. If they are fixed, the conics which osculate the curves $v = \text{const.}$ at the points in which they meet a fixed curve $u = \text{const.}$, will form a pencil, but we shall not attempt at present to deduce the analytical conditions for this special case.

We may of course obtain a formula similar to (97) for all of the conics which osculate the curves $u = \text{const.}$ in the vicinity of a given point (u, v). It is clear that these two systems of conics together with their Jacobians and their other covariants must give rise to a very large number of relations. The exhaustive discussion of these will not here be attempted. We are satisfied in so far as we have reduced the problem to a purely algebraic one.

§ 7. *Differential equations of the net in line-coördinates.*

Let

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3)$$

be the finite equations of the net, the triangle of reference being arbitrary. The equations of the lines tangent to the curves $v = \text{const.}$ and $u = \text{const.}$ at the point (u, v) will be

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y^{(1)} & y^{(2)} & y^{(3)} \\ y_u^{(1)} & y_u^{(2)} & y_u^{(3)} \end{vmatrix} = 0 \qquad \text{and} \qquad \begin{vmatrix} x_1 & x_2 & x_3 \\ y^{(1)} & y^{(2)} & y^{(3)} \\ y_v^{(1)} & y_v^{(2)} & y_v^{(3)} \end{vmatrix} = 0$$

respectively, or

$$Y^{(1)}x_1 + Y^{(2)}x_2 + Y^{(3)}x_3 = 0 \qquad \text{and} \qquad Z^{(1)}x_1 + Z^{(2)}x_2 + Z^{(3)}x_3 = 0,$$

where $Y^{(k)}$ and $Z^{(k)}$, the coördinates of these two lines, are binary determinants of the types

$$Y = |y, y_u|, \qquad Z = |y, y_v| \tag{102}$$

respectively. We propose to find the systems of partial differential equations determined by $Y^{(1)}$, $Y^{(2)}$, $Y^{(3)}$ and $Z^{(1)}$, $Z^{(2)}$, $Z^{(3)}$ respectively.

We find by differentiation

$$\begin{aligned} Y_u &= aY + bZ, \\ Y_v &= a'Y + b'Z - W, \\ Z_u &= a'Y + b'Z + W, \\ Z_v &= a''Y + b''Z, \end{aligned} \tag{103}$$

where

$$W = |y_u, y_v|. \tag{104}$$

We find further

$$\begin{aligned} W_u &= (a + b')W - c'Y + cZ, \\ W_v &= (a' + b'')W - c''Y + c'Z. \end{aligned} \tag{105}$$

Consequently we shall have

$$\begin{aligned} Y_{uu} &= (a_u + a^2 + a'b)Y + (b_u + ab + bb')Z + bW, \\ Y_{uv} &= (a_v + aa' + a''b)Y + (b_v + ab' + bb'')Z - aW \\ &= (a'_u + aa' + a'b' + c')Y + (b'_u + a'b + b'^2 - c)Z - aW, \\ Y_{vv} &= (a'_v + a'^2 + a''b' + c'')Y + (b'_v + a'b' + b'b'' - c')Z - (2a' + b'')W, \end{aligned} \tag{106}$$

the two values of Y_{uv} being identical on account of (5), and

$$\begin{aligned} Z_{uu} &= (a'_u + a'^2 + a'b' - c')Y + (b'_u + b'^2 + a'b + c)Z + (2a + b')W, \\ Z_{uv} &= (a'_v + a'^2 + a''b' - c'')Y + (b'_v + a'b' + b'b'' + c')Z + b''W \\ &= (a''_u + aa'' + a'b'')Y + (b''_u + a''b + b'b'')Z + b''W, \\ Z_{vv} &= (a''_v + a'a'' + a''b'')Y + (b''_v + a''b' + b''^2)Z - a''W. \end{aligned} \tag{107}$$

From (103) we have

$$\begin{aligned} bZ &= -aY + Y_u, \qquad a''Y = -b''Z + Z_v, \\ bW &= (a'b - ab')Y + b'Y_u - bY_v, \\ a''W &= (a'b'' - a''b')Z + a''Z_u - a'Z_v, \end{aligned} \tag{108}$$

which values substituted in (106) and (107) will give the desired two systems of partial differential equations. We find in this way:

$$Y_{uu} = \left(a + 2b' + \frac{b_u}{b}\right)Y_u - bY_v + \left(a_u - a\frac{b_u}{b} + 2a'b - 2ab'\right)Y,$$

$$Y_{uv} = \left(b'' + \frac{b_v}{b}\right) Y_u + a Y_v + \left(a_v - a\frac{b_v}{b} + a''b - ab''\right) Y,$$

$$Y_{vv} = \frac{1}{b}(b'_v - c' - a'b')\, Y_u + (2a' + b'')\, Y_v$$
$$+ \frac{1}{b}\left[ba'_v - ab'_v + b'(a''b - ab'') + bc'' + ac' + (a' + b'')(ab' - a'b)\right] Y.$$

If we assume that the system (4) *is written in its canonical form, we may write more simply*

$$Y_{uu} = \left(B' + \frac{B_u}{B}\right) Y_u - B Y_v + \left(-B'_u + B'\frac{B_u}{B} + 2A'B + 2B'^2\right) Y,$$

$$(109)\quad Y_{uv} = \left(-A' + \frac{B_v}{B}\right) Y_u - B' Y_v + \left(-B'_v + B'\frac{B_v}{B} + A''B - A'B'\right) Y,$$

$$Y_{vv} = -\frac{K}{B} Y_u + A' Y_v + \left(C'' + A'_v + A''B' + \frac{B'K}{B}\right) Y,$$

and similarly

$$Z_{uu} = B' Z_u - \frac{H}{A''} Z_v + \left(C' + B'_u + A'B - \frac{A'H}{A''}\right) Z,$$

$$(110)\quad Z_{uv} = -A' Z_u + \left(-B' + \frac{A''_u}{A''}\right) Z_v + \left(-A'_u + A'\frac{A''_u}{A''} + A''B - A'B'\right) Z,$$

$$Z_{vv} = -A'' Z_u + \left(A' + \frac{A''_v}{A''}\right) Z_v + \left(-A'_v + A'\frac{A''_v}{A''} + 2A''B' + 2A'^2\right) Z,$$

where, of course, A'' and B are assumed to be different from zero.

Neither of these systems is in its canonical form. According to the developments of section 2 we may reduce them to their canonical form by the substitutions

$$(111)\qquad Y = \sqrt[3]{B}\,\bar{Y}, \qquad Z = \sqrt[3]{A''}\,\bar{Z}.$$

Let us denote the coefficients of the new systems, i. e., the seminvariants of (109) and (110) by $\alpha^{(u)}$, $\beta^{(u)}$, $\gamma^{(u)}$, $\cdots$ $\gamma^{(u)''}$ and $\alpha^{(v)} \cdots \gamma^{(v)''}$ respectively. Then we shall have

$$\alpha^{(u)} = B' + \frac{1}{3}\frac{B_u}{B}, \qquad \alpha^{(u)'} = -A' + \frac{2}{3}\frac{B_v}{B}, \qquad \alpha^{(u)''} = -\frac{K}{B},$$

$$\beta^{(u)} = -B, \qquad \beta^{(u)'} = -B' - \frac{1}{3}\frac{B_u}{B}, \qquad \beta^{(u)''} = A' - \frac{2}{3}\frac{B_v}{B},$$

$$(112)\quad \gamma^{(u)} = 2A'B + 2B'^2 + \frac{4}{3}B'\frac{B_u}{B} - B'_u - \tfrac{1}{3}B_v + \frac{5}{9}\frac{B_u^2}{B^2} - \frac{1}{3}\frac{B_{uu}}{B},$$

$$\gamma^{(u)'} = A''B - A'B' - \tfrac{1}{3}A'\frac{B_u}{B} + \frac{2B'}{3}\frac{B_v}{B} - B'_v + \frac{5}{9}\frac{B_uB_v}{B^2} - \frac{1}{3}\frac{B_{uv}}{B},$$

$$\gamma^{(u)''} = C'' + A''B' + \frac{B'K}{B} + \tfrac{1}{3}A'\frac{B_v}{B} - \frac{1}{3}\frac{K}{B}\frac{B_u}{B} + A'_v + \frac{2}{9}\frac{B_v^2}{B^2} - \frac{1}{3}\frac{B_{vv}}{B};$$

and

$$\alpha^{(v)} = B' - \frac{2}{3}\frac{A''_u}{A''}, \qquad \alpha^{(v)'} = -A' - \frac{1}{3}\frac{A''_v}{A''}, \qquad \alpha^{(v)''} = -A'',$$

$$\beta^{(v)} = -\frac{H}{A''}, \qquad \beta^{(v)'} = -B' + \frac{2}{3}\frac{A''_u}{A''}, \qquad \beta^{(v)''} = A' + \frac{1}{3}\frac{A''_v}{A''},$$

$$(113)\quad \gamma^{(v)} = C + A'B - \frac{A'H}{A''} + \tfrac{1}{3}B'\frac{A''_u}{A''} - \frac{1}{3}\frac{H}{A''}\frac{A''_v}{A''} + B'_u + \frac{2}{9}\left(\frac{A''_u}{A''}\right)^2 - \frac{1}{3}\frac{A''_{uu}}{A''},$$

$$\gamma^{(v)'} = A''B - A'B' + \tfrac{2}{3}A'\frac{A''_u}{A''} - \tfrac{1}{3}B'\frac{A''_v}{A''} - A'_u + \frac{5}{9}\frac{A''_uA''_v}{(A'')^2} - \frac{1}{3}\frac{A''_{uv}}{A''},$$

$$\gamma^{(v)''} = 2A''B' + 2(A')^2 + \tfrac{4}{3}A'\frac{A''_v}{A''} - \tfrac{1}{3}A''_u - A'_v + \frac{5}{9}\left(\frac{A''_v}{A''}\right)^2 - \frac{1}{3}\frac{A''_{vv}}{A''}.$$

These equations will obviously be of service whenever we wish to characterize a net by properties of the tangents of its curves. By means of (109) the lines of the plane are assembled in two different ways; those for which v has a constant value c are tangent to the curve $v = c$ of our original net; those for which u has a constant value are tangent to a curve of the first Laplacian transformed net. By means of equations (110) the lines of the plane are grouped in a similar way with reference to the curves $u =$ const. of the original net and the minus first Laplacian transformed net.

In this whole theory we might, of course, adopt a dual interpretation. Instead of a net we should then have two one-parameter families of curves so related that, for every line of a certain domain, there exists one curve of each family tangent to it. We have seen instances of such a relation in the theory of the Laplacian transformation.

THE UNIVERSITY OF CHICAGO,
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NOTES AND ERRATA, VOLUMES 10, 11.

Volume 10.

Page 315. W. B. Fite: *Irreducible homogeneous linear groups in an arbitrary domain.*

The results of this article are embodied in the theorem, *A necessary and sufficient condition that any group of finite order be simply isomorphic with an irreducible group in any domain is that its central be cyclic.* But that this condition is not sufficient is shown by the following example to which Professor Burnside has kindly called my attention, namely the group defined by the operations A, P, Q, and R with the following relations:
$P^7 = 1$, $Q^7 = 1$, $R^n = 1$ (n any positive integer), $A^3 = 1$, $A^{-1}PA = P^2$, $A^{-1}QA = Q^2$, $A^{-1}RA = R$, and with P, Q, and R commutative among themselves.

The argument rests upon the supposed fact that $r_1 > 1$, and I can now see no reason why it might not equal unity.

I have been unable to determine what is both a sufficient and a necessary condition that a group of finite order may be simply isomorphic with an irreducible group. It is easy however to give sufficient conditions that are not necessary ones; for example, in order that a group of finite order be simply isomorphic with an irreducible group it is *sufficient* that its direct product with some group of finite order be simply isomorphic with an irreducible group.

Volume 11.

Page 489. G. E. Wahlin: *On the base of a relative number field, with an application to the composition of fields.*

The assertion that $O_{\tau\tau}^{(1)}$ is prime to I_τ is not true for every ϑ. I wish therefore to replace lines 7–19 of p. 489 by the following.

Since $\omega_{\tau\tau}^{(1)} = \bar{\omega}_\tau / D_k(\vartheta)$ is an integer in K, $\bar{\omega}_\tau$ is divisible by $D_k(\vartheta)$, and consequently also by O_τ. That is,

$$\bar{\omega}_\tau = O_{1\tau}^{(1)} + O_{2\tau}^{(1)}\vartheta + \cdots + O_{\tau\tau}^{(1)}\vartheta^{\tau-1} \equiv 0\,(O_\tau).$$

From the fact,* that if J and J_1 are two ideals in a number field, then there exists an ideal J' such that JJ' is a principal ideal and J' is prime to J_1, we know that there exists in k an ideal J_τ, prime to $[D_k(\vartheta)]$ and such that

* Dirichlet: *Vorlesungen über Zahlentheorie*, vierte Auflage, p. 559, Theorem XI.

$O_\tau \cdot J_\tau = (\alpha)$, a principal ideal. In the same way there exists an ideal J'_τ, prime to $[D_k(\vartheta)]$, such that $J_\tau \cdot J'_\tau = (\beta)$, a principal ideal, and since J_τ and J'_τ are both prime to $D_k(\vartheta)$, β is prime to $D_k(\vartheta)$.

Since $(\beta) = J_\tau \cdot J'_\tau$ and $(\alpha) = O_\tau \cdot J_\tau$ and since $\bar{\omega}_\tau$ is divisible by O_τ, it follows that $\beta\bar{\omega}_\tau$ is divisible by (α), and must therefore be a multiple of α, and

$$\frac{\beta\bar{\omega}_\tau}{\alpha} = \frac{\beta O^{(1)}_{1\tau} + \beta O^{(1)}_{2\tau}\vartheta + \cdots + \beta O^{(1)}_{\tau\tau}\vartheta^{\tau-1}}{\alpha}$$

is an integer in K.

But since $\bar{\omega}_\tau$ and $O^{(1)}_{\tau\tau}$ are both divisible by O_τ it follows that

$$X = O^{(1)}_{1\tau} + O^{(1)}_{2\tau}\vartheta + \cdots + O^{(1)}_{\tau-1,\tau}\vartheta^{\tau-2} = \bar{\omega}_\tau - O^{(1)}_{\tau\tau}\vartheta^{\tau-1}$$

must be divisible by O_τ and therefore, in the same way as above, we see that $\beta X/\alpha$ is an integer in K, and $R = \beta O^{(1)}_{\tau\tau}/\alpha$ an integer in k. We can now write

$$\frac{\beta\bar{\omega}_\tau}{\alpha} = \frac{\beta X + \beta O^{(1)}_{\tau\tau}\vartheta^{\tau-1}}{\alpha} = \frac{\beta X}{\alpha} + R\vartheta^{\tau-1}$$

and since O_τ is the highest common factor of $O^{(1)}_{\tau\tau}$ and $D_k(\vartheta)$, the ideals $O^{(1)}_{\tau\tau}/O_\tau$ and $I_\tau = D_k(\vartheta)/O_\tau$ are relatively prime, and since β is prime to I_τ, it follows that R is prime to I_τ and there exists in k an integer R' such that

$$R'R \equiv 1\,(I_\tau).$$

We have therefore

$$\frac{R'\beta\bar{\omega}_\tau}{\alpha} = \frac{R'\beta X}{\alpha} + R'R\vartheta^{\tau-1} \equiv \bar{\Omega}_\tau\,(I_\tau),$$

where

$$\bar{\Omega}_\tau = \frac{R'\beta O^{(1)}_{1\tau} + R'\beta O^{(1)}_{2\tau}\vartheta + \cdots + R'\beta O^{(1)}_{\tau-1\tau}\vartheta^{\tau-2}}{\alpha} + \vartheta^{\tau-1}.$$

Since $\bar{\omega}_\tau$ is divisible by $D_k(\vartheta)$, and α contains no other factor of $D_k(\vartheta)$ than O_τ, we see that the integer $\beta\bar{\omega}_\tau/\alpha$ must be divisible by I_τ and from the last congruence we have

$$\bar{\Omega}_\tau \equiv O\,(I_\tau).$$

While $\Omega_\tau = \bar{\Omega}_\tau/D_k(\vartheta)$ is not necessarily an integer in K, the products $\Omega_\tau \cdot O^{(1)}_{\tau\tau}$ and $\Omega_\tau \cdot D_k(\vartheta)$ are both integers in K. Moreover, since β is divisible by J_τ and since $D_k(\vartheta)$ and $O^{(1)}_{\tau\tau}$ are both divisible by O_τ, the coefficients $R'\beta O^{(1)}_{\tau\tau}$ when multiplied by $O^{(1)}_{\tau\tau}$ or $D_k(\vartheta)$ will be divisible by α and the integers $\Omega_\tau \cdot D_k(\vartheta)$ and $\Omega_\tau \cdot O^{(1)}_{\tau\tau}$ are therefore of the form $\omega^{(i)}_\tau$. The number Ω_τ thus found can therefore take the place of the Ω_τ in my original paper.

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